Republic of Iraq
Ministry of Higher Education
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Department of Physics

Electricity

(Lectures)

First stage

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Electricity

Chapter 1

Properties of Electric Charges

A number of simple experiments demonstrate the existence of electric forces and charges. For example, after running a comb through your hair on a dry day, you will find that the comb attracts bits of paper. The attractive force is often strong enough to suspend the paper. The same effect occurs when certain materials are rubbed together, such as glass rubbed with silk or rubber with fur.

Another simple experiment is to rub an inflated balloon with wool. The balloon then adheres to a wall, often for hours. When materials behave in this way, they are said to be ***electrified,*** or to have become electrically charged. You can easily electrify your body by vigorously rubbing your shoes on a wool rug. Evidence of the electric charge on your body can be detected by lightly touching (and startling) a friend.

Under the right conditions, you will see a spark when you touch, and both of you will feel a slight tingle. (Experiments such as these work best on a dry day because an excessive amount of moisture in the air can cause any charge you build up to “leak” from your body to the Earth.) **there are two kinds of electric charges, which were given the names positive and negative**

We identify negative charge as that type possessed by electrons and positive charge as that possessed by protons. To verify that there are two types of charge, suppose a hard rubber rod that has been rubbed with fur is suspended by a sewing thread, as shown in Figure 23.1. When a glass rod that has been rubbed with silk is brought near the rubber rod, the two attract each other (Fig. 23.1a). On the other hand, if two charged rubber rods (or two charged glass rods) are brought near each other, as shown in Figure 23.1b, the two repel each other. This observation shows that the rubber and glass have two different types of charge on them. On the basis of these observations, we conclude that charges of the same sign repel one another and charges with opposite signs attract one another.

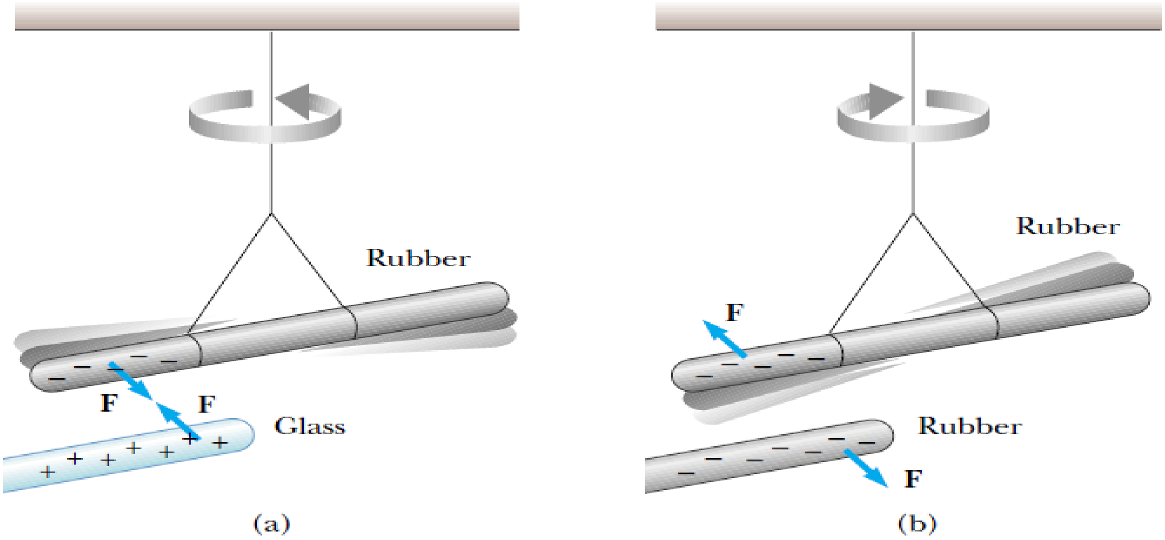


Figure 23.1 (a) A negatively charged rubber rod suspended by a thread is attracted to a positively charged glass rod. (b) A negatively charged rubber rod is repelled by another negatively charged rubber rod.

In addition to the existence of two types of charge, several other properties of charge have been discovered

• **Charge is quantized.** This means that electric charge comes in discrete amounts, and there is a smallest possible amount of charge that an object can have. In the SI system, this smallest amount is $e \equiv 1.602 \times 10{-}19$ C . No free particle can have less charge than this, and, therefore, the charge on any object—the charge on all objects—must be an integer multiple of this amount. All macroscopic, charged objects have charge because electrons have either been added or taken away from them, resulting in a net charge.

• **The magnitude of the charge is independent of the type.** Phrased another way, the smallest possible positive charge (to four significant figures) is $+1.602 \times 10{-}19$ C , and the smallest possible negative charge is $-1.602 \times 10{-}19$ C ; these values are exactly equal. This is simply how the laws of physics in our universe turned out.

• **Charge is conserved.** Charge can neither be created nor destroyed; it can only be transferred from place to place, from one object to another. Frequently, we speak of two charges "canceling"; this is verbal shorthand. It means that if two objects that have equal and opposite charges are physically close to each other, then the (oppositely directed) forces they apply on some other charged object cancel, for a net force of zero. It is important that you understand that the charges on the objects by no means disappear, however. The net charge of the universe is constant.

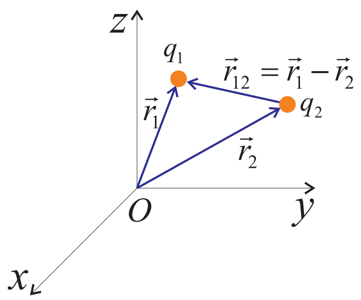
• **Charge is conserved in closed systems.** In principle, if a negative charge disappeared from your lab bench and reappeared on the Moon, conservation of charge would still hold. However, this never happens. If the total charge you have in your local system on your lab bench is changing, there will be a measurable flow of charge into or out of the system. Again, charges can and do move around, and their effects can and do cancel, but the net charge in your local environment (if closed) is conserved. The last two items are both referred to as the **law of conservation of charge**

Chapter 2

Electric Force & Electric Field

2.1 Electric Force

The electric force between two **charges** $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ can be described by **Coulomb's Law**.



$\vec{F}_{12} = Force$ on $\boldsymbol{q_1}$ exerted by $\boldsymbol{q_2}$

$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r_{12}^2} \cdot \hat{r}_{12}}$$

where $\hat{r}_{12} = \dfrac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the *unit vector* which locates particle 1 relative to particle 2.

i.e. $\boxed{\vec{r}_{12} = \vec{r}_1 - \vec{r}_2}$

- q_1, q_2 are electrical charges in units of *Coulomb*(C)
- Charge is *quantized*
  Recall 1 electron carries $1.602 \times 10^{-19} C$
- ϵ_0 = Permittivity of free space $= 8.85 \times 10^{-12} C^2/Nm^2$

<u>COULOMB'S LAW:</u>

(1) q_1, q_2 can be either positive or negative.

(2) If q_1, q_2 *are of same sign*, then the force experienced by q_1 is in direction *away from* q_2, that is, *repulsive*.

(3) Force on q_2 exerted by q_1:

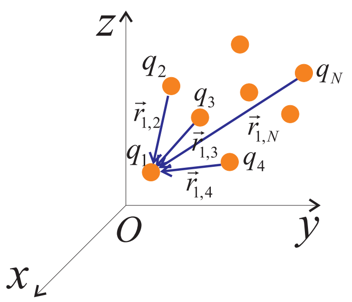
$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_2 q_1}{r_{21}^2} \cdot \hat{r}_{21}$$

BUT:

$$r_{12} = r_{21} = \text{distance between } q_1, q_2$$
$$\hat{r}_{21} = \frac{\vec{r}_{21}}{r_{21}} = \frac{\vec{r}_2 - \vec{r}_1}{r_{21}} = \frac{-\vec{r}_{12}}{r_{12}} = -\hat{r}_{12}$$

$$\therefore \boxed{\vec{F}_{21} = -\vec{F}_{12} \text{ Newton's 3rd Law}}$$

SYSTEM WITH MANY CHARGES:



The total force experienced by charge q_1 is the *vector sum* of the forces on q_1 exerted by other charges.

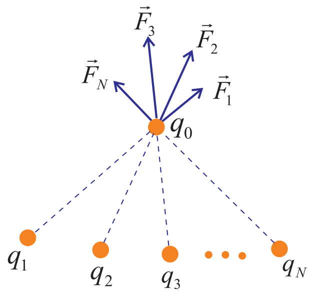
$$\begin{aligned} \vec{F}_1 &= \text{Force experienced by } q_1 \\ &= \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \cdots + \vec{F}_{1,N} \end{aligned}$$

PRINCIPLE OF SUPERPOSITION:

$$\vec{F}_1 = \sum_{j=2}^{N} \vec{F}_{1,j}$$

2.2 The Electric Field

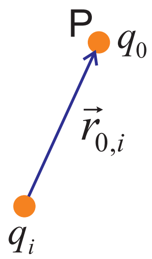
While we need two charges to quantify the **electric force**, we define the **electric field** for any single charge distribution to describe its effect on other charges.



Total force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$
The **electric field** is defined as

$$\boxed{\lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \vec{E}}$$

(a) E-field due to a single charge q_i:



From the definitions of **Coulomb's Law**, the force experienced at location of q_0 (point P)

$$\vec{F}_{0,i} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_0 q_i}{r_{0,i}^2} \cdot \hat{r}_{0,i}$$

where $\hat{r}_{0,i}$ is the unit vector along the direction *from charge* q_i *to* q_0,

$$\begin{aligned} \hat{r}_{0,i} &= \text{Unit vector from charge } q_i \text{ to point P} \\ &= \hat{r}_i \text{ (radical unit vector from } q_i) \end{aligned}$$

Recall $\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$
$\therefore$ E-field due to q_i at point P:

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_i}{r_i^2} \cdot \hat{r}_i$$

where $\vec{r}_i$ = Vector pointing from q_i to point P,
thus $\hat{r}_i$ = Unit vector pointing from q_i to point P
Note:

(1) E-field is a **vector**.

(2) Direction of E-field depends on **both** position of P and sign of q_i.

(b) E-field due to system of charges:

Principle of Superposition:
In a system with N charges, the **total** E-field due to all charges is the **vector sum** of E-field due to individual charges.

i.e. $$\boxed{\vec{E} = \sum_i \vec{E}_i = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r}_i}$$

(c) Electric Dipole

System of *equal and opposite* charges separated by a distance d.

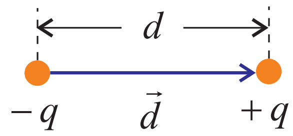
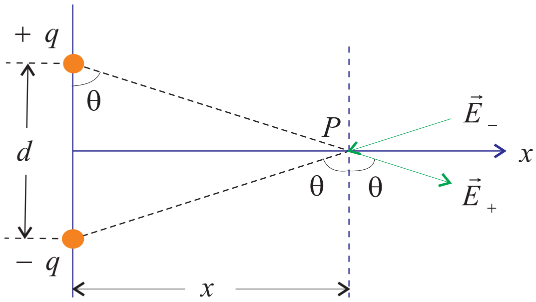


Figure 2.1: An electric dipole. (Direction of $\vec{d}$ from negative to positive charge)

Electric Dipole Moment
$$\vec{p} = q\vec{d} = qd\hat{d}$$

$$\boxed{p = qd}$$

<u>Example:</u> $\vec{E}$ due to dipole along x-axis



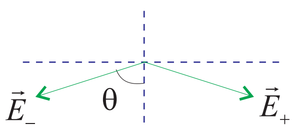
Consider point P at distance x along the perpendicular axis of the dipole $\vec{p}$:

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$\vec{E}_+$ ↑ (E-field due to $+q$)

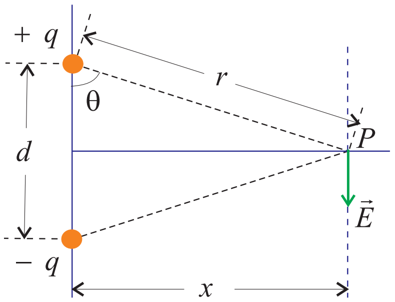
$\vec{E}_-$ ↑ (E-field due to $-q$)

Notice: Horizontal E-field components of $\vec{E}_+$ and $\vec{E}_-$ cancel out.



∴ Net E-field points along the axis opposite to the dipole moment vector.

Magnitude of E-field $= 2E_+ \cos\theta$



$$E_+ \text{ or } E_- \text{ magnitude!}$$

$$\therefore E = 2\left(\overbrace{\frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}}\right)\cos\theta$$

$$\text{But} \quad r = \sqrt{\left(\frac{d}{2}\right)^2 + x^2}$$

$$\cos\theta = \frac{d/2}{r}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0} \cdot \frac{p}{[x^2 + (\frac{d}{2})^2]^{\frac{3}{2}}}$$

$$(p = qd)$$

Special case: When $x \gg d$

$$[x^2 + (\frac{d}{2})^2]^{\frac{3}{2}} = x^3[1 + (\frac{d}{2x})^2]^{\frac{3}{2}}$$

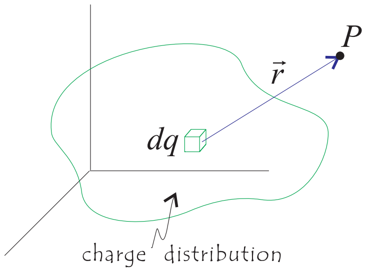
- Binomial Approximation:

$$(1+y)^n \approx 1 + ny \qquad \text{if } y \ll 1$$

$$\boxed{\text{E-field of dipole} \doteqdot \frac{1}{4\pi\epsilon_0} \cdot \frac{p}{x^3} \sim \frac{1}{x^3}}$$

- Compare with $\frac{1}{r^2}$ E-field for single charge
- Result also valid for point P along any axis with respect to dipole

2.3 Continuous Charge Distribution



E-field at point P due to dq:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \hat{r}$$

$\therefore$ E-field due to charge distribution:

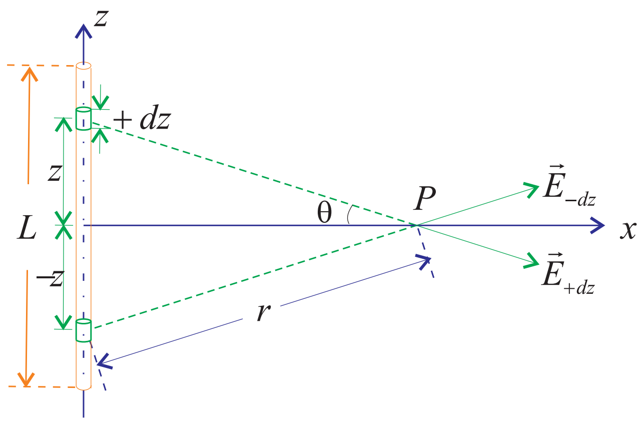
$$\vec{E} = \int_{Volume} d\vec{E} = \int_{Volume} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \hat{r}$$

(1) In many cases, we can take advantage of the *symmetry* of the system to simplify the integral.

(2) To write down the small charge element dq:

1-D	$dq = \lambda\, ds$	λ = linear charge density	ds = small length element
2-D	$dq = \sigma\, dA$	σ = surface charge density	dA = small area element
3-D	$dq = \rho\, dV$	ρ = volume charge density	dV = small volume element

Example 1: Uniform line of charge

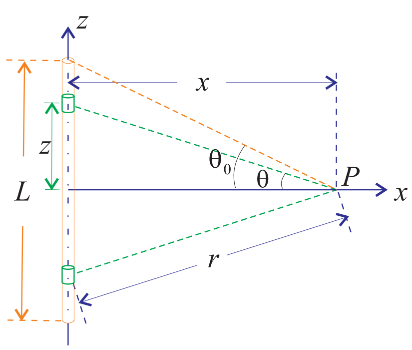


charge per unit length $= \lambda$

(1) Symmetry considered: The E-field from $+z$ and $-z$ directions *cancel along z-direction*, $\therefore$ Only horizontal E-field components need to be considered.

(2) For each element of length dz, charge $dq = \lambda dz$
$\therefore$ Horizontal E-field at point P due to element dz =

$$|d\vec{E}|\cos\theta = \underbrace{\frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2}}_{dE_{dz}} \cos\theta$$



$\therefore$ E-field due to entire line charge at point P

$$\begin{aligned} E &= \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2} \cos\theta \\ &= 2\int_0^{L/2} \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{dz}{r^2} \cos\theta \end{aligned}$$

To calculate this integral:

- First, notice that x is fixed, but $z,\ r,\ \theta$ all varies.
- Change of variable (from z to θ)

(1) $z = x\tan\theta \quad \therefore dz = x\sec^2\theta\ d\theta$
$x = r\cos\theta \quad \therefore r^2 = x^2\sec^2\theta$

(2) When $z = 0\ ,\quad \theta = 0^\circ$
$z = L/2 \quad \theta = \theta_0 \quad$ where $\tan\theta_0 = \dfrac{L/2}{x}$

$$
\begin{aligned}
E &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{x\sec^2\theta\ d\theta}{x^2\sec^2\theta}\cdot\cos\theta \\
&= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{1}{x}\cdot\cos\theta\ d\theta \\
&= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot(\sin\theta)\Big|_0^{\theta_0} \\
&= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\sin\theta_0 \\
&= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\frac{L/2}{\sqrt{x^2+(\frac{L}{2})^2}}
\end{aligned}
$$

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x\sqrt{x^2+(\frac{L}{2})^2}}} \qquad \text{along } x\text{-direction}$$

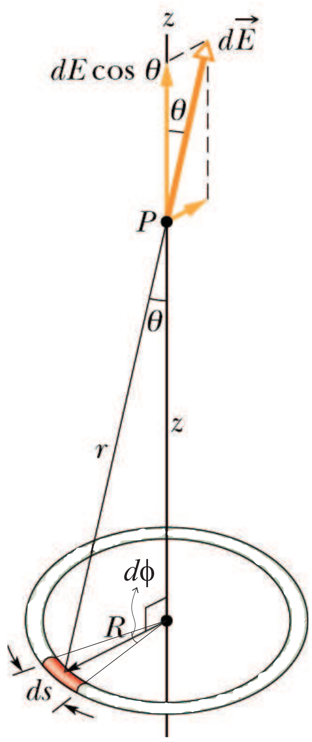
Important limiting cases:

1. $x \gg L: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x^2}$
   But λL = Total charge on rod
   $\therefore$ System behave like a point charge
2. $L \gg x: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x\cdot\frac{L}{2}}$

$$\boxed{E_x = \frac{\lambda}{2\pi\epsilon_0 x}}$$

ELECTRIC FIELD DUE TO INFINITELY LONG LINE OF CHARGE

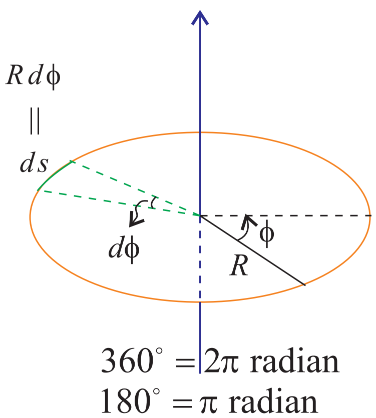
Example 2: Ring of Charge



E-field at a height z above a ring of charge of radius R

(1) Symmetry considered: For every charge element dq considered, there exists $dq\prime$ where the horizontal $\vec{E}$ field components cancel.
$\Rightarrow$ Overall E-field lies along z-direction.

(2) For each element of length dz, charge



$dq \quad = \quad \lambda \quad \cdot \quad ds$

λ ↑ Linear charge density

ds ↑ Circular length element

$dq = \lambda \cdot R\, d\phi$, where ϕ is the angle measured on the ring plane

$\therefore$ Net E-field along z-axis due to dq:

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \cos\theta$$

$$\begin{aligned}\text{Total E-field} &= \int dE \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\phi}{r^2} \cdot \cos\theta \qquad (\cos\theta = \frac{z}{r})\end{aligned}$$

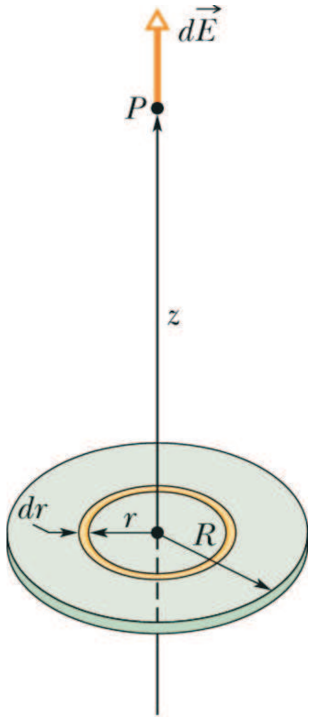
Note: Here in this case, θ, R and r are *fixed* as ϕ varies! BUT we want to convert r, θ to R, z.

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R z}{r^3} \int_0^{2\pi} d\phi$$

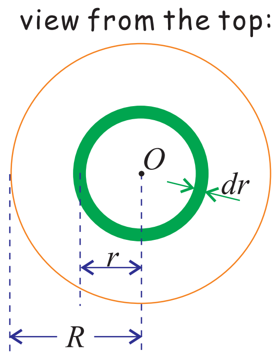
$$\boxed{E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}} \qquad \text{along } z\text{-axis}$$

BUT: $\lambda(2\pi R)$ = total charge on the ring

Example 3: E-field from a disk of surface charge density σ



We find the E-field of a disk by integrating concentric rings of charges.



Total charge of ring

$$dq = \sigma \cdot (\underbrace{2\pi r\, dr}_{\text{Area of the ring}})$$

Recall from Example 2:

$$\text{E-field from ring: } dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq\, z}{(z^2 + r^2)^{3/2}}$$

$$\begin{aligned} \therefore \quad E &= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma r\, dr \cdot z}{(z^2 + r^2)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^R 2\pi\sigma z \frac{r\, dr}{(z^2 + r^2)^{3/2}} \end{aligned}$$

- Change of variable:

$$\begin{aligned} & u = z^2 + r^2 && \Rightarrow \quad (z^2 + r^2)^{3/2} = u^{3/2} \\ \Rightarrow \quad & du = 2r\, dr && \Rightarrow \quad r\, dr = \tfrac{1}{2} du \end{aligned}$$

- Change of integration limit:

$$\begin{cases} r = 0 & , \quad u = z^2 \\ r = R & , \quad u = z^2 + R^2 \end{cases}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0} \cdot 2\pi\sigma z \int_{z^2}^{z^2+R^2} \frac{1}{2}\, u^{-3/2} du$$

BUT: $$\int u^{-3/2} du = \frac{u^{-1/2}}{-1/2} = -2u^{-1/2}$$

$$\begin{aligned} \therefore \ E &= \frac{1}{2\epsilon_0} \sigma z \left(-u^{-1/2}\right)\Big|_{z^2}^{z^2+R^2} \\ &= \frac{1}{2\epsilon_0} \sigma z \left(\frac{-1}{\sqrt{z^2 + R^2}} + \frac{1}{z} \right) \end{aligned}$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2 + R^2}} \right]}$$

VERY IMPORTANT LIMITING CASE:

If $R \gg z$, that is if we have an infinite sheet of charge with charge density σ:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &\simeq \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{R}\right] \end{aligned}$$

$$\boxed{E \approx \frac{\sigma}{2\epsilon_0}}$$

E-field is normal to the charged surface

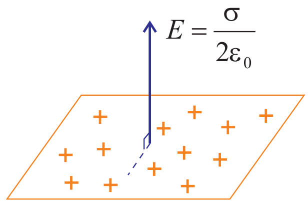
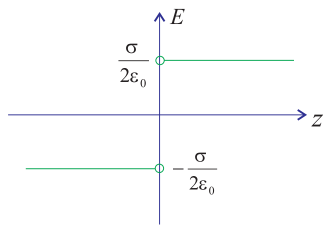


Figure 2.2: E-field due to an infinite sheet of charge, charge density = σ

Q: What's the E-field belows the charged sheet?

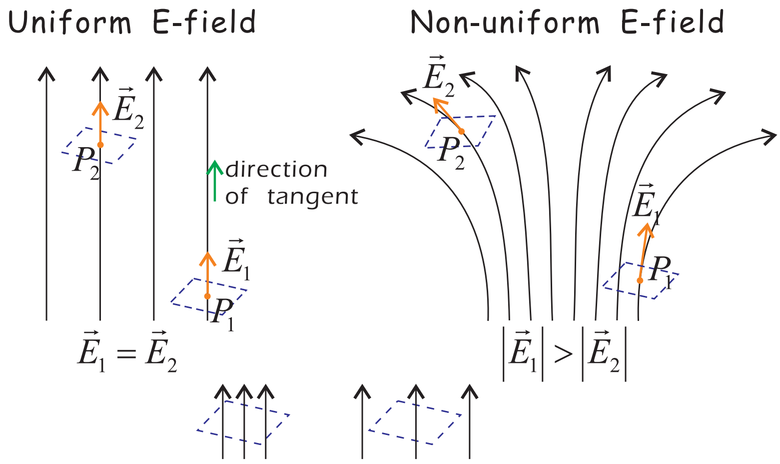


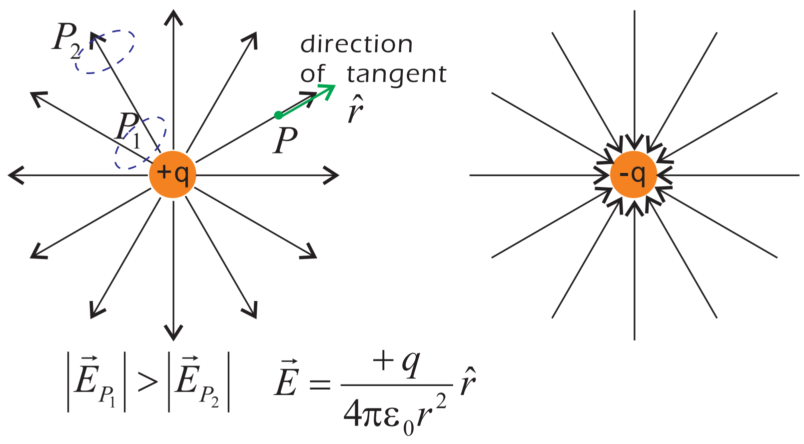
2.4 Electric Field Lines

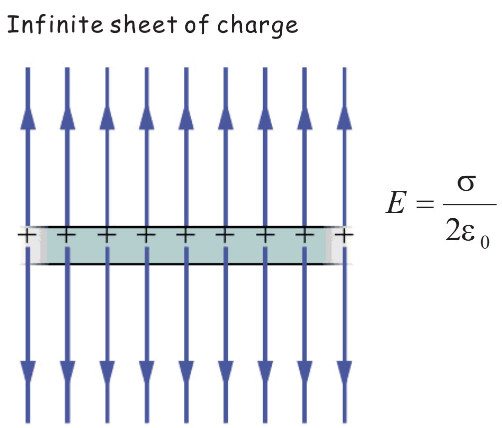
To visualize the electric field, we can use a graphical tool called the **electric field lines**.

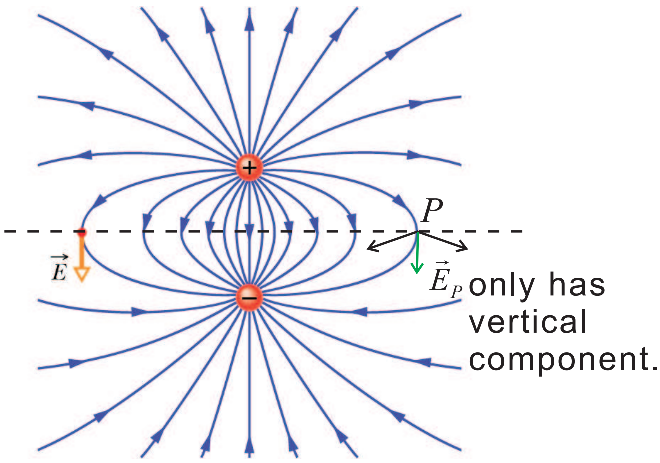
Conventions:

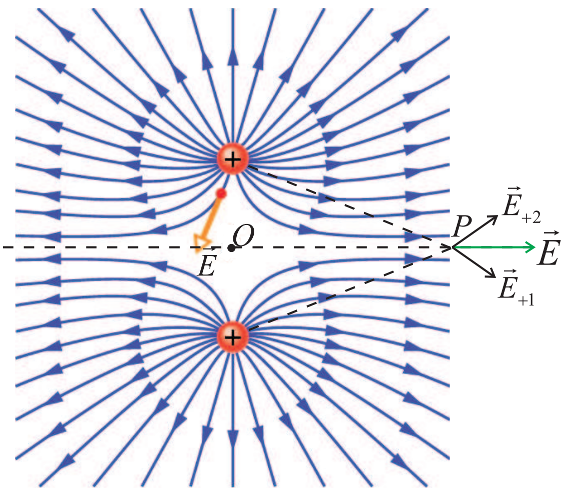
1. The start on position charges and end on negative charges.
2. *Direction* of E-field at any point is given by *tangent* of E-field line.
3. *Magnitude* of E-field at any point is proportional to *number of E-field lines per unit area perpendicular to the lines.*


Uniform E-field
Non-uniform E-field
direction
of tangent
$\vec{E}_1 = \vec{E}_2$
$|\vec{E}_1| > |\vec{E}_2|$


direction
of tangent
+q
-q
$|\vec{E}_{P_1}| > |\vec{E}_{P_2}|$
$\vec{E} = \frac{+q}{4\pi\varepsilon_0 r^2}\hat{r}$


Infinite sheet of charge
$E = \frac{\sigma}{2\varepsilon_0}$





$\vec{E}_{\text{at point } O} = 0$

2.5 Point Charge in E-field

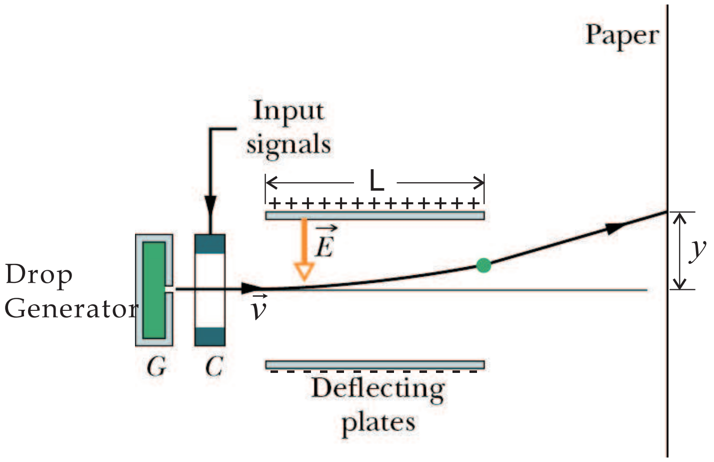
When we place a charge q in an E-field $\vec{E}$, the force experienced by the charge is

$$\boxed{\vec{F} = q\vec{E} = m\vec{a}}$$

Applications: *Ink-jet printer, TV cathoderay tube.*

Example:

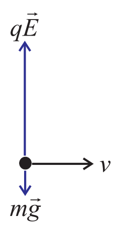
Ink particle has mass m, charge q ($q < 0$ here)



Assume that mass of inkdrop is small, what's the deflection y of the charge?

Solution:

First, the charge carried by the inkdrop is *negtive*, `i.e.` $q < 0$.



Note: $q\vec{E}$ points in opposite direction of $\vec{E}$.

Horizontal motion: Net force = 0

$$\therefore \; L = vt \tag{2.1}$$

Vertical motion: $|q\vec{E}| \gg |m\vec{g}|$, q is negative,

$$\therefore \text{ Net force} = -qE = ma \qquad \text{(Newton's 2nd Law)}$$

$$\therefore\ a = -\frac{qE}{m} \tag{2.2}$$

Vertical distance travelled:

$$y = \frac{1}{2}at^2$$

2.6 Dipole in E-field

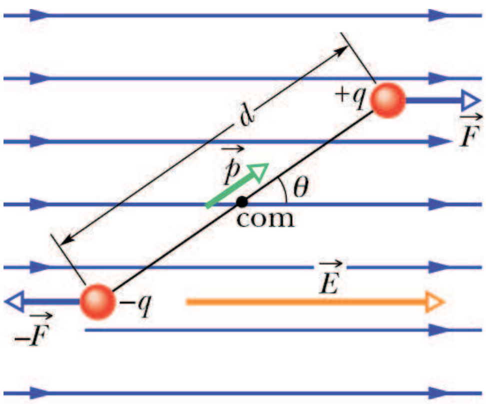
Consider the force exerted on the dipole in an *external* E-field:

Assumption: E-field from dipole doesn't affect the external E-field.

- Dipole moment:

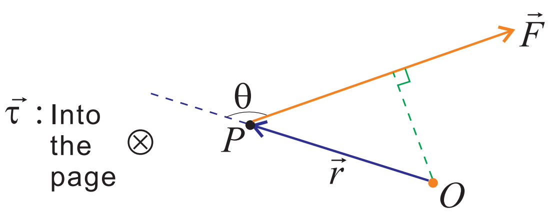
  $$\vec{p} = q\vec{d}$$

- Force due to the E-field on $+ve$ and $-ve$ charge are *equal and opposite in direction.* Total external force on dipole = 0.



BUT:	There is an external **torque** on the center of the dipole.

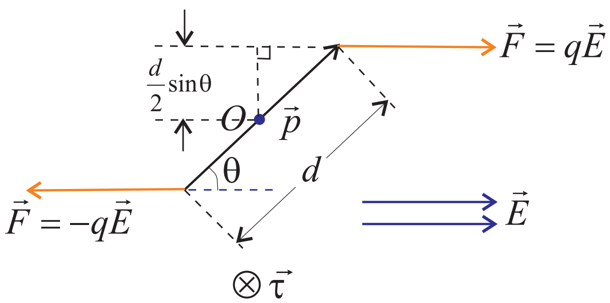
<u>Reminder</u>:



Force $\vec{F}$ exerts at point P. The force exerts a **torque** $\vec{\tau} = \vec{r} \times \vec{F}$ on point P with respect to point O.

Direction of the **torque vector** $\vec{\tau}$ is determined from the **right-hand rule**.

Reference: Halliday Vol.1 Chap 9.1 (Pg.175) *torque*
Chap 11.7 (Pg.243) *work done*



Net torque $\vec{\tau}$

- direction: clockwise torque
- magnitude:

$$\begin{aligned}\tau &= \tau_{+ve} + \tau_{-ve} \\ &= F \cdot \frac{d}{2}\sin\theta + F \cdot \frac{d}{2}\sin\theta \\ &= qE \cdot d\sin\theta \\ &= pE\sin\theta\end{aligned}$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

Energy Consideration:

When the dipole $\vec{p}$ rotates $d\theta$, the E-field does work.

Work done by external E-field on the dipole:

$$dW = -\tau\, d\theta$$

Negative sign here because torque by E-field acts to *decrease* θ.

BUT: Because E-field is a **conservative force field** [1] [2], we can define a **potential energy** (U) for the system, so that

$$\boxed{dU = -dW}$$

$\therefore$ For the dipole in external E-field:

$$dU = -dW = pE\sin\theta\, d\theta$$

$$\begin{aligned}\therefore\ U(\theta) = \int dU &= \int pE\sin\theta\, d\theta \\ &= -pE\cos\theta + U_0\end{aligned}$$

[1] more to come in Chap.4 of notes
[2] ref. Halliday Vol.1 Pg.257, Chap 12.1

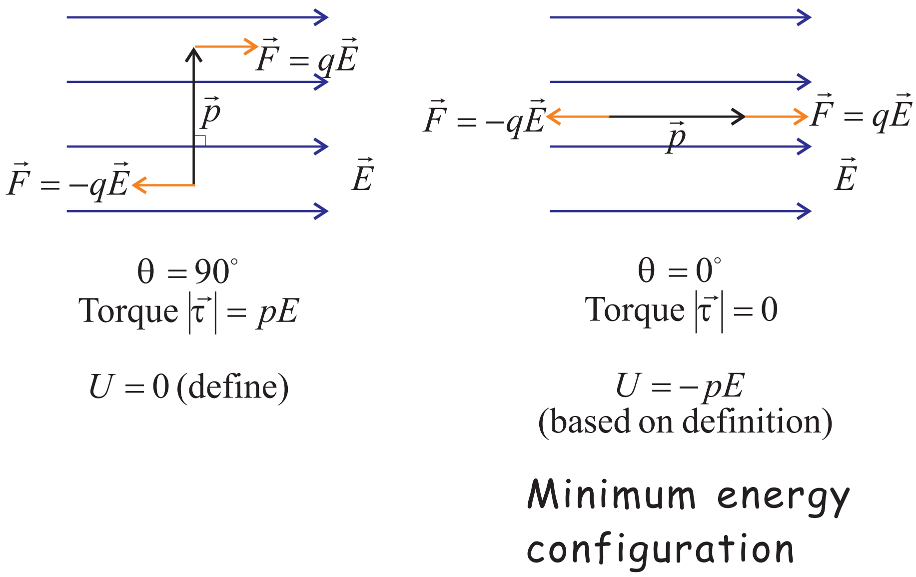
set $U(\theta = 90^\circ) = 0$,

$$\therefore\ 0 = -pE\cos 90^\circ + U_0$$

$$\therefore\ U_0 = 0$$

$\therefore$ Potential energy:

$$\boxed{U = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$



Chapter 3

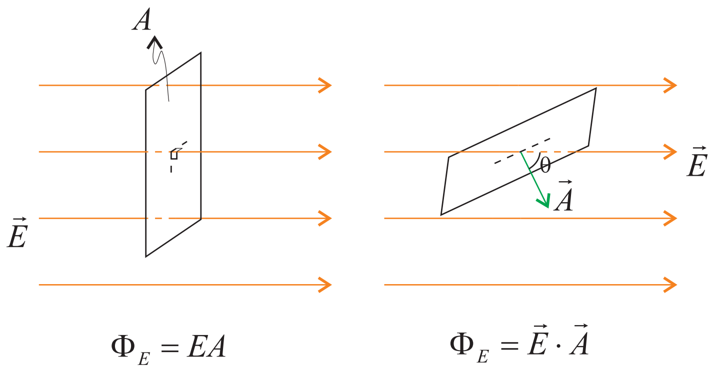
Electric Flux and Gauss' Law

3.1 Electric Flux

Latin: flux = "to flow"

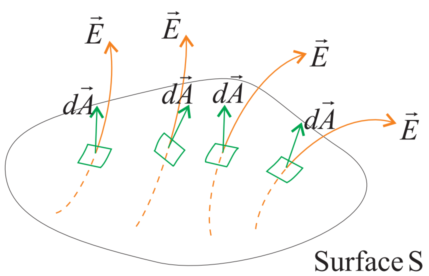
Graphically: Electric flux Φ_E represents the number of E-field lines crossing a surface.

Mathematically:



Reminder: Vector of the area $\vec{A}$ is perpendicular to the area A.

For non-uniform E-field & surface, direction of the area vector $\vec{A}$ is not uniform.



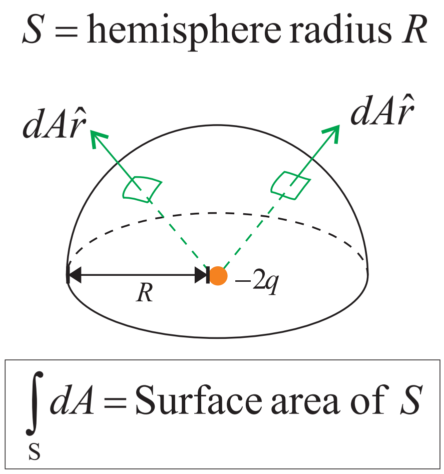
$d\vec{A}$ = Area vector for small area element dA

$$\therefore \text{ Electric flux} \qquad d\Phi_E = \vec{E} \cdot d\vec{A}$$

Electric flux of $\vec{E}$ through surface S: $\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{A}}$

$$\begin{aligned} \int_S &= \text{Surface integral over surface S} \\ &= \text{Integration of integral over all area elements on surface S} \end{aligned}$$

Example:

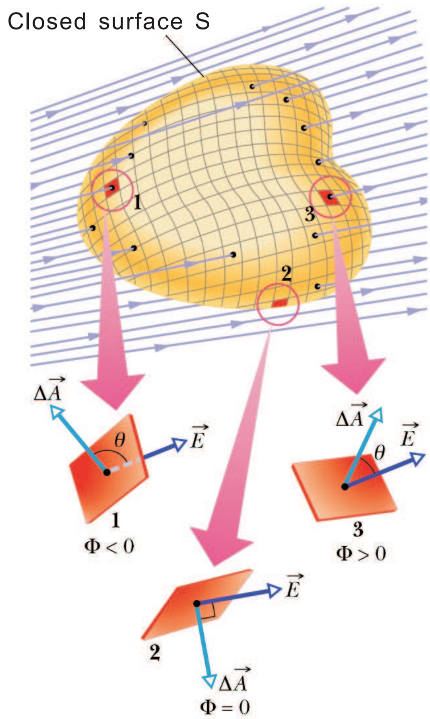


$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{-2q}{r^2}\,\hat{r} = \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}$$

For a hemisphere, $d\vec{A} = dA\,\hat{r}$

$$\begin{aligned} \Phi_E &= \int_S \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r} \cdot (dA\,\hat{r}) \qquad (\because \hat{r} \cdot \hat{r} = 1) \\ &= -\frac{q}{2\pi\epsilon_0 R^2} \underbrace{\int_S dA}_{2\pi R^2} \\ &= \frac{-q}{\epsilon_0} \end{aligned}$$

For a closed surface:

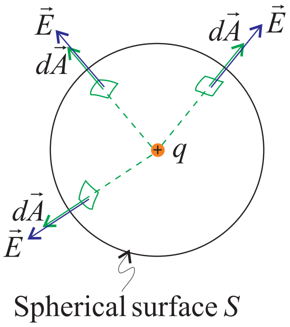


Recall: Direction of area vector $d\vec{A}$ goes *from inside to outside* of closed surface S.

Electric flux over closed surface S: $\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A}}$

$$\oint_S = \text{Surface integral over closed surface S}$$

Example:



Electric flux of charge q over closed spherical surface of radius R.

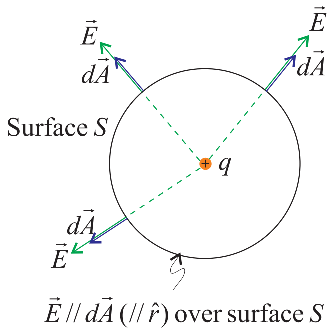
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\,\hat{r} = \frac{q}{4\pi\epsilon_0 R^2}\,\hat{r} \qquad \text{at the surface}$$

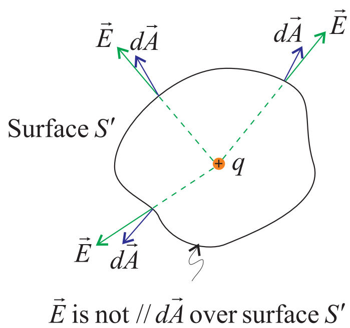
Again, $d\vec{A} = dA \cdot \hat{r}$

$$\begin{aligned} \therefore \Phi_E &= \oint_S \overbrace{\frac{q}{4\pi\epsilon_0 R^2}\,\hat{r}}^{\vec{E}} \cdot \overbrace{dA\,\hat{r}}^{d\vec{A}} \\ &= \frac{q}{4\pi\epsilon_0 R^2} \underbrace{\oint_S dA}_{\text{Total surface area of S} = 4\pi R^2} \\ \Phi_E &= \frac{q}{\epsilon_0} \end{aligned}$$

IMPORTANT POINT:

If we remove the spherical symmetry of closed surface S, *the total number of E-field lines crossing the surface remains the same.*
$\therefore$ The electric flux Φ_E





$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \oint_{S'} \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

3.2 Gauss' Law

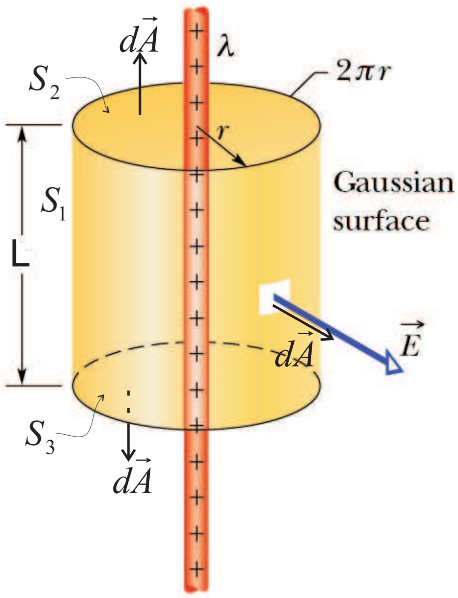
$$\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}} \quad \text{for any closed surface S}$$

And q is the net electric charge enclosed in closed surface S.

- Gauss' Law is valid for *all charge distributions and all closed surfaces. (Gaussian surfaces)*
- Coulomb's Law can be derived from Gauss' Law.
- For system with high order of *symmetry*, E-field can be easily determined if we construct *Gaussian surfaces with the same symmetry* and applies Gauss' Law

3.3 E-field Calculation with Gauss' Law

(A) Infinite line of charge



Linear charge density: λ
Cylindrical symmetry.
E-field directs radially outward from the rod.
Construct a Gaussian surface S in the shape of a **cylinder**, making up of a curved surface S_1, and the top and bottom circles S_2, S_3.

Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{\text{Total charge}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{A} = \underbrace{\int_{S_1} \vec{E}\cdot d\vec{A}}_{\vec{E}\parallel d\vec{A}} + \underbrace{\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A}}_{=0\ \because \vec{E}\perp d\vec{A}}$$

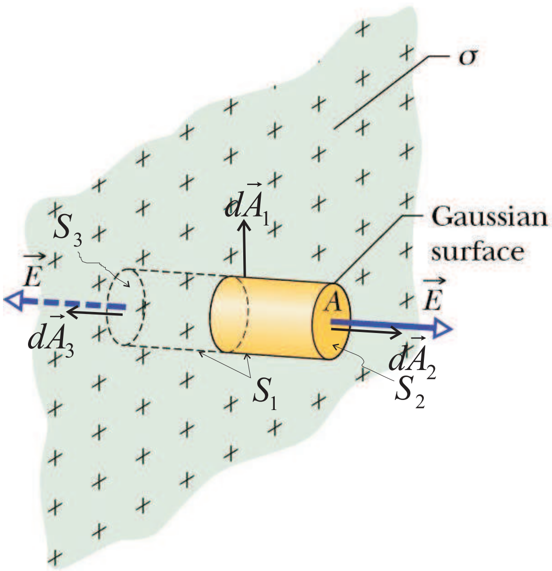
$$\therefore \quad E \underbrace{\int_{S_1} dA}_{\text{Total area of surface } S_1} = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\therefore \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}} \qquad \text{(Compare with Chapter 2 note)}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

(B) Infinite sheet of charge



Uniform surface charge density: σ
Planar symmetry.
E-field directs perpendicular to the sheet of charge.
Construct Gaussian surface S in the shape of a **cylinder (pill box)** of cross-sectional area A.

Gauss' Law: $$\oint_S \vec{E}\cdot d\vec{A} = \frac{A\sigma}{\epsilon_0}$$

$$\int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad \because \vec{E}\perp d\vec{A} \text{ over whole surface } S_1$$

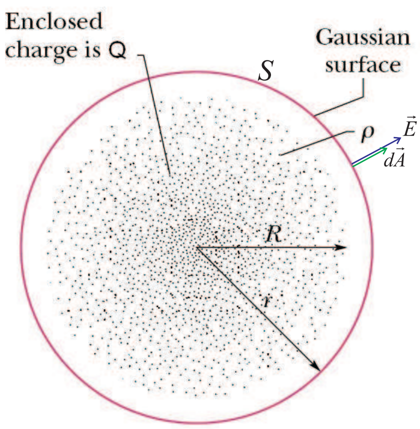
$$\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A} = 2EA \quad (\vec{E}\parallel d\vec{A}_2, \vec{E}\parallel d\vec{A}_3)$$

Note: For S_2, both $\vec{E}$ and $d\vec{A}_2$ point up
For S_3, both $\vec{E}$ and $d\vec{A}_3$ point down

$$\therefore \quad 2EA = \frac{A\sigma}{\epsilon_0} \quad \Rightarrow \quad \boxed{E = \frac{\sigma}{2\epsilon_0}} \quad \text{(Compare with Chapter 2 note)}$$

(C) Uniformly charged sphere
Total charge $= Q$
Spherical symmetry.

(a) For $r > R$:



Consider a spherical Gaussian surface S of radius r:

$$\vec{E} \parallel d\vec{A} \parallel \hat{r}$$

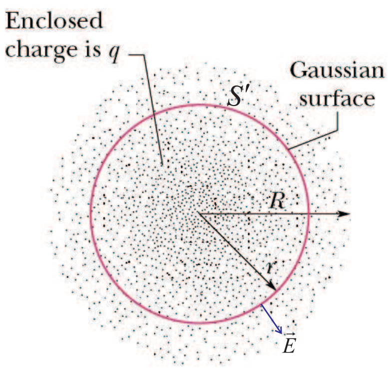
Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_S E \cdot dA = \frac{Q}{\epsilon_0}$$

$$E \underbrace{\oint_S dA}_{\text{surface area of } S \,=\, 4\pi r^2} = \frac{Q}{\epsilon_0}$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}\,; \qquad \text{for } r > R$$

(b) For $r < R$:



Consider a spherical Gaussian surface S' of radius $r < R$, then total charge included q is *proportional to the volume included by* S'

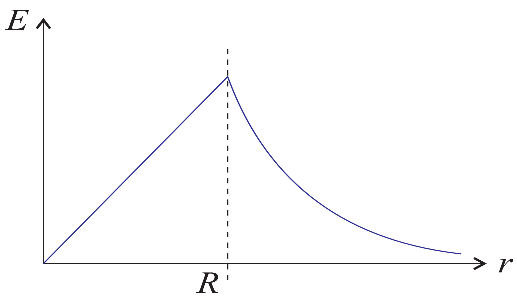
$$\therefore \quad \frac{q}{Q} = \frac{\text{Volume enclosed by } S'}{\text{Total volume of sphere}}$$

$$\frac{q}{Q} = \frac{4/3\,\pi r^3}{4/3\,\pi R^3} \quad \Rightarrow \quad q = \frac{r^3}{R^3}\,Q$$

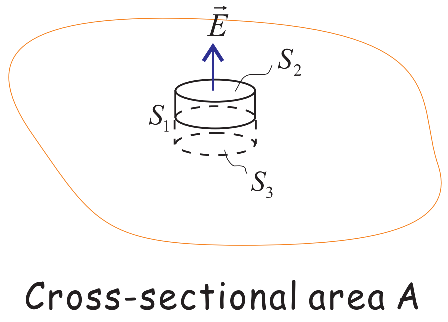
Gauss' Law: $\displaystyle\oint_{S'} \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$

$$E \underbrace{\oint_{S'} dA}_{\text{surface area of } S' \,=\, 4\pi r^2} = \frac{r^3}{R^3}\frac{1}{\epsilon_0}\cdot Q$$

$$\therefore \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R^3}\,r\,\hat{r}\,; \qquad \text{for } r \leq R$$



3.4 Gauss' Law and Conductors



For *isolated* conductors, charges are free to move until *all charges lie outside the surface of the conductor.* Also, the *E-field at the surface of a conductor is perpendicular to its surface.* (`Why?`)

Consider Gaussian surface S of shape of cylinder:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

$$\text{BUT} \quad \int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad (\because\ \vec{E} \perp d\vec{A}\,)$$

$$\int_{S_3} \vec{E}\cdot d\vec{A} = 0 \quad (\because\ \vec{E} = 0 \text{ inside conductor })$$

$$\int_{S_2} \vec{E}\cdot d\vec{A} = E \underbrace{\int_{S_2} dA}_{\text{Area of } S_2} \qquad (\because\ \vec{E} \parallel d\vec{A}\,)$$

$$= EA$$

$$\therefore \text{ Gauss' Law} \quad \Rightarrow \quad EA = \frac{\sigma A}{\epsilon_0}$$

$$\therefore \boxed{\text{On conductor's surface} \quad E = \frac{\sigma}{\epsilon_0}}$$

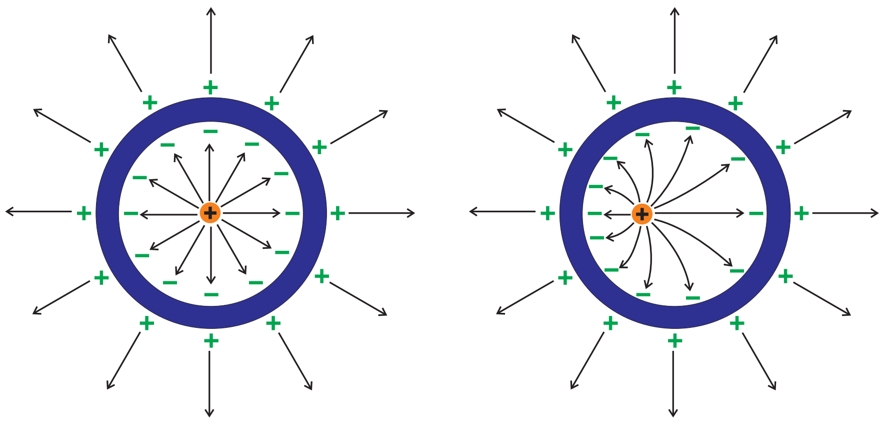
BUT, there's no charge inside conductors.

$$\therefore \boxed{\text{Inside conductors} \quad E = 0} \quad \textit{Always!}$$

Notice: Surface charge density on a conductor's surface is *not uniform.*

Example: Conductor with a charge inside

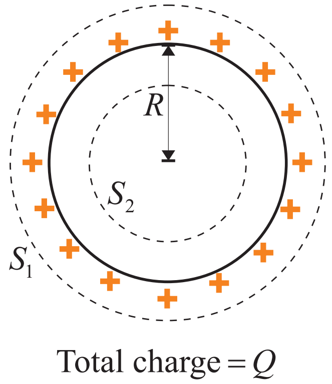
Note: This is not an isolated system (because of the charge inside).



Note: In BOTH cases, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}$ outside

Example:

I. Charge sprayed on a conductor sphere:



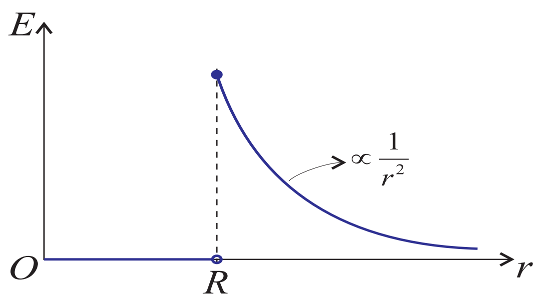
First, we know that charges all move to the *surface* of conductors.

(i) For $r < R$:
Consider Gaussian surface S_2

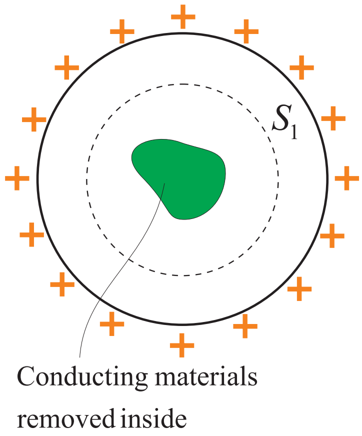
$$\oint_{S_2} \vec{E} \cdot d\vec{A} = 0 \quad (\because \text{ no charge inside })$$
$$\Rightarrow \quad E = 0 \quad \text{everywhere.}$$

(ii) For $r \geq R$:
Consider Gaussian surface S_1:

$$\oint_{S_1} \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$E \underbrace{\oint_{S_1} d\vec{A}}_{4\pi r^2} = \frac{Q}{\epsilon_0} \qquad (\overbrace{\vec{E} \parallel d\vec{A}}^{\text{For a conductor}} \underbrace{\parallel \hat{r}}_{\text{Spherically symmetric}})$$
$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$



II. Conductor sphere with hole inside:

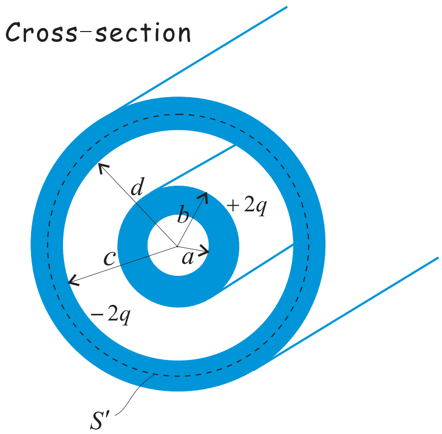


Consider Gaussian surface S_1: Total charge included = 0

$$\therefore \text{ E-field} = 0 \text{ inside}$$

The E-field is identical to the case of a solid conductor!!

III. A long hollow cylindrical conductor:



Example:
Inside hollow cylinder ($+2q$)

$$\begin{cases} \text{Inner radius} & \text{a} \\ \text{Outer radius} & \text{b} \end{cases}$$

Outside hollow cylinder ($-3q$)

$$\begin{cases} \text{Inner radius} & \text{c} \\ \text{Outer radius} & \text{d} \end{cases}$$

Question: Find the charge on each surface of the conductor.

For the inside hollow cylinder, charges distribute only on the surface.
$\therefore$ Inner radius a surface, charge $= 0$
and Outer radius b surface, charge $= +2q$

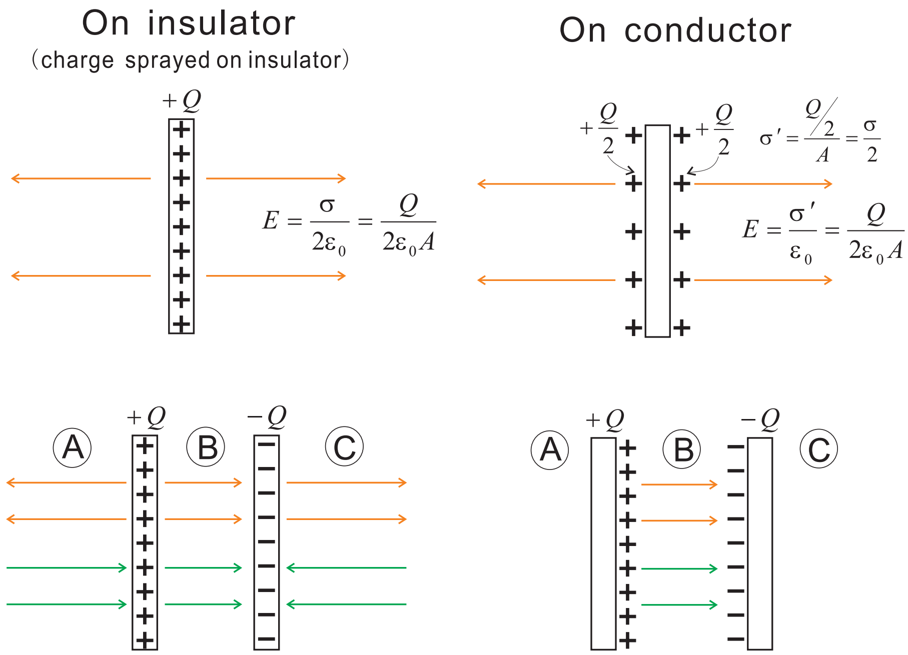
For the outside hollow cylinder, charges do not distribute only on outside.
$\because$ It's not an isolated system. (There are charges inside!)

Consider Gaussian surface S' inside the conductor:
E-field always $= 0$
$\therefore$ Need charge $-2q$ on radius c surface to balance the charge of inner cylinder.
So charge on radius d surface $= -q$. (`Why?`)

IV. Large sheets of charge:
Total charge Q on sheet of area A,

$$\therefore \quad \text{Surface charge density } \sigma = \frac{Q}{A}$$

By principle of superposition



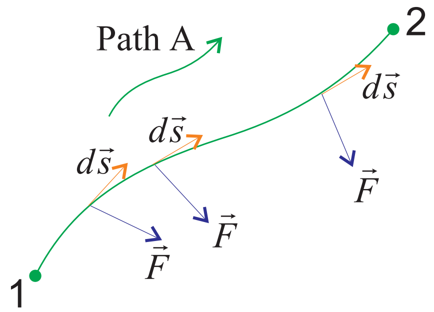
	On insulator	On conductor
Region A:	$E = 0$	$E = 0$
Region B:	$E = \dfrac{Q}{\epsilon_0 A}$	$E = \dfrac{Q}{\epsilon_0 A}$
Region C:	$E = 0$	$E = 0$

Chapter 4

Electric Potential

4.1 Potential Energy and Conservative Forces

(Read Halliday Vol.1 Chap.12)
Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a charge moves an infinitesimal distance $d\vec{s}$ along *Path A* $= dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of *Path A*.

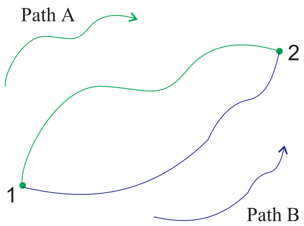
$$dW = \vec{F} \cdot d\vec{s}$$

$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \int_{1\ \text{Path A}}^{2} \vec{F} \cdot d\vec{s}}$$

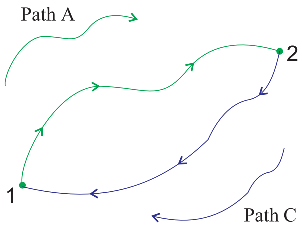
$$\begin{aligned} \int_{1\ \text{Path A}}^{2} &= \textit{Path Integral} \\ &= \text{Integration over Path A from Point 1 to Point 2.} \end{aligned}$$

DEFINITION: A force is **conservative** *if* the work done on a particle by the force is *independent of the path taken.*



$\therefore$ For conservative forces,

$$\underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} = \underset{\text{Path B}}{\int_1^2 \vec{F}\cdot d\vec{s}}$$



Let's consider a path starting at point 1 to 2 through *Path A* and from 2 to 1 through *Path C*

$$\begin{aligned}\text{Work done} &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} + \underset{\text{Path C}}{\int_2^1 \vec{F}\cdot d\vec{s}} \\ &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} - \underset{\textbf{Path B}}{\int_{\mathbf{1}}^{\mathbf{2}} \vec{F}\cdot d\vec{s}}\end{aligned}$$

DEFINITION: The work done by a **conservative force** on a particle when it *moves around a closed path returning to its initial position* is *zero.*

MATHEMATICALLY, $\vec{\nabla}\times\vec{F} = 0$ everywhere for conservative force $\vec{F}$

Conclusion: Since the work done by a conservative force $\vec{F}$ is *path-independent*, we can define a quantity, **potential energy**, that depends only on the *position* of the particle.

Convention: We define **potential energy** U such that

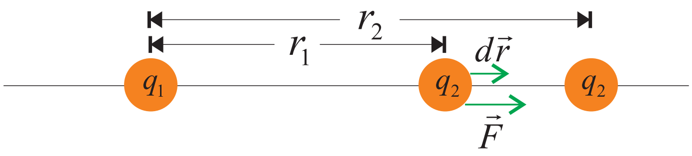
$$\boxed{dU = -W = -\int \vec{F}\cdot d\vec{s}}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}\cdot d\vec{s}$$

where U_1, U_2 are **potential energy** at position 1, 2.

Example:



Suppose charge q_2 moves from point 1 to 2.

$$
\begin{aligned}
\text{From definition:}\quad U_2 - U_1 &= -\int_1^2 \vec{F}\cdot d\vec{r} \\
&= -\int_{r_1}^{r_2} F\,dr \quad (\because\ \vec{F} \parallel d\vec{r}\,) \\
&= -\int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r^2}\,dr \\
(\because \int \frac{dr}{r^2} = -\frac{1}{r} + C\ ) \qquad &= \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r}\bigg|_{r_1}^{r_2} \\
-\Delta W = \Delta U &= \frac{1}{4\pi\epsilon_0}\,q_1 q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)
\end{aligned}
$$

Note:

(1) This result is generally true for 2-Dimension or 3-D motion.

(2) If q_2 moves away from q_1,
then $r_2 > r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U < 0$, $\Delta W > 0$
  (ΔW = Work done by electric *repulsive* force)
- If q_1, q_2 are of *different* sign,
  then $\Delta U > 0$, $\Delta W < 0$
  (ΔW = Work done by electric *attractive* force)

(3) If q_2 moves towards q_1,
then $r_2 < r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U \quad 0$, $\Delta W \quad 0$
- If q_1, q_2 are of *different* sign,
  then $\Delta U \quad 0$, $\Delta W \quad 0$

(4) Note: It is the *difference* in potential energy that is important.

$$\text{REFERENCE POINT:} \quad U(r=\infty)=0$$

$$\therefore \; U_\infty - U_1 = \frac{1}{4\pi\epsilon_0}\, q_1 q_2 \left(\underset{\substack{\downarrow \\ \infty}}{\frac{1}{r_2}} - \frac{1}{r_1}\right)$$

$$\boxed{U(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 q_2}{r}}$$

If q_1, q_2 *same sign*, then $U(r) > 0$ for all r
If q_1, q_2 *opposite sign*, then $U(r) < 0$ for all r

(5) Conservation of Mechanical Energy:
For a system of charges with no external force,

$$E = \underset{\swarrow}{K} + \underset{\searrow}{U} = \text{Constant}$$

(Kinetic Energy) (Potential Energy)

$$\text{or} \quad \boxed{\Delta E = \Delta K + \Delta U = 0}$$

Potential Energy of A System of Charges

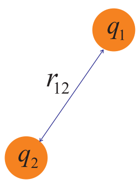
Example: P.E. of 3 charges q_1, q_2, q_3

Start: q_1, q_2, q_3 all at $r=\infty, U=0$

Step1:

Move q_1 from ∞ to its position $\Rightarrow\; U=0$

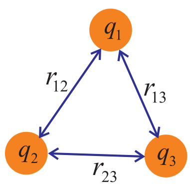
Step2:



Move q_2 from ∞ to new position $\Rightarrow$

$$U = \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r_{12}}$$

Step3:



Move q_3 from ∞ to new position $\Rightarrow$ Total P.E.

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Step4: What if there are 4 charges?

4.2 Electric Potential

Consider a charge q at center, we consider its effect on test charge q_0

DEFINITION: We define electric potential V so that

$$\boxed{\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}}$$

($\therefore$ V is the P.E. per unit charge)

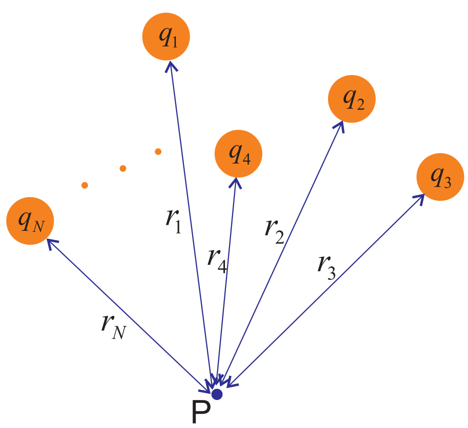
- Similarly, we take $V(r=\infty)=0$.
- Electric Potential is a **scalar**.
- Unit: $Volt(V) = Joules/Coulomb$
- For a single point charge:

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r}}$$

- Energy Unit: $\Delta U = q\Delta V$

$$electron - Volt(eV) = \underbrace{1.6\times 10^{-19}}_{\text{charge of electron}}\ J$$

Potential For A System of Charges



For a total of N point charges, the potential V at any point P can be derived from the **principle of superposition**.

Recall that potential due to q_1 at point P: $V_1 = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1}{r_1}$

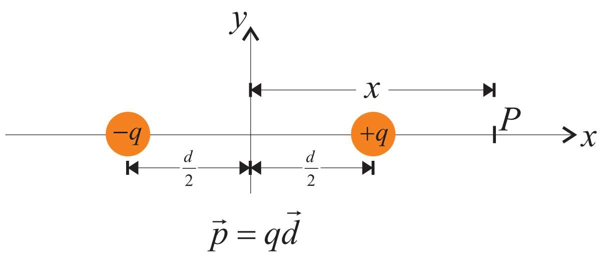
$\therefore$ Total potential at point P due to N charges:

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_N \quad \text{(principle of superposition)} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right] \end{aligned}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}}$$

Note: For $\vec{E}, \vec{F}$, we have a sum of vectors
For V, U, we have a sum of scalars

Example: Potential of an electric dipole



Consider the potential of point P at distance $x > \frac{d}{2}$ from dipole.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x-\frac{d}{2}} + \frac{-q}{x+\frac{d}{2}}\right]$$

Special Limiting Case: $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x}\cdot\frac{1}{1\mp\frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x}\left[1+\frac{d}{2x} - (1-\frac{d}{2x})\right]$$

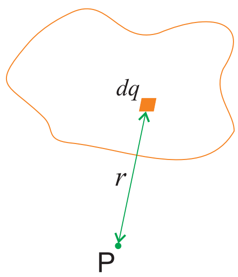
$$V = \frac{p}{4\pi\epsilon_0 x^2} \qquad \text{(Recall } p = qd\text{)}$$

For a point charge $E \propto \frac{1}{r^2}$ $V \propto \frac{1}{r}$

For a dipole $E \propto \frac{1}{r^3}$ $V \propto \frac{1}{r^2}$

For a quadrupole $E \propto \frac{1}{r^4}$ $V \propto \frac{1}{r^3}$

Electric Potential of Continuous Charge Distribution



For any charge distribution, we write the electrical potential dV due to infinitesimal charge dq:

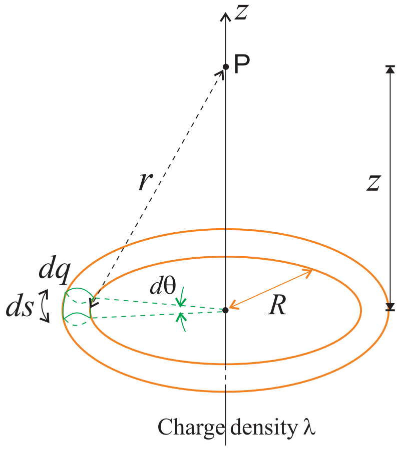
$$dV = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r}$$

$$\therefore \quad \boxed{V = \int\limits_{\substack{\text{charge} \\ \text{distribution}}} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}}$$

Similar to the previous examples on E-field, for the case of *uniform* charge distribution:

1-D	$\Rightarrow$	long rod	$\Rightarrow$	$dq = \lambda\, dx$
2-D	$\Rightarrow$	charge sheet	$\Rightarrow$	$dq = \sigma\, dA$
3-D	$\Rightarrow$	uniformly charged body	$\Rightarrow$	$dq = \rho\, dV$

Example (1): Uniformly-charged ring



Length of the infinitesimal ring element $= ds = Rd\theta$

$$\therefore \quad \text{charge } dq = \lambda\, ds = \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

The integration is around the entire ring.

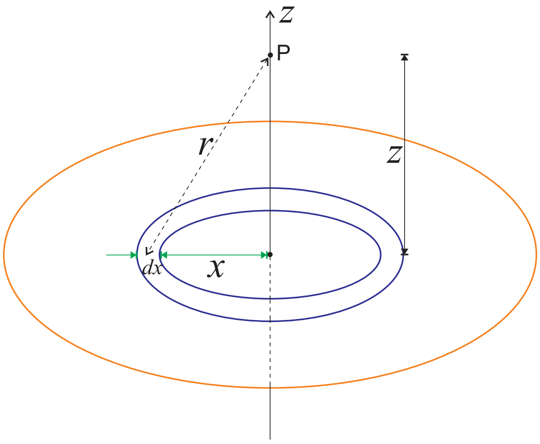
$$\begin{aligned} \therefore \quad V &= \int\limits_{\text{ring}} dV \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2+z^2}} \\ &= \frac{\lambda R}{4\pi\epsilon_0\sqrt{R^2+z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \end{aligned}$$

Total charge on the ring $= \lambda \cdot (2\pi R)$

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{R^2+z^2}}$$

LIMITING CASE: $z \gg R \;\Rightarrow\; V = \dfrac{Q}{4\pi\epsilon_0\sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0|z|}$

Example (2): Uniformly-charged disk



Using the **principle of superposition**, we will find the potential of a disk of uniform charge density by integrating the potential of *concentric rings.*

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0} \int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x: $dq = \sigma\, dA = \sigma\,(2\pi x dx)$

$$\begin{aligned}
\therefore \quad V &= \int_0^R \frac{1}{4\pi\epsilon_0} \cdot \frac{\sigma 2\pi x\, dx}{\sqrt{x^2+z^2}} \\
&= \frac{\sigma}{4\epsilon_0} \int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}} \\
V &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - \sqrt{z^2}\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - |z|\right)
\end{aligned}$$

Recall: $|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

Limiting Case:

(1) If $|z| \gg R$

$$\begin{aligned}
\sqrt{z^2+R^2} &= \sqrt{z^2\Big(1+\frac{R^2}{z^2}\Big)} \\
&= |z| \cdot \Big(1+\frac{R^2}{z^2}\Big)^{\frac{1}{2}} \qquad (\,(1+x)^n \approx 1+nx \text{ if } x \ll 1\,) \\
&\simeq |z| \cdot \Big(1+\frac{R^2}{2z^2}\Big) \qquad (\,\frac{|z|}{z^2} = \frac{1}{|z|}\,)
\end{aligned}$$

$\therefore$ At large z, $V \simeq \frac{\sigma}{2\epsilon_0} \cdot \frac{R^2}{2|z|} = \frac{Q}{4\pi\epsilon_0 |z|}$ (like a point charge)

where Q = total charge on disk $= \sigma \cdot \pi R^2$

(2) If $|z| \ll R$

$$\begin{aligned}\sqrt{z^2+R^2} &= R\cdot\left(1+\frac{z^2}{R^2}\right)^{\frac{1}{2}} \\ &\simeq R\left(1+\frac{z^2}{2R^2}\right)\end{aligned}$$

$$\therefore \quad V \simeq \frac{\sigma}{2\epsilon_0}\left[R-|z|+\frac{z^2}{2R}\right]$$

At $z=0$, $V=\dfrac{\sigma R}{2\epsilon_0}$; Let's call this V_0

$$\begin{aligned}\therefore \quad V(z) &= \frac{\sigma R}{2\epsilon_0}\left[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\right] \\ V(z) &= V_0\left[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\right]\end{aligned}$$

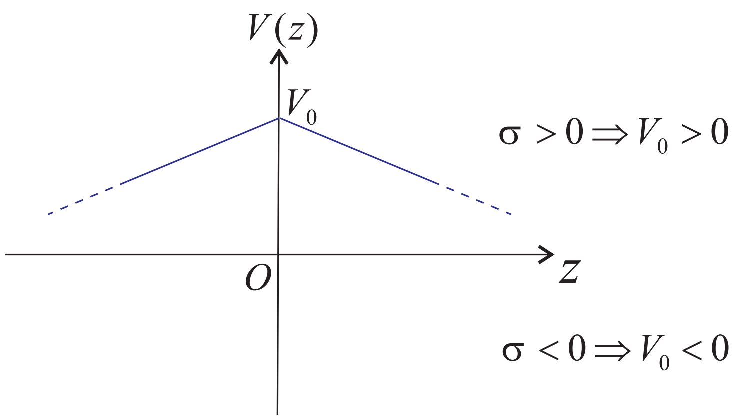
The *key* here is that it is the difference between potentials of two points that is important.

$\Rightarrow$ A convenience reference point to compare in this example is the potential of the charged disk.

$\therefore$ The important quantity here is

$$V(z)-V_0 = -\frac{|z|}{R}V_0 + \underbrace{\frac{z^2}{2R^2}V_0}_{\text{neglected as } z\ll R}$$

$$\boxed{V(z)-V_0 = -\frac{V_0}{R}|z|}$$



4.3 Relation Between Electric Field E and Electric Potential V

(A) To get V from $\vec{E}$:
Recall our definition of the potential V:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

where ΔU is the change in P.E.; W_{12} is the work done in bringing charge q_0 from point 1 to 2.

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F}\cdot d\vec{s}}{q_0}$$

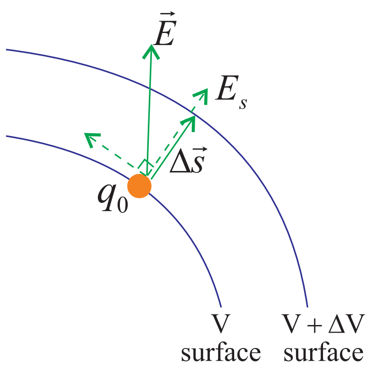
However, the definition of E-field: $\vec{F} = q_0\vec{E}$

$$\therefore \quad \boxed{\Delta V = V_2 - V_1 = -\int_1^2 \vec{E}\cdot d\vec{s}}$$

Note: The integral on the right hand side of the above can be calculated *along any path from point 1 to 2.* (**Path-Independent**)

Convention: $V_\infty = 0 \quad \Rightarrow \quad \boxed{V_P = -\int_\infty^P \vec{E}\cdot d\vec{s}}$

(B) To get $\vec{E}$ from V:



(i.e. Potential = V on the surface)

Again, use the definition of V:

$$\Delta U = q_0\Delta V = \underbrace{-W}_{\text{Work done}}$$

However,

$$W = \underbrace{q_0\vec{E}}_{\text{Electric force}} \cdot \Delta\vec{s} = q_0\, E_s\, \Delta s$$

where E_s is the E-field component along the path $\Delta\vec{s}$.

$$\therefore \quad q_0\Delta V = -q_0 E_s \Delta s$$

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs,

$$\therefore \quad \boxed{E_s = -\frac{dV}{ds}}$$

Note: (1) Therefore the E-field component along *any direction* is the negtive derivative of the potential *along the same direction.*

(2) If $d\vec{s} \perp \vec{E}$, then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally, for a potential $V(x, y, z)$, the relation between $\vec{E}(x, y, z)$ and V is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivative**s

For $\frac{\partial}{\partial x} V(x, y, z)$, everything y, z are treated like a *constant* and we only take derivative with respect to x.

Example: If $V(x, y, z) = x^2 y - z$

$$\frac{\partial V}{\partial x} =$$

$$\frac{\partial V}{\partial y} =$$

$$\frac{\partial V}{\partial z} =$$

For other co-ordinate systems

(1) Cylindrical:

$$V(r, \theta, z) \qquad \begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\ E_\theta = -\dfrac{1}{r} \cdot \dfrac{\partial V}{\partial \theta} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$$

(2) Spherical:

$$V(r,\theta,\phi) \qquad \begin{cases} E_r &= -\dfrac{\partial V}{\partial r} \\ E_\theta &= -\dfrac{1}{r}\cdot\dfrac{\partial V}{\partial \theta} \\ E_\phi &= -\dfrac{1}{r\sin\theta}\cdot\dfrac{\partial V}{\partial \phi} \end{cases}$$

Note: Calculating V involves summation of *scalars*, which is easier than adding *vectors* for calculating E-field.

$\therefore$ To find the E-field of a general charge system, we first calculate $\boldsymbol{V}$, and then derive $\vec{\boldsymbol{E}}$ from the partial derivative.

Example: Uniformly charged disk

From potential calculations:

$$V = \frac{\sigma}{2\epsilon_0}(\sqrt{R^2+z^2} - |z|\,)$$ for a point along the z-axis

For $z > 0, \quad |z| = z$

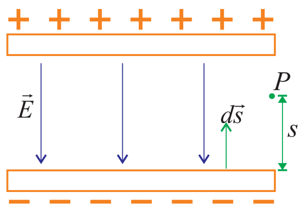
$$\therefore \quad E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\Big[1 - \frac{z}{\sqrt{R^2+z^2}}\Big]$$ (Compare with Chap.2 notes)

Example: Uniform electric field

(e.g. Uniformly charged $+ve$ and $-ve$ plates)

Consider a path going from the $-ve$ plate to the $+ve$ plate

Potential at point P, V_P can be deduced from definition.



$$\text{i.e.} \qquad V_P - V_- = -\int_0^s \vec{E}\cdot d\vec{s}$$ (V_- = Potential of $-ve$ plate)

$$= -\int_0^s (-E\,ds)$$ $\because$ $\vec{E}, d\vec{s}$ pointing opposite directions

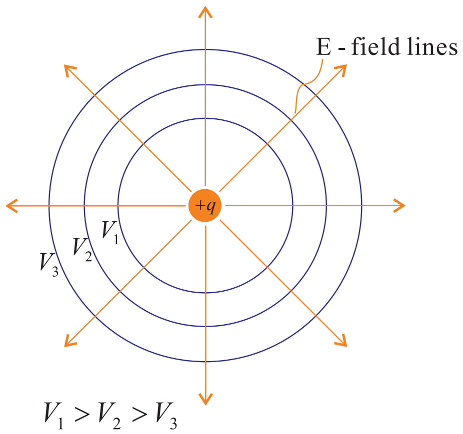
$$= E\int_0^s ds = Es$$

Convenient reference: $V_- = 0$

$$\therefore \quad \boxed{V_P = E\cdot s}$$

4.4 Equipotential Surfaces

Equipotential surface is a surface on which the *potential is constant.*
$\Rightarrow \quad (\Delta V = 0)$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

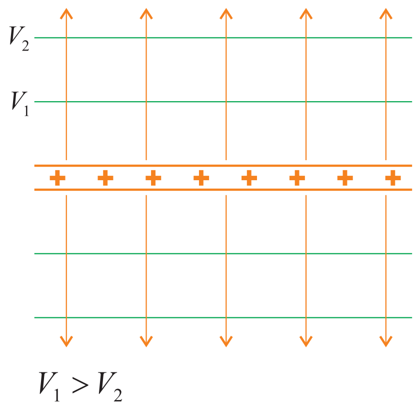
$\Rightarrow \quad r = const$

$\Rightarrow$ Equipotential surfaces are *circles/spherical surfaces*

Note: (1) A charge can move freely on an equipotential surface without any work done.

(2) The **electric field lines** must be *perpendicular* to the **equipotential surfaces**. `(Why?)`
On an equipotential surface, $V = constant$
$\Rightarrow \ \Delta V = 0 \ \Rightarrow \ \vec{E} \cdot d\vec{l} = 0$, where $d\vec{l}$ is *tangent* to equipotential surface
$\therefore \quad \vec{E}$ must be *perpendicular* to equipotential surfaces.

Example: Uniformly charged surface (infinite)

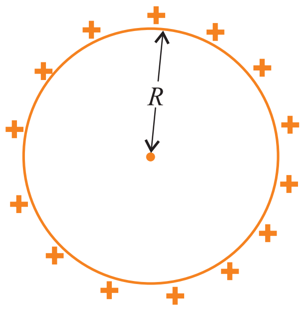


Recall $\quad V = V_0 - \frac{\sigma}{2\epsilon_0}|z|$
$\uparrow$
Potential at $z = 0$

Equipotential surface means

$$V = const \ \Rightarrow \ V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$
$$\Rightarrow \ |z| = constant$$

Example: Isolated spherical charged conductors



Recall:

(1) E-field inside = 0

(2) charge distributed on the *outside* of conductors.

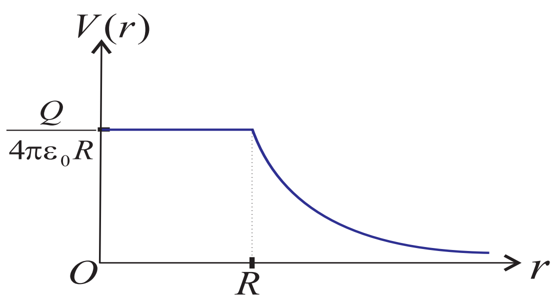
(i) Inside conductor:

$$\begin{aligned} E = 0 \quad &\Rightarrow \quad \Delta V = 0 \text{ everywhere in conductor} \\ &\Rightarrow \quad V = \textit{constant} \text{ everywhere in conductor} \\ &\Rightarrow \quad \text{The entire conductor is at the same potential} \end{aligned}$$

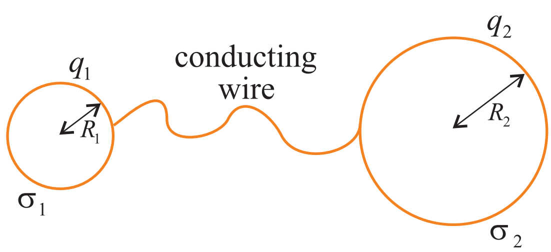
(ii) Outside conductor:

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

$\because$ Spherically symmetric (Just like a point charge.)
BUT not true for conductors of arbitrary shape.



Example: Connected conducting spheres



Two conductors connected can be seen as a *single conductor*

$\therefore$ Potential everywhere is identical.

Potential of radius R_1 sphere $V_1 = \dfrac{q_1}{4\pi\epsilon_0 R_1}$

Potential of radius R_2 sphere $V_2 = \dfrac{q_2}{4\pi\epsilon_0 R_2}$

$$\mathbf{V_1 = V_2}$$

$$\Rightarrow \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

Surface charge density

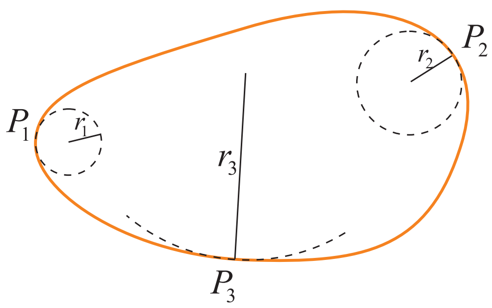
$$\sigma_1 = \underbrace{\frac{q_1}{4\pi R_1^2}}_{\text{Surface area of radius } R_1 \text{ sphere}}$$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2} \cdot \frac{R_2^2}{R_1^2} = \frac{R_2}{R_1}$$

$$\therefore \quad \text{If } R_1 < R_2, \text{ then } \sigma_1 > \sigma_2$$

And the surface electric field $E_1 > E_2$

For arbitrary shape conductor:



$E_3 < E_2 < E_1$

At every point on the conductor, we fit a *circle*. The radius of this circle is the *radius of curvature*.

Note: Charge distribution on a conductor does **not** have to be uniform.

Chapter 5

Capacitance and DC Circuits

5.1 Capacitors

A **capacitor** is a system of *two conductors* that carries *equal and opposite charges*. A capacitor *stores charge and energy* in the form of electro-static field.

We define **capacitance** as

$$\boxed{C = \frac{Q}{V}} \qquad \text{Unit: Farad(F)}$$

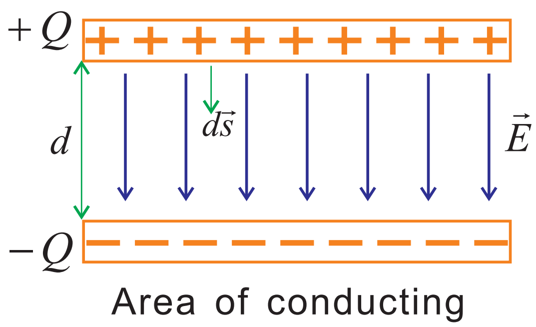
where

$$Q = \text{Charge on one plate}$$
$$V = \text{Potential difference between the plates}$$

Note: The C of a capacitor is a *constant* that depends only on its shape and material.
`i.e.` If we increase V for a capacitor, we can increase Q stored.

5.2 Calculating Capacitance

5.2.1 Parallel-Plate Capacitor



(1) Recall from Chapter 3 note,

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

(2) Recall from Chapter 4 note,

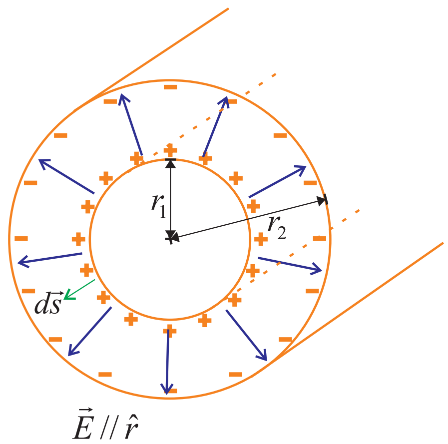
$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{s}$$

Again, notice that this integral is independent of the path taken.
$\therefore$ We can take the path that is parallel to the $\vec{E}$-field.

$$\begin{aligned}\therefore \quad \Delta V &= \int_+^- \vec{E}\cdot d\vec{s} \\ &= \int_+^- E\cdot ds \\ &= \frac{Q}{\epsilon_0 A}\underbrace{\int_+^- ds}_{\text{Length of path taken}} \\ &= \frac{Q}{\epsilon_0 A}\cdot d\end{aligned}$$

$$(3)\ \therefore \quad \boxed{C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}}$$

5.2.2 Cylindrical Capacitor



Consider two concentric cylindrical wire of innner and outer radii r_1 and r_2 respectively. The length of the capacitor is L where $r_1 < r_2 \ll L$.

(1) Using Gauss' Law, we determine that the E-field between the conductors is (`cf.` Chap3 note)

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \cdot \frac{\lambda}{r}\,\hat{r} = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{Lr}\,\hat{r}$$

where λ is charge per unit length

(2)

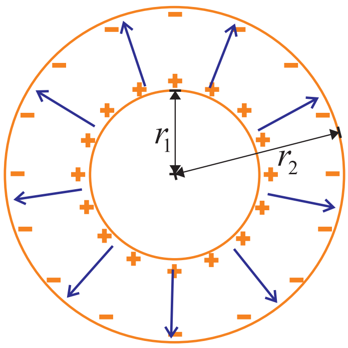
$$\Delta V = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

Again, we choose the path of integration so that $d\vec{s} \parallel \hat{r} \parallel \vec{E}$

$$\therefore \quad \Delta V = \int_{r_1}^{r_2} E\,dr = \frac{Q}{2\pi\epsilon_0 L} \underbrace{\int_{r_1}^{r_2} \frac{dr}{r}}_{\ln(\frac{r_2}{r_1})}$$

$$\therefore \quad \boxed{C = \frac{Q}{\Delta V} = 2\pi\epsilon_0 \frac{L}{\ln(r_2/r_1)}}$$

5.2.3 Spherical Capacitor



For the space between the two conductors,

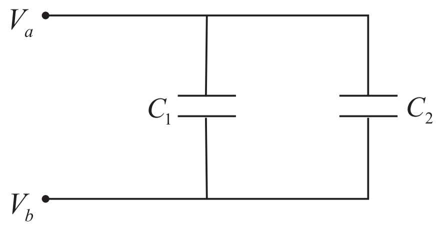
$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}; \qquad r_1 < r < r_2$$

$$\begin{aligned} \Delta V &= \int_{+}^{-} \vec{E} \cdot d\vec{s} \\ \text{Choose } d\vec{s} \parallel \hat{r} \qquad &= \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\,dr \\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \end{aligned}$$

$$\boxed{C = 4\pi\epsilon_0 \left[\frac{r_1 r_2}{r_2 - r_1}\right]}$$

5.3 Capacitors in Combination

(a) Capacitors in Parallel



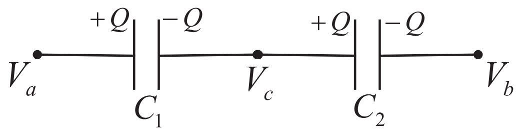
In this case, it's the *potential difference* $V = V_a - V_b$ that is the same across the capacitor.

BUT: *Charge* on each capacitor different

$$\begin{aligned} \text{Total charge} \quad Q &= Q_1 + Q_2 \\ &= C_1 V + C_2 V \\ Q &= \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}} V \end{aligned}$$

$\therefore$ For capacitors in parallel: $\boxed{C = C_1 + C_2}$

(b) Capacitors in Series



The *charge across capacitors* are the same.

BUT: *Potential difference* (P.D.) across capacitors different

$$\begin{aligned} \Delta V_1 &= V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1 \\ \Delta V_2 &= V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2 \end{aligned}$$

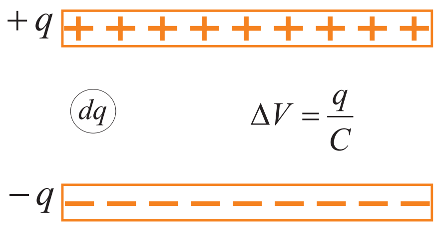
$\therefore$ Potential difference

$$\begin{aligned} \Delta V &= V_a - V_b \\ &= \Delta V_1 + \Delta V_2 \\ \Delta V &= Q\,(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C} \end{aligned}$$

where C is the **Equivalent Capacitance**

$$\therefore \quad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

5.4 Energy Storage in Capacitor



In charging a capacitor, *positive charge* is being moved from the *negative plate* to the *positive plate*.
$\Rightarrow$ NEEDS WORK DONE!

Suppose we move charge dq from $-ve$ to $+ve$ plate, *change in potential energy*

$$dU = \Delta V \cdot dq = \frac{q}{C}\, dq$$

Suppose we keep putting in a total charge Q to the capacitor, the *total potential energy*

$$U = \int dU = \int_0^Q \frac{q}{C}\, dq$$

$$\therefore \quad \boxed{U = \frac{Q^2}{2C} = \frac{1}{2} C \Delta V^2} \qquad (\because Q = C\Delta V)$$

The energy stored in the capacitor is stored in the **electric field** between the plates.

Note : In a parallel-plate capacitor, the *E-field is constant between the plates.*

$\therefore$ We can consider the E-field energy

$$\text{density u} = \frac{\text{Total energy stored}}{\text{Total volume with E-field}}$$

$$\therefore \quad u = \frac{U}{\underbrace{Ad}_{\text{Rectangular volume}}}$$

Recall

$$\begin{cases} C &= \dfrac{\epsilon_0 A}{d} \\ E &= \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

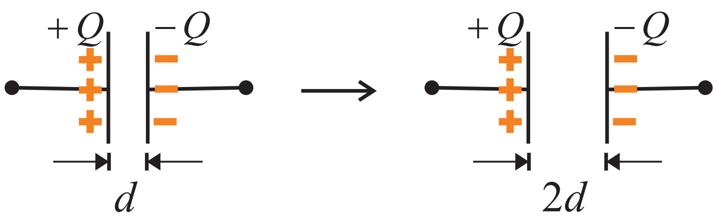
$$\therefore \quad u = \frac{1}{2} (\overbrace{\frac{\epsilon_0 A}{d}}^{C}) \cdot (\overbrace{Ed}^{(\Delta V)^2})^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{Volume}}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$ Energy per unit volume of the electrostatic field

↑

can be generally applied

Example : Changing capacitance



(1) Isolated Capacitor:

Charge on the capacitor plates remains *constant*.

BUT: $C_{new} = \frac{\epsilon_0 A}{2d} = \frac{1}{2} C_{old}$

$\therefore \quad U_{new} = \frac{Q^2}{2C_{new}} = \frac{Q^2}{2C_{old}/2} = 2U_{old}$

$\therefore$ In pulling the plates apart, work done $W > 0$

Summary :

	Q	$\rightarrow$	Q	C	$\rightarrow$	$C/2$	
$(V = \frac{Q}{C}) \Rightarrow$	V	$\rightarrow$	$2V$	E	$\rightarrow$	E	$(E = \frac{V}{d})$
$\frac{1}{2}\epsilon_0 E^2 =$	u	$\rightarrow$	u	U	$\rightarrow$	$2U$	$(U = u \cdot volume)$

(2) Capacitor connected to a battery:

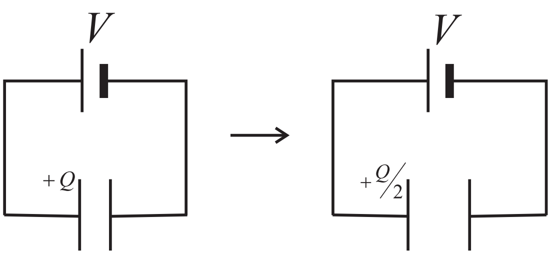
Potential difference between capacitor plates remains *constant*.

$U_{new} = \frac{1}{2} C_{new} \Delta V^2 = \frac{1}{2} \cdot \frac{1}{2} C_{old} \Delta V^2 = \frac{1}{2} U_{old}$

$\therefore$ In pulling the plates apart, work done by battery < 0

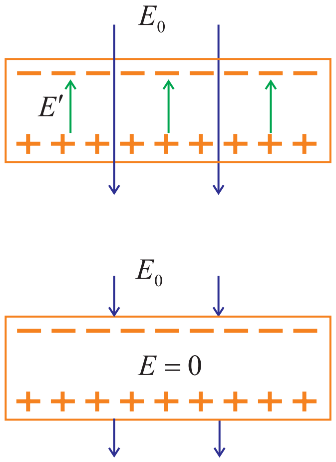
Summary :

Q	$\rightarrow$	$Q/2$	C	$\rightarrow$	$C/2$
V	$\rightarrow$	V	E	$\rightarrow$	$E/2$
u	$\rightarrow$	$u/4$	U	$\rightarrow$	$U/2$



5.5 Dielectric Constant

We first recall the case for a *conductor* being placed in an *external E-field* E_0.



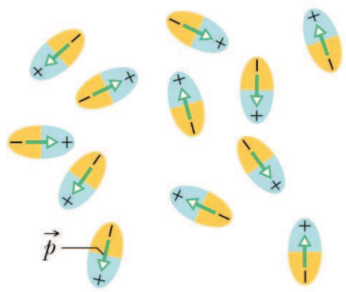
In a conductor, charges are free to move inside so that the *internal E-field* E' set up by these charges

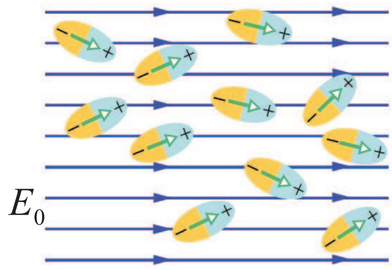
$$E' = -E_0$$

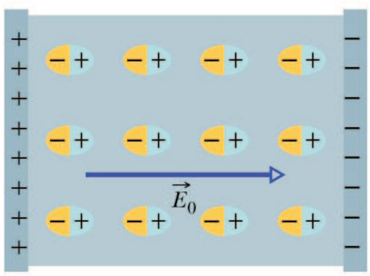
so that E-field inside conductor = 0.

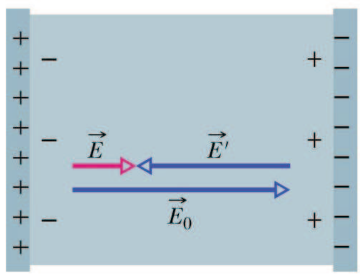
Generally, for **dielectric**, the atoms and molecules behave like a **dipole** in an E-field.

Or, we can envision this so that in the absence of E-field, the *direction of dipole in the dielectric are* randomly distributed.









The aligned dipoles will generate an *induced E-field* E', where $|E'| < |E_0|$.
We can observe the aligned dipoles in the form of *induced surface charge*.

Dielectric Constant : When a dielectric is placed in an external E-field E_0, the E-field inside a dielectric is *induced*.
E-field in dielectric

$$\boxed{E = \frac{1}{K_e} E_0}$$

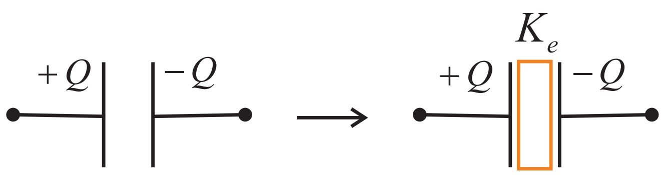
$$K_e = \textbf{dielectric constant} \quad \geq 1$$

Example :

Vacuum	$K_e = 1$
Porcelain	$K_e = 6.5$
Water	$K_e \sim 80$
Perfect conductor	$K_e = \infty$
Air	$K_e = 1.00059$

5.6 Capacitor with Dielectric

Case I :



Again, the *charge remains constant* after dielectric is inserted.

BUT: $E_{new} = \dfrac{1}{K_e} E_{old}$

$\therefore \quad \Delta V = Ed \quad \Rightarrow \quad \Delta V_{new} = \dfrac{1}{K_e} \Delta V_{old}$

$\therefore \quad C = \dfrac{Q}{\Delta V} \quad \Rightarrow \quad C_{new} = K_e C_{old}$

For a parallel-plate capacitor with dielectric:

$$\boxed{C = \frac{K_e \epsilon_0 A}{d}}$$

We can also write $C = \dfrac{\epsilon A}{d}$ in general with

$$\epsilon = K_e \, \epsilon_0 \qquad \text{(called \textbf{permittivity of dielectric})}$$

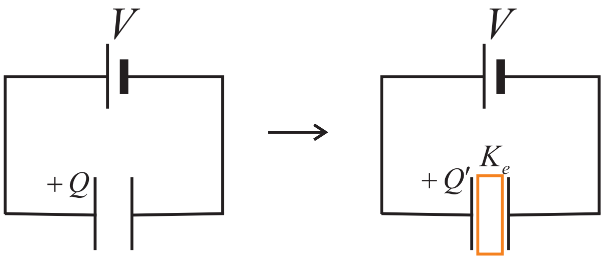
(Recall ϵ_0 = **Permittivity of free space**)

Energy stored $U = \dfrac{Q^2}{2C}$;

$$\therefore \quad U_{new} = \frac{1}{K_e} U_{old} \quad < \quad U_{old}$$

$$\boxed{\therefore \quad \text{Work done in inserting dielectric} < 0}$$

Case II : Capacitor connected to a battery



Voltage across capacitor plates *remains constant* after insertion of dielectric.
In both scenarios, the E-field inside capacitor remains constant
($\because \; E = V/d$)

BUT: How can E-field remain constant?
ANSWER: By having extra charge on capacitor plates.

Recall: For conductors,

$$E = \frac{\sigma}{\epsilon_0} \qquad \text{(Chapter 3 note)}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A} \qquad (\sigma = \text{charge per unit area} = Q/A)$$

After insertion of dielectric:

$$E' = \frac{E}{K_e} = \frac{Q'}{K_e \epsilon_0 A}$$

But E-field remains constant!

$$\therefore \quad E' = E \;\Rightarrow\; \frac{Q'}{K_e \epsilon_0 A} = \frac{Q}{\epsilon_0 A}$$

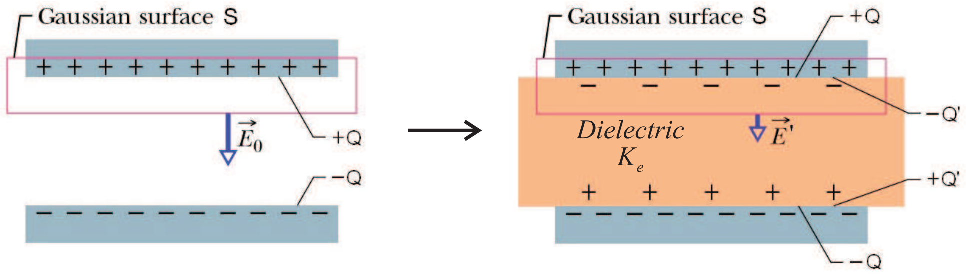
$$\Rightarrow \; Q' = K_e Q \; > \; Q$$

$\therefore$ Capacitor $C = Q/V \Rightarrow C' \to K_e C$
Energy stored $U = \frac{1}{2}CV^2 \Rightarrow U' \to K_e U$
(i.e. $U_{new} > U_{old}$)

$$\boxed{\therefore \text{ Work done to insert dielectric} > 0}$$

5.7 Gauss' Law in Dielectric

The Gauss' Law we've learned is applicable in *vacuum only*. Let's use the capacitor as an example to examine Gauss' Law in dielectric.



Free charge on plates	$\pm Q$	$\pm Q$
Induced charge on dielectric	0	$\mp Q'$

Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad E_0 = \frac{Q}{\epsilon_0 A} \qquad (1)$$

Gauss' Law:

$$\oint_S \vec{E}'\cdot d\vec{A} = \frac{Q-Q'}{\epsilon_0}$$

$$\therefore \quad E' = \frac{Q-Q'}{\epsilon_0 A} \qquad (2)$$

However, we define $\quad E' = \dfrac{E_0}{K_e} \qquad (3)$

From (1), (2), (3) $\quad \therefore \dfrac{Q}{K_e\epsilon_0 A} = \dfrac{Q}{\epsilon_0 A} - \dfrac{Q'}{\epsilon_0 A}$

$$\therefore \quad \boxed{\text{Induced charge density } \sigma' = \frac{Q'}{A} = \sigma\left(1 - \frac{1}{K_e}\right) < \sigma}$$

where σ is free charge density.

Recall Gauss' Law in Dielectric:

$$\epsilon_0 \oint_S \vec{E}'\cdot d\vec{A} = Q - Q'$$

$\uparrow$ E-field in dielectric; $\uparrow$ free charge; $\uparrow$ induced charge

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q\left[1 - \frac{1}{K_e}\right]$$
$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{K_e}$$

$$\boxed{\oint_S K_e \vec{E}' \cdot d\vec{A} = \frac{Q}{\epsilon_0}}$$ Gauss' Law in dielectric

Note :

(1) This goes back to the Gauss' Law in vacuum with $E = \dfrac{E_0}{K_e}$ for dielectric

(2) Only *free charges* need to be considered, even for dielectric where there are *induced charges*.

(3) Another way to write:
$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$$
where $\vec{E}$ is E-field in dielectric, $\epsilon = K_e\epsilon_0$ is Permittivity

Energy stored with dielectric:

Total energy stored: $U = \dfrac{1}{2}CV^2$

With dielectric, recall $C = \dfrac{K_e\epsilon_0 A}{d}$

$V = Ed$

$\therefore$ Energy stored per unit volume:

$$\boxed{u_e = \frac{U}{Ad} = \frac{1}{2}K_e\epsilon_0 E^2}$$

$$\text{and } u_{dielectric} = K_e u_{vacuum}$$

$\therefore$ More energy is stored per unit volume in dielectric than in vacuum.

5.8 Ohm's Law and Resistance

ELECTRIC CURRENT is defined as the flow of electric charge through a cross-sectional area.

$$\boxed{i = \frac{dQ}{dt}}$$ Unit: Ampere (A) = C/second

Convention :

(1) Direction of current is the direction of *flow of positive charge.*

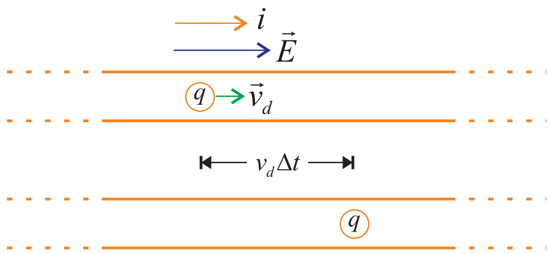
(2) Current is NOT a vector, but the **current density** is a **vector**.

$\vec{j}$ = charge flow per unit time per unit area

$$\boxed{i = \int \vec{j} \cdot d\vec{A}}$$

Drift Velocity :

Consider a current i flowing through a cross-sectional area A:



$\therefore$ In time Δt, total charges passing through segment:

$$\Delta Q = q \underbrace{A(V_d \Delta t)}_{\text{Volume of charge passing through}} n$$

where q is charge of the current carrier, n is density of charge carrier per unit volume

$$\therefore \quad \text{Current:} \quad \boxed{i = \frac{\Delta Q}{\Delta t} = nqAv_d}$$

$$\text{Current Density:} \quad \boxed{\vec{j} = nq\vec{v}_d}$$

Note : For metal, the charge carriers are the free electrons inside.
$\therefore$ $\vec{j} = -ne\vec{v}_d$ for metals
$\therefore$ Inside metals, $\vec{j}$ and $\vec{v}_d$ are in *opposite direction.*

We define a general property, **conductivity** ($\boldsymbol{\sigma}$), of a material as:

$$\boxed{\vec{j} = \boldsymbol{\sigma}\vec{E}}$$

Note : In general, σ is NOT a constant number, but rather a *function of position and applied E-field.*

A more commonly used property, **resistivity** ($\boldsymbol{\rho}$), is defined as $\rho = \dfrac{1}{\sigma}$

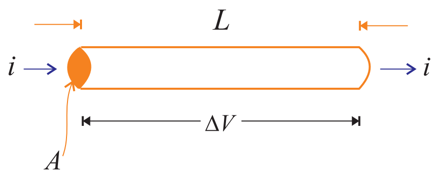
$$\therefore \quad \vec{E} = \rho \vec{j}$$

Unit of ρ : Ohm-meter (Ωm)
where Ohm (Ω) = Volt/Ampere

<u>OHM'S LAW:</u>
Ohmic materials have resistivity that are *independent of the applied electric field.*
`i.e.` metals (in not too high E-field)

Example :

Consider a **resistor** (ohmic material) of length L and cross-sectional area A.



$\therefore$ Electric field inside conductor:

$$\Delta V = \int \vec{E} \cdot d\vec{s} = E \cdot L \quad \Rightarrow \quad E = \frac{\Delta V}{L}$$

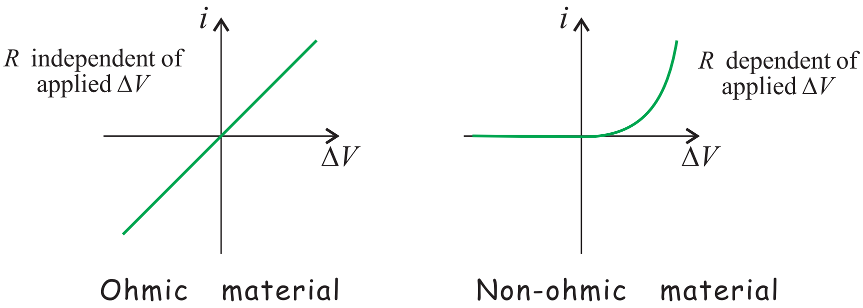
Current density: $\quad j = \dfrac{i}{A}$

$$\therefore \quad \rho = \frac{E}{j}$$
$$\rho = \frac{\Delta V}{L} \cdot \frac{1}{i/A}$$

$$\boxed{\frac{\Delta V}{i} = R = \rho \frac{L}{A}}$$

where R is the **resistance** of the conductor.

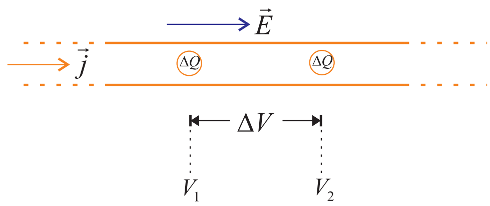
Note: $\Delta V = iR$ is NOT a statement of Ohm's Law. It's just a definition for resistance.



(Read Chap. 29-4 of Halliday Vol 2)

ENERGY IN CURRENT:

Assuming a charge ΔQ enters with potential V_1 and leaves with potential V_2 :



$\therefore$ Potential energy lost in the wire:

$$\Delta U = \Delta Q\, V_2 - \Delta Q\, V_1$$
$$\Delta U = \Delta Q (V_2 - V_1)$$

$\therefore$ Rate of energy lost per unit time

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t}(V_2 - V_1)$$

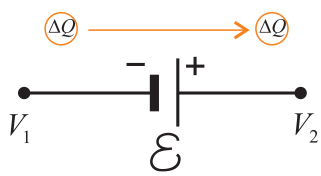
Joule's heating $P = i \cdot \Delta V =$ Power dissipated in conductor

For a resistor R, $P = i^2 R = \dfrac{\Delta V^2}{R}$

5.9 DC Circuits

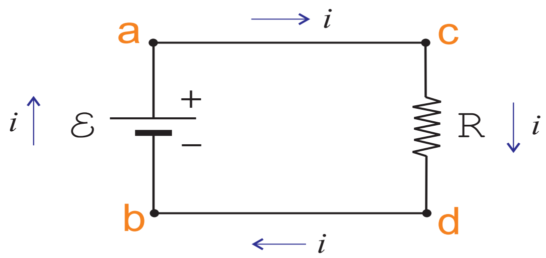
A **battery** is a device that *supplies electrical energy* to maintain a current in a circuit.

In moving from point 1 to 2, electric potential energy increase by $\Delta U = \Delta Q(V_2 - V_1) =$ Work done by $\mathcal{E}$



Define $\mathcal{E} =$ Work done/charge $= V_2 - V_1$

Example :



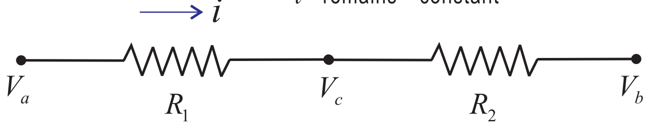
$$\left.\begin{array}{rcl} V_a & = & V_c \\ V_b & = & V_d \end{array}\right\} \quad \text{assuming}^{(1)} \text{ perfect conducting wires.}$$

$$\text{By Definition:} \quad \begin{array}{rcl} V_c - V_d & = & iR \\ V_a - V_b & = & \mathcal{E} \end{array}$$

$$\therefore \quad \mathcal{E} = iR \quad \Rightarrow \quad i = \frac{\mathcal{E}}{R}$$

Also, we have assumed$^{(2)}$ zero resistance inside battery.

Resistance in combination :



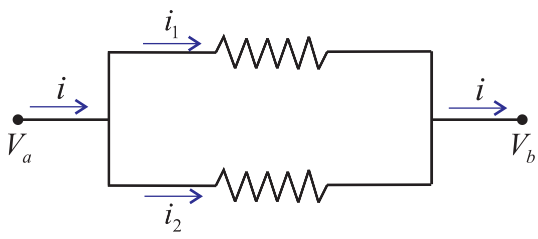
Potential differece (P.D.)

$$\begin{array}{rcl} V_a - V_b & = & (V_a - V_c) + (V_c - V_b) \\ & = & iR_1 + iR_2 \end{array}$$

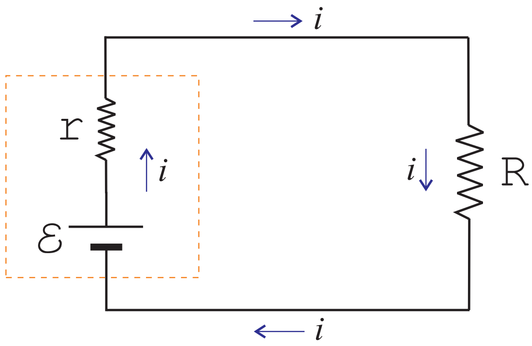
$\therefore$ Equivalent Resistance

$$R = R_1 + R_2 \qquad \text{for resistors in series}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{for resistors in parallel}$$



Example :



For real battery, there is an **internal resistance** that we cannot ignore.

$$\therefore \quad \mathcal{E} = i(R+r)$$
$$i = \frac{\mathcal{E}}{R+r}$$

Joule's heating in resistor R :

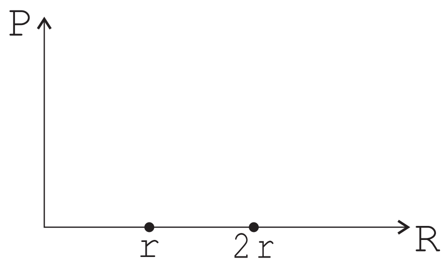
$$P = i \cdot (\text{P.D. across resistor } R)$$
$$= i^2 R$$
$$P = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

Question: What is the value of R to obtain *maximum* Joule's heating?

Answer: We want to find R to *maximize* P.

$$\frac{dP}{dR} = \frac{\mathcal{E}^2}{(R+r)^2} - \frac{\mathcal{E}^2\, 2R}{(R+r)^3}$$

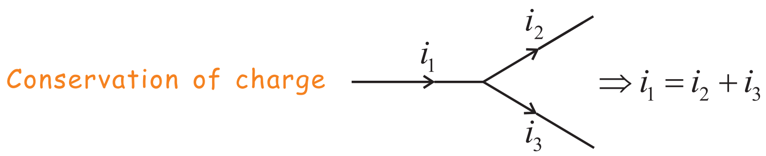
$$\text{Setting} \quad \frac{dP}{dR} = 0 \quad \Rightarrow \quad \frac{\mathcal{E}^2}{(R+r)^3}\left[(R+r) - 2R\right] = 0$$
$$\Rightarrow \quad r - R = 0$$
$$\Rightarrow \quad R = r$$



ANALYSIS OF COMPLEX CIRCUITS:

KIRCHOFF'S LAWS:

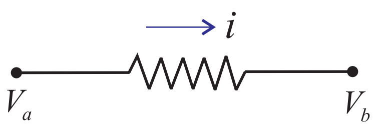
(1) First Law (Junction Rule):
Total current entering a junction equal to the total current leaving the junction.



(2) Second Law (Loop Rule):
The sum of potential differences around a complete circuit loop is zero.

Convention :

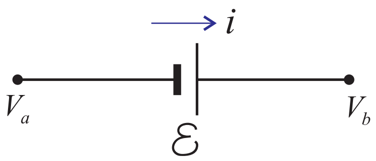
(i)



$$V_a > V_b \quad \Rightarrow \quad \text{Potential difference} = -iR$$

i.e. Potential *drops* across resistors

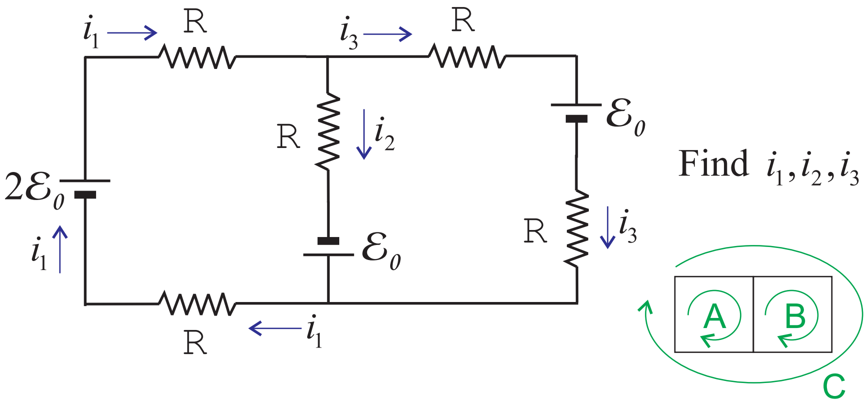
(ii)



$$V_b > V_a \quad \Rightarrow \quad \text{Potential difference} = +\mathcal{E}$$

i.e. Potential *rises* across the negative plate of the battery.

Example :



By junction rule:

$$i_1 = i_2 + i_3 \tag{5.1}$$

By loop rule:

$$\text{Loop A} \Rightarrow 2\mathcal{E}_0 - i_1 R - i_2 R + \mathcal{E}_0 - i_1 R = 0 \tag{5.2}$$

$$\text{Loop B} \Rightarrow -i_3 R - \mathcal{E}_0 - i_3 R - \mathcal{E}_0 + i_2 R = 0 \tag{5.3}$$

$$\text{Loop C} \Rightarrow 2\mathcal{E}_0 - i_1 R - i_3 R - \mathcal{E}_0 - i_3 R - i_1 R = 0 \tag{5.4}$$

BUT: (5.4) = (5.2) + (5.3)
General rule: Need only 3 equations for 3 current

$$i_1 = i_2 + i_3 \tag{5.1}$$

$$3\mathcal{E}_0 - 2i_1 R - i_2 R = 0 \tag{5.2}$$

$$-2\mathcal{E}_0 + i_2 R - 2i_3 R = 0 \tag{5.3}$$

Substitute (5.1) into (5.2) :

$$3\mathcal{E}_0 - 2(i_2 + i_3)R - i_2 R = 0$$

$$\Rightarrow \quad 3\mathcal{E}_0 - 3i_2 R - 2i_3 R = 0 \tag{5.4}$$

Subtract (5.3) from (5.4), `i.e.` (5.4)−(5.3)

$$3\mathcal{E}_0 - (-2\mathcal{E}_0) - 3i_2 R - i_2 R = 0$$

$$\Rightarrow \quad \boxed{i_2 = \frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}}$$

Substitute i_2 into (5.3) :

$$-2\mathcal{E}_0 + \left(\frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}\right) R - 2i_3 R = 0$$

$$\Rightarrow \qquad \boxed{i_3 = -\frac{3}{8}\cdot\frac{\mathcal{E}_0}{R}}$$

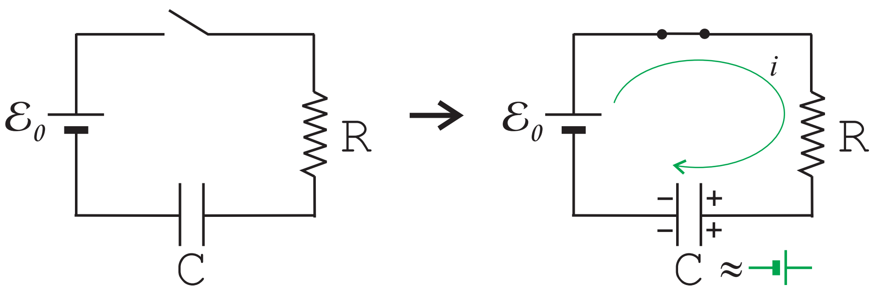
Substitute i_2, i_3 into (5.1) :

$$\boxed{i_1 = \left(\frac{5}{4} - \frac{3}{8}\right)\frac{\mathcal{E}_0}{R} = \frac{7}{8}\cdot\frac{\mathcal{E}_0}{R}}$$

Note: A *negative* current means that it is flowing in *opposite direction* from the one assumed.

5.10 RC Circuits

(A) *Charging* a capacitor with battery:



Using the loop rule:

$$+\mathcal{E}_0 \quad - \underbrace{iR}_{\substack{\text{P.D.}\\ \text{across R}}} \quad - \underbrace{\frac{Q}{C}}_{\substack{\text{P.D.}\\ \text{across C}}} \quad = 0$$

Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *increasing*.

$$\therefore \quad \mathcal{E} = R\overbrace{\frac{dQ}{dt}}^{i} + \frac{Q}{C} \qquad \text{1st order differential eqn.}$$

$$\Rightarrow \quad \frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= 0$

$$\int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC}$$

$$
\begin{aligned}
& -\ln(\mathcal{E}C - Q)\Big|_0^Q = \frac{t}{RC}\Big|_0^t \\
\Rightarrow\ & -\ln(\mathcal{E}C - Q) + \ln(\mathcal{E}C) = \frac{t}{RC} \\
\Rightarrow\ & \ln\Big(\frac{1}{1-\frac{Q}{\mathcal{E}C}}\Big) = \frac{t}{RC} \\
\Rightarrow\ & \frac{1}{1-\frac{Q}{\mathcal{E}C}} = e^{t/RC} \\
\Rightarrow\ & \frac{Q}{\mathcal{E}C} = 1 - e^{-t/RC} \\
\Rightarrow\ & \boxed{Q(t) = \mathcal{E}C(1 - e^{-t/RC})}
\end{aligned}
$$

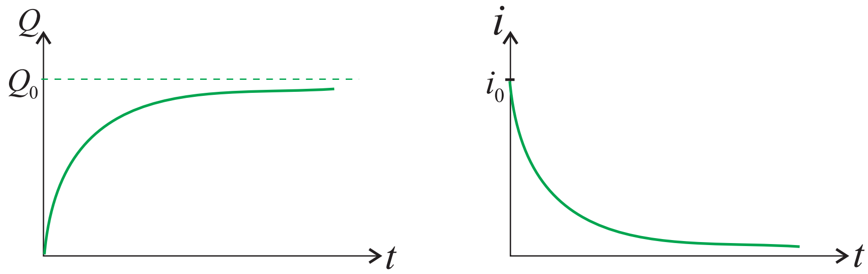
Note: (1) At $t = 0$, $Q(t=0) = \mathcal{E}C(1-1) = 0$

(2) As $t \to \infty$, $Q(t \to \infty) = \mathcal{E}C(1-0) = \mathcal{E}C$
$=$ Final charge on capacitor (Q_0)

(3) Current:

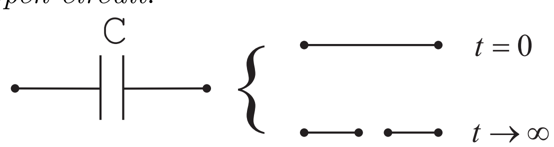
$$
\begin{aligned}
i &= \frac{dQ}{dt} \\
&= \mathcal{E}C\Big(\frac{1}{RC}\Big)e^{-t/RC} \\
i(t) &= \frac{\mathcal{E}}{R}e^{-t/RC}
\end{aligned}
$$

$$
\begin{cases}
i(t=0) = \frac{\mathcal{E}}{R} = \text{Initial current} = i_0 \\
i(t \to \infty) = 0
\end{cases}
$$



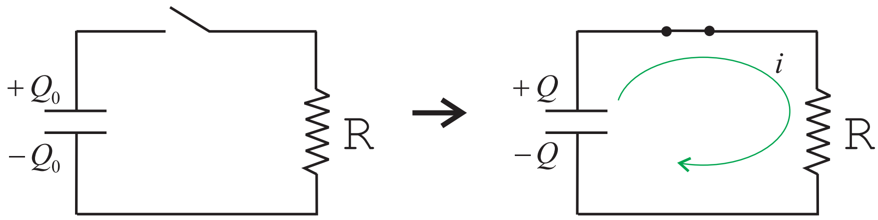
(4) At time $= 0$, the capacitor acts like *short circuit* when there is *zero charge on the capacitor*.

(5) As time $\to \infty$, the capacitor is *fully charged* and current $= 0$, it acts like a *open circuit*.



(6) $\tau_c = RC$ is called the **time constant**. It's the time it takes for the charge to reach $(1 - \frac{1}{e})\, Q_0 \simeq 0.63 Q_0$

(B) *Discharging* a charged capacitor:



Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *decreasing*.

$$\therefore \quad i = -\frac{dQ}{dt}$$

Loop Rule:

$$\begin{aligned} & V_c - iR = 0 \\ \Rightarrow \quad & \frac{Q}{C} + \frac{dQ}{dt} R = 0 \\ \Rightarrow \quad & \frac{dQ}{dt} = -\frac{1}{RC}\, Q \end{aligned}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= Q_0$

$$\begin{aligned} & \int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC} \int_0^t dt \\ \Rightarrow \quad & \ln Q - \ln Q_0 = -\frac{t}{RC} \\ \Rightarrow \quad & \ln\Big(\frac{Q}{Q_0}\Big) = -\frac{t}{RC} \\ \Rightarrow \quad & \frac{Q}{Q_0} = e^{-t/RC} \\ \Rightarrow \quad & Q(t) = Q_0\, e^{-t/RC} \\ (i = -\frac{dQ}{dt}) \qquad \Rightarrow \quad & i(t) = \frac{Q_0}{RC}\, e^{-t/RC} \\ (\text{At } t = 0) \qquad \Rightarrow \quad & i(t=0) = \frac{1}{R} \cdot \underbrace{\frac{Q_0}{C}}_{\text{Initial P.D. across capacitor}} \end{aligned}$$

$$i_0 = \frac{V_0}{R}$$

At $\quad t = RC = \tau \qquad\qquad Q(t = RC) \;=\; \frac{1}{e}\,Q_0 \;\simeq\; 0.37Q_0$