Lecture Note On Mathematical Statistics I

B.Sc. in Mathematics

Fourth Stage

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استمارة انجاز الخطة التدريسية للمادة

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اسم المادة	الاحصاء الرياضي١
مقرر الفصل	
اهداف المادة	١ - ان يتعلم الطالب كيفية ايجاد توزيع حاصل جمع او ضرب او قسمة او طرح متغيرين او اكثر. ٢ - ان يتعلم ويتعرف على مفاهيم المعاينة والاحصاءة المرتبة والتقارب التصادفي ٣ - ان يفهم الطالب موضوع التقدير والتقدير بنقطة من خلال استخدام طرق التقدير . ٤ - ان يدرس الطالب كيفية اختيار المقدر الجيد من بين مجموعة من المقدرات من خلال دراسة ومعرفة خواص المقدرات.
التفاصيل الاساسية للمادة	**Chapter 1** **Additional Topics in Probability** **Special Distribution Functions** : The Binomial Probability Distribution , Poisson Probability Distribution , Uniform Probability Distribution , Normal Probability Distribution , Gamma Probability Distribution ,**Distributions of Functions of random Variables** (Transformation technique, Distribution Function technique, Moment generating function technique), **LIMIT THEOREMS**: CHEBYSHEV'S THEOREM, LAW OF LARGE NUMBERS, CENTRAL LIMIT THEOREM (CLT) **Chapter 2** **Sampling Distributions** SAMPLING DISTRIBUTIONS ASSOCIATED WITH NORMAL POPULATIONS , Distribution of $\overline{X}$ and $\frac{nS^2}{\sigma^2}$, Chi-Square Distribution, Student t-Distribution, F-Distribution, Distributions of Order statistics, Large sample

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Approximations: The Normal Approximation to the Binomial Distribution,), Limiting Distribution : Stochastic Convergence, Limiting of moment generating functions, Theorems on Limiting distributions.

<u>Chapter 3</u>

<u>Point Estimation</u>

The Method of Moments, The Method of Maximum Likelihood, SOME DESIRABLE PROPERTIES OF POINT ESTIMATORS, Unbiased Estimators, Sufficiency, Consistency, Efficiency, Minimal Sufficiency and Minimum-Variance Unbiased Estimation, Cramér–Rao procedure to test for efficiency,
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CHAPTER 1

Additional Topics in Probability

1-1 Probability Distributions

We present some common probability distributions that are useful in mathematical methods that will be used in this lecture note. We give the density function, mean, variance and moment generating function (mgf).

Name	pfd	Mean	Variance	mgf
Bernoulli distribution	$f(x,p)=\begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{otherwise.} \end{cases}$ $0<p<1$	p	$p(1-p)$	$q+pe^t$, $q=1-p$
Binomial	$f(x,n,p)=\binom{n}{x}p^x q^{n-x}$, $x=0,1,\ldots,n$	np	npq	$(q+pe^t)^n$
Geometric	$f(x,p)=q^{x-1}\ p, x=1,2,\ldots$ $0<p\le 1$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
Hyper-geometric	$f(x,N,m,n)=\frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$, $N=0,1,2,\ldots,m=0,1,\ldots,N$, $n=0,1,\ldots,N$	$\frac{nm}{N}$	$\frac{n\left(\frac{m}{N}\right)\left(1-\frac{m}{N}\right)(N-n)}{N-1}$	

Name	pfd	Mean	Variance	mgf
Negative binomial	$f(x, r, p) = \binom{x+r-1}{x} p^r q^x$ $x = 0, 1, 2, \ldots$	$r\frac{q}{p}$	$r\frac{q}{p^2}$	$\left(\frac{p}{1-qe^t}\right)^r$
Poisson	$f(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$	λ	λ	$\exp(\lambda(e^t - 1))$
Beta	$f(x, \alpha, \beta) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Chi-square	$f(x, \nu) = \frac{2^{\nu/2} x^{\nu-1} e^{-x^2/2}}{\Gamma(x/2)}$, $x \geq 0, \nu > 0$(degrees of freedom)	$\sqrt{2}\frac{\Gamma((\boldsymbol{\nu}+1)/2)}{\Gamma(\boldsymbol{\nu}/2)}$	$\nu - \mu^2$	
Exponential	$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$ $\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(1 - \frac{t}{\lambda}\right)^{-1}$
Gamma	$f(x, \alpha, \beta) = x^{\alpha-1}\frac{\beta^\alpha e^{-\beta^x}}{\Gamma(\alpha)}$, $x > 0, \alpha > 0, \beta > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(1 - \frac{t}{\beta}\right)^{-\alpha}$, $t < \beta$
Laplace	$f(x, \mu, \sigma) = \frac{1}{2\sigma}\exp\left(-\frac{\lvert x-\mu\rvert}{\sigma}\right)$, $-\infty < x, \mu$	μ	$2\sigma^2$	
Normal	$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right)$, $-\infty < x, \mu < \infty, \sigma > 0$	μ	σ^2	$\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$
Uniform	$f(x, a, b) = \frac{1}{b-a}$, $a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$

1-2 Distributions of Functions of random Variables

In many statistical applications, given the probability distribution of a univariate random variable X, one would like to know the probability distribution of another univariate random variable $Y = \phi(X)$, where ϕ is some known function. For example, if we know the probability distribution of the random variable X, we would like know the distribution of $Y = \ln(X)$. For univariate random variable X, some commonly used transformed random variable Y of X are: $Y = X^2$, $Y = |X|$, $Y = \sqrt{|X|}$, $Y = \ln(X)$, $Y = \frac{X-\mu}{\sigma}$, and $Y = \left(\frac{X-\mu}{\sigma}\right)^2$. Similarly for a bivariate random variable (X, Y), some of the most common transformations of X and Y are $X + Y$, XY, $\frac{X}{Y}$, $\min\{X, Y\}$, $\max\{X, Y\}$ or $\sqrt{X^2 + Y^2}$.

In this section we discuss the methods of finding the probability distribution of a function of a random variable X. We are given the distribution of X, and we are required to find the distribution of $g(X)$. There are many physical problems that call for the derivation of the distribution of a function of a random variable. The following is one of the classical examples. The velocity V of a gas molecule (Maxwell–Boltzmann law) behaves as a gamma-distributed random variable. We would like to derive the distribution of $E = mV^2$, the kinetic energy of the gas molecule. Because the value of the velocity is the outcome of a random experiment, so is the value of E. This is a problem of finding the distribution of a function of a random variable $E = g(V)$. We now illustrate various techniques for finding the distribution of $g(X)$ by means of examples.

1-2-1 Distribution Functiom Method

Basically the *method of distribution functions* is as follows. If X is a random variable with pdf $f_X(x)$ and if Y is some function of X, then we can find the cdf $F_Y(y) = P(Y \leq y)$ directly by integrating $f_X(x)$ over the region for which $\{Y \leq y\}$. Now, by differentiating $F_Y(y)$, we get the probability density function $f_Y(y)$ of Y. In general, if Y is a function of random variables $X_1, \ldots, X_n$, say $g(X_1, \ldots, X_n)$, then we can summarize the method of distribution function as follows.

PROCEDURE TO FIND CDF OF A FUNCTION OF R.V. USING THE METHOD OF DISTRIBUTION FUNCTIONS

1. Find the region $\{Y \leq y\}$ in the $(x_1, x_2, \ldots, x_n)$ space, that is find the set of $(x_1, x_2, \ldots, x_n)$ for which $g(x_1, \ldots, x_n) \leq y$.
2. Find $F_Y(y) = P(Y \leq y)$ by integrating $f(x_1, x_2, \ldots, x_n)$ over the region $\{Y \leq y\}$.
3. Find the density function $f_Y(y)$ by differentiating $F_Y(y)$.

Example 1-1

Let $X \sim N(0, 1)$. Using the cdf of X, find the pdf of $Y = e^X$.

Solution :

Note that the pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad ; -\infty < x < \infty$$

Then the cumulative distribution function of Y for a given y > 0 is

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(e^X \leq y) \\ &= P(x \leq \ln y) \\ &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \end{aligned}$$

Hence, by differentiating $F_Y(y)$,we obtain the probability density function as

$$f(y) = \begin{cases} \frac{1}{y\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} & ; 0 < y \\ 0 & other\ wise \end{cases}$$

Example 1-2

Let $X \sim N(0, 1)$. Using the cdf of X, find the pdf of X^2.

Solution

Let $Y = X^2$. Note that the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

Then the cumulative distribution function of Y for a given $y \geq 0$ is

$$\begin{aligned} F(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= 2 \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad \text{(by the symmetry of } e^{-x^2/2}\text{)}. \end{aligned}$$

Hence, by differentiating $F(y)$, we obtain the probability density function as

$$\begin{aligned} f_Y(y) &= \frac{2}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} \\ &= \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This is a χ^2-distribution with 1 degree of freedom.

Example 1-3

Suppose that the random variable X has a Poisson probability distribution

$$f(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function of $Y = aX + b$.

Solution

The cdf of Y is given by

$$F(y) = P(Y \le y) = P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right) = \sum_{x=0}^{\left[\frac{y-b}{a}\right]} \frac{e^{-\lambda}\lambda^x}{x!},$$

where $[x]$ is the largest integer less than or equal to x. Therefore,

$$F(y) = \begin{cases} 0, & y < b \\ \sum_{x=0}^{\left[\frac{y-b}{a}\right]} \frac{e^{-\lambda}\lambda^x}{x!}, & y \ge b. \end{cases}$$

It should be noted here that the pmf, $f_Y(y)$ of Y, can be found from the equation

$$f_Y(y) = F_Y(y) - F_Y(y-1), \quad \text{for } y = an + b, \quad n = 0, 1, 2, \dots$$

1-2-2 Transformation Method

Theorem .1. Let X be a continuous random variable with probability density function $f(x)$. Let $y = T(x)$ be an increasing (or decreasing) function. Then the density function of the random variable $Y = T(X)$ is given by

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y))$$

where $x = W(y)$ is the inverse function of $T(x)$.

Proof: Suppose $y = T(x)$ is an increasing function. The distribution function $G(y)$ of Y is given by

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(T(X) \leq y) \\ &= P(X \leq W(y)) \\ &= \int_{-\infty}^{W(y)} f(x)\, dx. \end{aligned}$$

Then, differentiating we get the density function of Y, which is

$$\begin{aligned} g(y) &= \frac{dG(y)}{dy} \\ &= \frac{d}{dy}\left(\int_{-\infty}^{W(y)} f(x)\, dx\right) \\ &= f(W(y))\, \frac{dW(y)}{dy} \\ &= f(W(y))\, \frac{dx}{dy} \qquad \text{(since} \quad x = W(y)). \end{aligned}$$

On the other hand, if $y = T(x)$ is a decreasing function, then the distribution function of Y is given by

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(T(X) \leq y) \\ &= P(X \geq W(y)) \qquad \text{(since } T(x) \text{ is decreasing)} \\ &= 1 - P(X \leq W(y)) \\ &= 1 - \int_{-\infty}^{W(y)} f(x)\, dx. \end{aligned}$$

As before, differentiating we get the density function of Y, which is

$$\begin{aligned} g(y) &= \frac{dG(y)}{dy} \\ &= \frac{d}{dy}\left(1 - \int_{-\infty}^{W(y)} f(x)\,dx\right) \\ &= -f(W(y))\,\frac{dW(y)}{dy} \\ &= -f(W(y))\,\frac{dx}{dy} \qquad \text{(since} \quad x = W(y)). \end{aligned}$$

Hence, combining both the cases, we get

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y))$$

and the proof of the theorem is now complete.

Example 1-4 Let $Z = \frac{X-\mu}{\sigma}$. If $X \sim N(\mu, \sigma^2)$, what is the probability density function of Z?

Answer:

$$z = U(x) = \frac{x-\mu}{\sigma}.$$

Hence, the inverse of U is given by

$$\begin{aligned} W(z) &= x \\ &= \sigma\, z + \mu. \end{aligned}$$

Therefore

$$\frac{dx}{dz} = \sigma.$$

Hence, by Theorem 10.1, the density of Z is given by

$$\begin{aligned} g(z) &= \left|\frac{dx}{dz}\right| f(W(y)) \\ &= \sigma\,\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{1}{2}\left(\frac{W(z)-\mu}{\sigma}\right)^2} \\ &= \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{z\sigma+\mu-\mu}{\sigma}\right)^2} \\ &= \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}z^2}. \end{aligned}$$

Example 1-5 Let $Z = \frac{X-\mu}{\sigma}$. If $X \sim N(\mu, \sigma^2)$, then show that Z^2 is chi-square with one degree of freedom, that $Z^2 \sim \chi^2(1)$.

Answer:

$$y = T(x) = \left(\frac{x-\mu}{\sigma}\right)^2.$$

$$x = \mu + \sigma\sqrt{y}.$$

$$W(y) = \mu + \sigma\sqrt{y}, \qquad y > 0.$$

$$\frac{dx}{dy} = \frac{\sigma}{2\sqrt{y}}.$$

The density of Y is

$$\begin{aligned}
g(y) &= \left|\frac{dx}{dy}\right| f(W(y)) \\
&= \sigma \frac{1}{2\sqrt{y}} f(W(y)) \\
&= \sigma \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{W(y)-\mu}{\sigma}\right)^2} \\
&= \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}\left(\frac{\sqrt{y}\sigma+\mu-\mu}{\sigma}\right)^2} \\
&= \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}y} \\
&= \frac{1}{2\sqrt{\pi}\sqrt{2}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \\
&= \frac{1}{2\Gamma\left(\frac{1}{2}\right)\sqrt{2}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}.
\end{aligned}$$

Hence $Y \sim \chi^2(1)$.

Theorem 2. Let X and Y be two continuous random variables with joint density $f(x,y)$. Let $U = P(X,Y)$ and $V = Q(X,Y)$ be functions of X and Y. If the functions $P(x,y)$ and $Q(x,y)$ have single valued inverses, say $X = R(U,V)$ and $Y = S(U,V)$, then the joint density $g(u,v)$ of U and V is given by

$$g(u,v) = |J| \ f(R(u,v), S(u,v)),$$

where J denotes the Jacobian and given by

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Example 1-6

Let X and Y be independent random variables with common pdf $f(x)=e^{-x}$, (x > 0). Find the joint pdf of
U = X/(X + Y) , V = X + Y.

Solution :

$$f(x,y)=f(x).f(y)=e^{-x}e^{-y}$$

We have U =X/(X+Y)=X/V. Hence, X=UV and Y =V −X=V −UV=V(1−U). Thus, the Jacobian

$$J=\begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}=v(1-u)-(-vu)=v$$

.

Then |J| = v . Note that $0 \le u \le 1$, $0 < v < \infty$.

$$f(u,v)=f(h_1^{-1}(u,v),h_2^{-1}(u,v)) \cdot |J|$$
$$= e^{-vu}\, e^{-v(1-u)}\, v = ve^{-v} \qquad , 0 \le u \le 1,\ 0 < v < \infty.$$

Suppose we want the marginal $f(v)$ and $f(u)$, that is,

$$f(u)=\int_0^\infty ve^{-v}\,dv=1 \qquad \Rightarrow \qquad f(u)=1 \quad ; 0 \le u \le 1$$

$$f(v)=\int_0^1 ve^{-v}\,du=ve^{-v} \quad\Rightarrow\quad f(v)=ve^{-v} \quad ; 0\le v<\infty$$

Sometimes the expressions for two variables, *U* and *V*, may not be given. Only one expression is available. In that case, call the given expression of *X* and *Y* as *U*, and define *V* = *Y*. Then, we can use the previous method to first find the joint density and then find the marginal to obtain the pdf of *U*. The following example demonstrates the method .

If we talking about discrete random variables the Jacobian determined is dropped .

Example 1- 7

Let X and Y be independent random variables uniformly distributed on [0, 1]. Find the distribution of X+Y.

Solution :

$$f(x,y)=f(x).f(y)=1$$

Let U = X+Y and let V=Y ,Hence X = U-V and Y = V .

Thus, the Jacobian

$$J=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1$$

.

Then |J| = 1 .

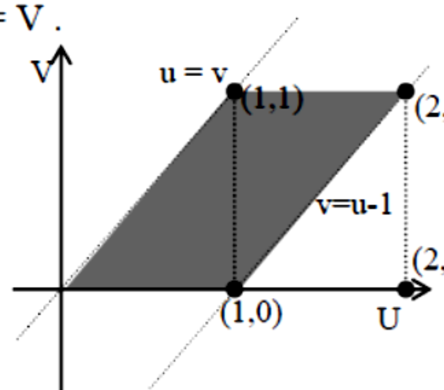


Figure 1 :The regions of integration.

If x=0 then 0=u-v i.e v=u

If x=1 then 1=u-v i.e v=u-1

$$f(u,v)=f(h_1^{-1}(u,v),h_2^{-1}(u,v))\cdot|J| =\begin{cases}1 & ; 0\le u-v\le 1 \;,\; 0\le v\le 1\\ 0 & \text{other wise}\end{cases}$$

Because V is the variable we introduced, to get the pdf of U, we just need to find the marginal pdf from the joint pdf. From Figure 1, the regions of integration are $0 \le u \le 1$, and $0 \le u \le 2$. That is,

$$f(v)=\begin{cases} \int_0^u 1du = u & ; \quad 0 \le u \le 1 \\ \int_{v-1}^1 1du = 2-u & ; \quad 0 \le u \le 2 \end{cases}$$

Example 1- 8

Let X and Y be independent random variables such that $X \sim P(x,\lambda_1)$ and $Y \sim P(y,\lambda_2)$. Find the distribution of X+Y

Solution :

since $X \sim P(x,\lambda_1) \Rightarrow P(x)=e^{-\lambda_1}\frac{\lambda_1^x}{x!}$, also $Y \sim P(y,\lambda_2) \Rightarrow P(y)=e^{-\lambda_2}\frac{\lambda_2^y}{y!}$

since X and Y are independent

$$P(x,y)=P(x).P(y)=e^{-\lambda_1}\frac{\lambda_1^x}{x!}e^{-\lambda_2}\frac{\lambda_2^y}{y!}$$

Space of X and Y

$\Omega=\{(x,y);\ x=0,1,2,3,..... \quad , y=0,1,2,3,......... \}$

Let U = X+Y and let V = Y

So

u = x + y $=h_1(x,y)$ and v = y = $h_2(x,y)$, such that h_1, h_2 are a (1-1) trans. Map. from a space Ω onto the space β of u , v .

and

x = u – v = $h_1^{-1}(u,v)$ and y = v = $h_2^{-1}(u,v)$, such that h_1^{-1}, h_2^{-1} are a (1-1) trans. Map. from a space β onto the space Ω of x , y .

$$x \ge 0 \Rightarrow u-v \ge 0 \Rightarrow u \ge v$$

$$y \ge 0 \Rightarrow v \ge 0 \Rightarrow v=0,1,.....,u$$

$$\beta=\{(u,v)\ ;\ u\geq v\ ,\ v=0,1,....,u\ \}$$

The joint pmf of U and V is

$$g(u,v)=g[h_1^{-1},h_2^{-1}]=e^{-(\lambda_1+\lambda_2)}\frac{\lambda_1^{u-v}\ \lambda_2^{v}}{(u-v)!\ v!}\qquad ;u,v\in\beta$$

The pmf of U is

$$g(u)=\sum_{v=0}^{u}e^{-(\lambda_1+\lambda_2)}\frac{\lambda_1^{u-v}\ \lambda_2^{v}}{(u-v)!\ v!}=e^{-(\lambda_1+\lambda_2)}\sum_{v=0}^{u}\frac{\lambda_1^{u-v}\ \lambda_2^{v}}{(u-v)!\ v!}$$

$$=e^{-(\lambda_1+\lambda_2)}\sum_{v=0}^{u}\frac{u!}{u!}\frac{\lambda_1^{u-v}\ \lambda_2^{v}}{(u-v)!\ v!}=e^{-(\lambda_1+\lambda_2)}\frac{1}{u!}\sum_{v=0}^{u}u!\frac{\lambda_1^{u-v}\ \lambda_2^{v}}{(u-v)!\ v!}$$

$$=e^{-(\lambda_1+\lambda_2)}\frac{1}{u!}\sum_{v=0}^{u}\binom{u}{v}\lambda_1^{u-v}\ \lambda_2^{v}$$

$$=e^{-(\lambda_1+\lambda_2)}\frac{1}{u!}(\lambda_1+\lambda_2)^{u}$$

$$\therefore\ g(u=x+y)=e^{-(\lambda_1+\lambda_2)}\frac{1}{(x+y)!}(\lambda_1+\lambda_2)^{x+y}\qquad ;x+y=0,1,.........$$

Thus ,

$X+Y\sim\ P(x+y\ .\lambda_1+\lambda_2)$

Example 1- 9

Let $X\sim f(x)=2x\ ;0<x<1$. Find the distribution of $Y=8X^3$.

Solution :

Space of X is

$\Omega=\{x\ ;\ x\in(0,1)\ \}$

Space of Y is

$\beta=\{\,y\ ;\ \ 0<y<8\ \}$

$$y=8x^3=h(x)\quad\Rightarrow x=\frac{\sqrt[3]{y}}{2}=h^{-1}(y)$$

$$|J|=\left|\frac{dx}{dy}\right|=\frac{1}{2}\ \frac{1}{3}y^{\frac{-1}{3}}=\frac{1}{6}y^{\frac{-2}{3}}$$

$$g(y)=f[h^{-1}(y)]\ |J|=2\left(\frac{\sqrt[3]{y}}{2}\right)\left(\frac{1}{6}y^{\frac{-2}{3}}\right)=\frac{1}{6}y^{\frac{-1}{3}}\qquad ;0<y<8$$

Example 1- 10

Let X and Y be independent random variables uniformly distributed on [0, 1]. Find the distribution of U where U= X+Y and V = X – Y

Solution :

$\because X \sim U(0,1) \Rightarrow f(x)=1 \quad also \;\; Y \sim U(0,1) \Rightarrow f(y)=1$

$\therefore f(x,y)=f(x).f(y)=1$

$\Omega=\{(x,y);0<x<1\,,\,0<y<1\}$

$$u=x+y=h_1(x,y) \quad \Rightarrow x=\frac{1}{2}(u+v)=h_1^{-1}(u,v)$$

$$v=x-y=h_2(x,y) \quad \Rightarrow y=\frac{1}{2}(u-v)=h_2^{-1}(u,v)$$

Thus, the Jacobian

$$J=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}=-\frac{1}{2}$$

Then $|J| = \frac{1}{2}$.

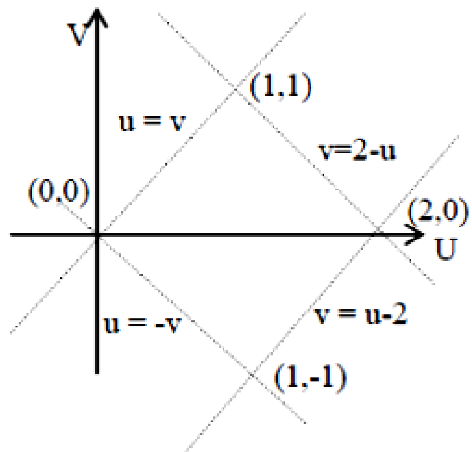


Figure 2 :The regions of integration.

$$If\;\; x=0 \;\Rightarrow 0=\frac{1}{2}(u+v)\Rightarrow u=-v$$

$$If\;\; x=1 \;\Rightarrow 1=\frac{1}{2}(u+v)\Rightarrow u=2-v$$

$$If\;\; y=0 \;\Rightarrow 0=\frac{1}{2}(u-v)\Rightarrow u=v$$

$$If\;\; y=1 \;\Rightarrow 1=\frac{1}{2}(u-v)\Rightarrow v=u-2$$

$\beta_1=\{(u,v)\,;\,-u<v<u \;\;,0<u<1\}$

$\beta_2=\{(u,v)\,;\,u-2<v<2-u \;\;,1<u<2\}$

$\beta=\beta_1\cup\beta_2$

$$g(u,v)=f(h_1^{-1}(u,v),h_2^{-1}(u,v))\cdot|J|=\frac{1}{2}$$

$$g(u)=\begin{cases}\displaystyle\int_{-u}^{u}\frac{1}{2}dv=u & ;0<u<1\\ \displaystyle\int_{u-2}^{2-u}\frac{1}{2}dv=2 & ;1<u<2\end{cases}$$

Example 1- 11

Let $X \sim f(x)=1 \quad , 0<x<1$, Let Y= $-4\ln X$, Define the pdf of Y .

Solution :

Space of X is

$\Omega=\{x;0<x<1 \ \}$

Space of Y is

$\beta=\{y;0<x<\infty \ \}$

$$y=-4\ln x=h(x) \quad \Rightarrow x=e^{\frac{-y}{4}}=h^{-1}(y)$$

$$|J|=\left|\frac{dx}{dy}\right|=\left|\frac{-1}{4}e^{\frac{-y}{4}}\right| = \frac{1}{4}e^{\frac{-y}{4}}$$

$$g(y)=f[h^{-1}(y)] \ |J|= \ (1)\left(\frac{1}{4}e^{\frac{-y}{4}}\right)=\frac{1}{4}e^{\frac{-y}{4}}$$

$$g(y) = \frac{1}{4}e^{\frac{-y}{4}} \qquad ;0<y<\infty$$

Hence , $Y \sim EXP(4)$

OR

$$g(y) = \frac{1}{4\Gamma(1)}y^{1-1}e^{\frac{-y}{4}} \qquad ;0<y<\infty \qquad since \ \ \Gamma(1) = (1-1)!=0!=1$$

Thus , $Y \sim Gamma(1,4)$

OR

$Y \sim \chi^2_{(4)}$

Exampl 1-12 . Let each of the independent random variables X and Y have the density function

$$f(x) = \begin{cases} e^{-x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the joint density of $U = X$ and $V = 2X + 3Y$ and the domain on which this density is positive?

Answer: Since

$$\left.\begin{aligned} U &= X \\ V &= 2X + 3Y, \end{aligned}\right\}$$

we get by solving for X and Y

$$\left.\begin{aligned} X &= U \\ Y &= \frac{1}{3}V - \frac{2}{3}U. \end{aligned}\right\}$$

Hence, the Jacobian of the transformation is given by

$$\begin{aligned} J &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= 1 \cdot \left(\frac{1}{3}\right) - 0 \cdot \left(-\frac{2}{3}\right) \\ &= \frac{1}{3}. \end{aligned}$$

The joint density function of U and V is

$$\begin{aligned} g(u, v) &= |J| \; f(R(u, v), \, S(u, v)) \\ &= \left|\frac{1}{3}\right| f\left(u, \; \frac{1}{3}v - \frac{2}{3}u\right) \\ &= \frac{1}{3}e^{-u} \; e^{-\frac{1}{3}v + \frac{2}{3}u} \\ &= \frac{1}{3}\, e^{-\left(\frac{u+v}{3}\right)}. \end{aligned}$$

Since

$$\begin{aligned} 0 < x < \infty \\ 0 < y < \infty, \end{aligned}$$

we get

$$\begin{aligned} 0 < u < \infty \\ 0 < v < \infty, \end{aligned}$$

Further, since $v = 2u + 3y$ and $3y > 0$, we have

$$v > 2u.$$

Hence, the domain of $g(u, v)$ where nonzero is given by

$$0 < 2u < v < \infty.$$

The joint density $g(u, v)$ of the random variables U and V is given by

$$g(u,v) = \begin{cases} \frac{1}{3}\, e^{-\left(\frac{u+v}{3}\right)} & \text{for } 0 < 2u < v < \infty \\ 0 & \text{otherwise.} \end{cases}$$

1-2-3 Moment generating function Method

Notes :

- If $Y = aX + b$ then $m_Y(t) = e^{bt}\ m_X(at)$
- $m_{Y+X}(t) = m_Y(at)\ m_X(t)$

Theorem 1-1 :
Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

- **Summary of the Moment-Generating Function Method :**
Let U be a function of the random variables $Y_1, Y_2, \ldots\ldots, Y_n$.
1. Find the moment-generating function for U, $m_U(t)$.
2. Compare $m_U(t)$. with other well-known moment-generating functions. If $m_U(t) = m_V(t)$ for all values of t, Theorem 6.1 implies that U and V have identical distributions.

This

Example 1- 13
Let X and Y be independent random variables Gamma distributed on [α, 1]. Find the distribution of X+Y.

Solution :

$m_x(t)=(1-t)^{-\alpha}$, $m_y(t)=(1-t)^{-\alpha}$

$$m_{x+y}(t)=E\left(e^{t(x+y)}\right)=E\left(e^{tx+ty}\right)=E\left(e^{tx}\right)E\left(e^{ty}\right)=m_x(t)\, m_y(t)$$

$$=(1-t)^{-\alpha}\,.\,(1-t)^{-\alpha}$$

$$=(1-t)^{-2\alpha}$$

That is a moment generating of Gamma(2α ,1),Thus

X+Y ~ Gamma(2α ,1) .

In general if a r.v's $X_i \sim G(\alpha_i , \beta)$, $\forall i=1,2,...,n$, Find the pdf of Y = $\sum_{i=1}^{n} X_i$

$m_{x_i}(t)=(1-\beta t)^{-\alpha_i}$, $\forall i=1,2,...,n$

$$m_y(t)=E\left(e^{ty}\right)=E\left(e^{t(x_1+x_2+....+x_n)}\right)=E\left(e^{tx_1+tx_2+....+tx_n}\right)$$

$$=E\left(e^{tx_1}\right)E\left(e^{tx_2}\right)...E\left(e^{tx_n}\right)=m_{x_1}(t)\, m_{x_2}(t).....\, m_{x_n}(t)$$

$$=(1-\beta t)^{-\alpha_1}\,.(1-\beta t)^{-\alpha_2}....\,(1-\beta t)^{-\alpha_n}$$

$$=(1-\beta t)^{-\sum_{i=1}^{n}\alpha_i}$$

That is a moment generating of Gamma($\sum_{i=1}^{n}\alpha_i$,1),Thus

Y ~ Gamma($\sum_{i=1}^{n}\alpha_i$,1) .

Example 1- 14

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables such that $X_i \sim N(\mu_i , \sigma_i^2)$, $\forall i=1,2,...,n$, Find the pdf of Y = $\sum_{i=1}^{n} a_i X_i$, a is constant .

Solution :

$$m_{x_i}(t)=EXP\left\{\ \mu_i t+\frac{1}{2}\sigma_i^2 t^2\right\}\quad ; i=1,2,\dots,n$$

$$m_y(t)=E\left(e^{ty}\right)=E\left(e^{t(a_1x_1+a_2x_2+\dots+a_nx_n)}\right)=E\left(e^{ta_1x_1+ta_2x_2+\dots+ta_nx_n}\right)$$

$$=E\left(e^{ta_1x_1}\right)E\left(e^{ta_2x_2}\right)\dots E\left(e^{ta_nx_n}\right)=m_{x_1}(a_1t)\,m_{x_2}(a_2t)\dots\dots\ m_{x_n}(a_nt)$$

$$=EXP\left\{\ \mu_1 a_1 t+\frac{1}{2}\sigma_1^2 a_1^2 t^2\right\}.EXP\left\{\ \mu_1 a_1 t+\frac{1}{2}\sigma_2^2 a_2^2 t^2\right\}\dots\dots EXP\left\{\ \mu_n t+\frac{1}{2}\sigma_n^2 a_n^2 t^2\right\}$$

$$=EXP\left\{\left(\sum_{i=1}^n \mu_i a_i\right)t+\frac{1}{2}\left(\sum_{i=1}^n \sigma_i^2 a_i^2\right)t^2\right\}$$

That is a moment generating function of $N\left(\left(\sum_{i=1}^n \mu_i a_i\right),\left(\sum_{i=1}^n \sigma_i^2 a_i^2\right)\right)$,Thus

$Y \sim N\left(\left(\sum_{i=1}^n \mu_i a_i\right),\left(\sum_{i=1}^n \sigma_i^2 a_i^2\right)\right)$.

For a special case that is if X,Y $\sim N(\mu,\sigma^2)$ then X-Y $\sim N(\mu-\mu,\sigma^2+\sigma^2)$

i.e X-Y $\sim N(0,2\sigma^2)$

Example 1- 15

Let $Y_1, Y_2, \dots, Y_n$ be independent and identically distributed random variables such that for $0 < p < 1$, $P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1 - p$. (Such random variables are called Bernoulli random variables.)

Let $W = Y_1 + Y_2, \dots + Y_n$, What is the distribution of W?

Solution :

$$m_y(t)=\left(e^t p+q\right)$$

$$m_W(t)=E\left(e^{tW}\right)=E\left(e^{t(y_1+y_2+\dots+y_n)}\right)=E\left(e^{ty_1+ty_2+\dots+ty_n}\right)$$

$$=E\left(e^{ty_1}\right)E\left(e^{ty_2}\right)\dots E\left(e^{ty_n}\right)=m_{y_1}(t)\,m_{y_2}(t)\dots\dots\ m_{y_n}(t)$$

$$=\underbrace{\left(e^t p+q\right).\left(e^t p+q\right)\dots\left(e^t p+q\right)}_{n-times}$$

$$=\left(e^t p+q\right)^n$$

That is a moment generating of b(n,p) ,Thus

W ~ b(n,p).

Example 1- 16

If X~N(0,1) then $X^2 \sim \chi^2_{(1)}$?

Solution :

Let $Y = X^2$

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad ,-\infty<x<\infty$$

$$m_y(t)=E\left(e^{ty}\right)=E\left(e^{tx^2}\right)=\int_{-\infty}^{\infty}e^{tx^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}+tx^2}\,dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}(1-2t)}\,dx=\frac{1}{(1-2t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{(1-2t)^{\frac{1}{2}}}{\sqrt{2\pi}}e^{-\frac{x^2}{2}(1-2t)}\,dx$$

$$=(1-2t)^{-\frac{1}{2}}$$

That is a moment generating of $\chi^2_{(1)}$,Thus

$X^2 \sim \chi^2_{(1)}$.

In general If $X_1^2, X_2^2, \ldots\ldots, X_n^2 \sim N(0,1)$, then $X_1^2+X_2^2+\ldots\ldots+X_n^2 \sim \chi^2_{(n)}$

Example 1- 16

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables such that $X_i \sim N(\mu,\sigma^2)$, $\forall i=1,2,\ldots,n$, Find the pdf of $Y = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$

Solution :

$$m_y(t)=E\left(e^{ty}\right)=E\left(e^{t\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}\right)=E\left(e^{t\left(\frac{x_1-\mu}{\sigma}\right)^2+t\left(\frac{x_2-\mu}{\sigma}\right)^2+\ldots+t\left(\frac{x_n-\mu}{\sigma}\right)^2}\right)$$

$$=E\left(e^{t\left(\frac{x_1-\mu}{\sigma}\right)^2}\right)E\left(e^{t\left(\frac{x_2-\mu}{\sigma}\right)^2}\right)\ldots E\left(e^{t\left(\frac{x_n-\mu}{\sigma}\right)^2}\right)=\left(E\left(e^{t\left(\frac{x-\mu}{\sigma}\right)^2}\right)\right)^n$$

$$= \left(\int_{-\infty}^{\infty} e^{t\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right)^n = \left(\int_{-\infty}^{\infty} e^{t\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right)^n$$

$$= \left(\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}+t\left(\frac{x-\mu}{\sigma}\right)^2} \right)^n = \left(\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}(1-2t)} \right)^n$$

$$Let \quad \left(\frac{x-\mu}{\sigma}\right)\sqrt{(1-2t)} = y \Rightarrow x = \sigma\frac{1}{\sqrt{(1-2t)}} y + \mu \Rightarrow dx = \sigma\frac{1}{\sqrt{(1-2t)}} dy$$

$$\therefore \left(\frac{1}{(1-2t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right)^n = \left(\frac{1}{(1-2t)^{\frac{1}{2}}} \overbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}^{pdf\ of\ N(0,1)\ so\ it\ is\ equal 1} \right)^n$$

$$= (1-2t)^{\frac{-n}{2}}$$

That is a moment generating function of $\chi^2_{(n)}$,Thus
$Y \sim \chi^2_{(n)}$.

EXERCISES :

1-1- Let $X_1, X_2 \sim N(0,1)$ prove that the pdf of $Y = \frac{X_1}{X_2}$ is Cauchy distribution .

1-2- Let X , Y be two indep. R.v. 's such that $X \sim G(\alpha_1, 1)$, $Y \sim G(\alpha_2, 1)$, Let $W_1 = X + Y$ and $W_2 = \frac{X}{X+Y}$. Find the pdf of W_1 and W_2 .

1-3- Let $X \sim f(x) = 1$, $0 < x < 1$, Let Y= -2 lnX , Define the pdf of Y .

1-4- Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with pdf $f(x) = \frac{1}{\lambda} e^{\frac{-x}{\lambda}}$, $x > 0$, $\lambda > 0$. Find the pdf of $\sum_{i=1}^{n} X_i$.

1-5- The joint pdf of X and Y is $f(x,y) = \theta e^{-(x+\theta y)}$, $x > 0, \theta > 0$. Find the pdf of XY.

1-6- If Y_1 and Y_2 are independent and identically distributed normal random variables with mean μ and variance σ^2, find the probability density function for $U = (1/2)(Y_1 - 3Y_2)$.

1-7- Let $X \sim G(\alpha, \beta)$, Show that $Y = \frac{2X}{\beta} \sim \chi^2_{(2\alpha)}$.

1-8- The joint pdf of *(X, Y)* is

$$f(x,y) = \frac{1}{\theta^2} e^{-\frac{x+y}{\theta}} \qquad x, y > 0 \qquad , \theta > 0 .$$

Find the pdf of $U = X - Y$.

1-9- If the joint pdf of *(X, Y)* is

$$f(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2} e^{-\frac{x^2+y^2}{4\sigma_1^2\sigma_2^2}} \quad , -\infty < x < \infty \; , \; -\infty < y < \infty \; ; \; \sigma_1, \sigma_2 > 0$$

find the pdf of $X^2 + Y^2$.

Chapter 2

Sampling Distributions

We call the probability distribution of a sample statistic its *sampling distribution*. Sampling distributions provide the link between probability theory and statistical inference. The ability to determine the distribution of a statistic is a critical part in the construction and evaluation of statistical procedures. It is important to observe that there is a difference between the distribution of population from which the sample was taken and the distribution of the sample statistic. In general, a population has a distribution called a population distribution, which is usually unknown, whereas a statistic has a sampling distribution, which is usually different from the population distribution. *The sampling distribution of a statistic provides a theoretical model of the relative frequency histogram for the likely values of the statistic that one would observe through repeated sampling*.

Definition 2-1 A sample is a set of observable random variables X1, . . . , Xn. The number n is called the sample size.

In most of the inferential procedures that we study in this book, we are dealing with random samples. We call the random variables $X_1, \ldots, X_n$ identically distributed if every X_i has the same probability distribution.

Definition 2-2 A random sample of size n from a population is a set of n independent and identically distributed (iid) observable random variables $X_1, \ldots, X_n$.

Definition 2-3 A function T of observable random variables $X_1, \ldots, X_n$ that does not depend on any unknown parameters is called a **statistic**.

For example, suppose that we want to estimate a population mean μ. If we obtain a random sample of n observations, $x_1, x_2, \ldots, x_n$, it seems reasonable to estimate μ with the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The goodness of this estimate depends on the behavior of the random variables $X_1, X_2, \ldots, X_n$ and the effect that this behavior has on $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Notice that the random variable $\bar{x}$ is a function of (only) the random variables $X_1, X_2, \ldots, X_n$ and the (constant) sample size n. The random variable X is therefore an example of a ***statistic.***

Definition 2-4 The probability distribution of a sample statistic is called the sampling distribution.

2 1- The Distribution of $\overline{X}$:

Let $X_1, X_2, ...and\ X_n$ are independent and identically distributed normal random variables with mean μ and variance σ^2, then the way to find the Distribution of $\overline{X}$ is

$$m_x(t) = EXP\left\{ \mu t + \frac{1}{2}\sigma^2 t^2 \right\}$$

$$m_{\overline{X}}(t) = E\left(e^{t\overline{X}}\right) = E\left(e^{\frac{t}{n}(x_1 + x_2 + ... + x_n)} \right) = E\left(e^{\frac{t}{n}x_1 + \frac{t}{n}x_2 + ... + \frac{t}{n}x_n} \right)$$

$$= E\left(e^{\frac{t}{n}x_1} \right) E\left(e^{\frac{t}{n}x_2} \right) ....E\left(e^{\frac{t}{n}x_n} \right)$$

$$= m_{x_1}(\frac{t}{n})\ m_{x_2}(\frac{t}{n}) .....\ m_{x_n}(\frac{t}{n})$$

$$= \underbrace{EXP\left\{ \mu\frac{t}{n} + \frac{1}{2n^2}\sigma^2 t^2 \right\}.\ EXP\left\{ \mu\frac{t}{n} + \frac{1}{2n^2}\sigma^2 t^2 \right\} ......EXP\left\{ \mu\frac{t}{n} + \frac{1}{2n^2}\sigma^2 t^2 \right\}}_{n-times}$$

$$= EXP\left\{ n\mu\frac{t}{n} + n\frac{1}{2n^2}\sigma^2 t^2 \right\} = EXP\left\{ \mu t + \frac{1}{2n}\sigma^2 t^2 \right\}$$

That is a moment generating function of $N\left(\mu, \frac{\sigma^2}{n} \right)$,Thus

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n} \right)$.

That is

$$f(\overline{X}) = \sqrt{\frac{n}{2\pi}}\ \frac{1}{\sigma} e^{-\frac{n(\overline{X}-\mu)^2}{2\sigma^2}} \qquad ; -\infty < \overline{X} < \infty$$

To drive the mean and var. of $\overline{X}$:

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i \right) = \frac{1}{n}\left(E(x_1) + E(x_2) + ... + E(x_n) \right) = \frac{1}{n}\left(\underbrace{\mu + \mu + ... + \mu}_{n-times} \right) = \frac{1}{n} n\mu = \mu$$

$$\therefore E(\overline{X}) = \mu$$

$$\operatorname{var}(\overline{X})=\operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)=\frac{1}{n^2}(\operatorname{var}(x_1)+\operatorname{var}(x_2)+......+\operatorname{var}(x_n))=\frac{1}{n}\left(\underbrace{\sigma^2+\sigma^2+........+\sigma^2}_{n-times}\right)$$

$$=\frac{1}{n^2}n\sigma^2=\frac{1}{n}\sigma^2$$

$$\therefore \operatorname{var}(\overline{X})=\frac{1}{n}\sigma^2$$

Example 2-1: Let $X_1, X_2, ...and\ X_n$ are independent and identically distributed $G(\alpha,\beta)$, Find the Distribution of $\overline{X}$?

Solution :

$$m_x(t)=(1-\beta t)^{-\alpha}$$

$$m_{\overline{X}}(t)=E\left(e^{t\overline{X}}\right)=E\left(e^{\frac{t}{n}(x_1+x_2+....+x_n)}\right)=E\left(e^{\frac{t}{n}x_1+\frac{t}{n}x_2+....+\frac{t}{n}x_n}\right)$$

$$=E\left(e^{\frac{t}{n}x_1}\right)E\left(e^{\frac{t}{n}x_2}\right)....E\left(e^{\frac{t}{n}x_n}\right)$$

$$=m_{x_1}(\frac{t}{n})\ m_{x_2}(\frac{t}{n}).....\ m_{x_n}(\frac{t}{n})$$

$$=\underbrace{\left(1-\beta\frac{t}{n}\right)^{-\alpha}.\left(1-\beta\frac{t}{n}\right)^{-\alpha}.......\left(1-\beta\frac{t}{n}\right)^{-\alpha}}_{n-times}.$$

$$=\left(1-\frac{\beta}{n}t\right)^{-n\alpha}.$$

That is a moment generating function of $G\left(n\alpha, \frac{\beta}{n}\right)$,Thus

$\overline{X} \sim G\left(n\alpha, \frac{\beta}{n}\right)$.

That is

$$f(\overline{X}) = \frac{n^{\alpha}}{\beta^{\alpha}\,\Gamma(n\alpha)}(\overline{x})^{n\alpha-1} e^{-\frac{n\overline{x}}{\beta}} \qquad ; 0 < \overline{X} < \infty$$

To drive the mean and var. of $\overline{X}$:

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\left(E(x_1) + E(x_2) + \ldots\ldots + E(x_n)\right) = \frac{1}{n}\left(\underbrace{\alpha\beta + \alpha\beta + \ldots\ldots\ldots + \alpha\beta}_{n-times}\right) = \frac{1}{n} n\alpha\beta = \alpha\beta$$

$$\therefore E(\overline{X}) = \alpha\beta$$

$$\operatorname{var}(\overline{X}) = \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\left(\operatorname{var}(x_1) + \operatorname{var}(x_2) + \ldots\ldots + \operatorname{var}(x_n)\right) = \frac{1}{n}\left(\underbrace{\alpha\beta^2 + \alpha\beta^2 + \ldots\ldots\ldots + \alpha\beta^2}_{n-times}\right)$$

$$= \frac{1}{n^2} n\alpha\beta^2 = \frac{1}{n}\alpha\beta^2$$

$$\therefore \operatorname{var}(\overline{X}) = \frac{1}{n}\alpha\beta^2$$

2-2- The Distribution of S^2 :

we want to find The Distribution of S^2 , so we must be first find the distr. of $\frac{nS^2}{\sigma^2}$.

Let $X_1 , X_2 , ...and\ X_n$ are independent and identically distributed normal random variables with mean μ and variance σ^2

we have

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}\left(x_i-\overline{X}+\overline{X}-\mu\right)^2=\sum_{i=1}^{n}\left\{\left(x_i-\overline{X}\right)^2+2\left(x_i-\overline{X}\right)\left(\overline{X}-\mu\right)+\left(\overline{X}-\mu\right)^2\right\}$$

$$=\sum_{i=1}^{n}\left(x_i-\overline{X}\right)^2+2\sum_{i=1}^{n}\left(x_i-\overline{X}\right)\left(\overline{X}-\mu\right)+\sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2$$

$$=\sum_{i=1}^{n}\left(x_i-\overline{X}\right)^2+0+\sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2$$

$$\therefore\ \sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}\left(x_i-\overline{X}\right)^2+n\left(\overline{X}-\mu\right)^2$$

$$\Rightarrow\ \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}=\frac{n}{\sigma^2}\sum_{i=1}^{n}\frac{\left(x_i-\overline{X}\right)^2}{n}+n\frac{\left(\overline{X}-\mu\right)^2}{\sigma^2}$$

$$\Rightarrow\frac{n}{\sigma^2}\sum_{i=1}^{n}\frac{\left(x_i-\overline{X}\right)^2}{n}=\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}-n\frac{\left(\overline{X}-\mu\right)^2}{\sigma^2}$$

$$\Rightarrow\frac{n}{\sigma^2}S^2=\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}-n\frac{\left(\overline{X}-\mu\right)^2}{\sigma^2}$$

We have $\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\ \sim\ \chi^2_{(n)}$ and $n\frac{\left(\overline{X}-\mu\right)^2}{\sigma^2}\ \sim\ \chi^2_{(1)}$

Thus ,

$$\frac{n}{\sigma^2}S^2\ \sim\ \chi^2_{(n-1)}$$

Hence ,

$$S^2\ \sim\frac{\sigma^2}{n}\chi^2_{(n-1)}$$

What about the mean and var. of S^2 ? I will use the properties of mean and var. to find them .

Since $\frac{n}{\sigma^2}S^2\ \sim\ \chi^2_{(n-1)}$, then $E\left(\frac{n}{\sigma^2}S^2\right)=n-1$

$$\Rightarrow \frac{n}{\sigma^2} E(S^2) = n-1 \Rightarrow E(S^2) = \frac{\sigma^2(n-1)}{n}$$

Also ,

$$\text{var}\left(\frac{n}{\sigma^2} S^2\right) = 2(n-1) \Rightarrow \frac{n^2}{\sigma^4} \text{var}(S^2) = 2(n-1)$$

$$\Rightarrow \text{var}(S^2) = \frac{2(n-1)\,\sigma^4}{n^2}$$

Exercises :

2-1- Let $X_1, X_2, \ldots, X_n$ denote a R.S. of size n $(n > 2)$ from N(1,2) , Find the pdf of $\overline{X}$ and Drive mean , var. and mgf of it .

2-2- If $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ Show that :

a) $\left(\frac{\overline{X} - \mu}{\sigma / \sqrt{n}}\right) \sim N(0,1)$

b) $\left(\frac{\overline{X} - \mu}{\sigma / \sqrt{n}}\right)^2 \sim \chi^2_{(1)}$

c) $\sum_{i=1}^{n} \left(\frac{\overline{X} - \mu}{\sigma / \sqrt{n}}\right)^2 \sim \chi^2_{(n)}$.

2-3-Let $X_1, X_2, \ldots, X_n$ denote a R.S. of size n $(n > 2)$ from G(2,2) , Find the pdf of $\overline{X}$ and Drive mean , var. and mgf of it .

Now we introduce some distributions that can be derived from a normal distribution. These distributions play a very important role in inferential problems.

Since the normal population is very important in statistics, the sampling distributions associated with the normal population are very important. The most important sampling distributions which are associated with the normal

2-3 Chi-Square Distribution

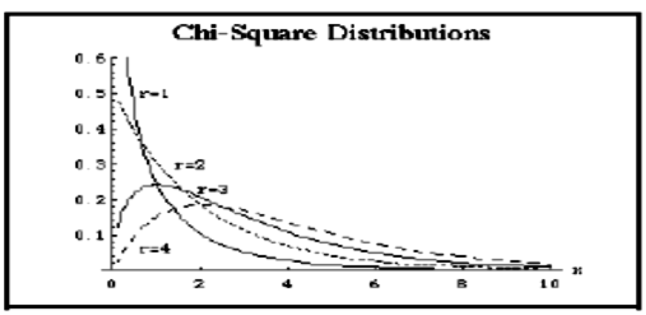
A chi-square distribution is used in many inferential problems, for example, in inferential problems dealing with the variance. Recall that the chi-square distribution is a special case of a gamma distribution with $\alpha = n/2$ and $\beta = 2$. If n is a positive integer, then the parameter n is called the ***degrees of freedom***. However, if n is not an integer, but $\beta = 2$, we still refer to this distribution as a chi-square. The mgf of a χ^2- random variable is $M(t) = (1-2t)^{-n/2}$. The mean and variance of a chi-square distribution are $\mu = n$ and $\sigma^2 = 2n$, respectively. That is, the mean of a $\chi^2(n)$ random variable is equal to its degree of freedom and the variance is twice the degree of freedom. We now give some useful results for χ^2- random variables.

Definition 2- 5:

A continuous random variable X is said to have a chi-square distribution with r degrees of freedom if its probability density function is of the form

$$f(x;r) = \begin{cases} \frac{1}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}}\, x^{\frac{r}{2}-1}\, e^{-\frac{x}{2}} & \text{if } 0 \le x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $r > 0$. If X has chi-square distribution, then we denote it by writing $X \sim \chi^2(r)$. Recall that a gamma distribution reduces to chi-square distribution if $\alpha = \frac{r}{2}$ and $\theta = 2$. The mean and variance of X are r and $2r$, respectively.



Thus, chi-square distribution is also a special case of gamma distribution. Further, if $r \to \infty$, then chi-square distribution tends to normal distribution.

Theorem 4.2.3 *Let $X_1, \ldots, X_k$ be independent χ^2- random variables with $n_1, \ldots, n_k$ degrees of freedom, respectively. Then the sum $V = \sum_{i=1}^{k} X_i$ is chi-square distributed with $n_1 + n_2 + \cdots + n_k$ degrees of freedom.*

Proof. The mgf of V is

$$M_V(t) = \prod_{i=1}^{k} (1-2t)^{-n_i/2} = (1-2t)^{-\left(\sum_{i=1}^{k} n_i\right)/2}.$$

This implies that $V \sim \chi^2\left(\sum_{i=1}^{k} n_i\right)$.

Theorem 4.2.5 *If a random variable X has a gamma distribution with parameters α and β, then*

$$Y = \frac{2X}{\beta} \sim \chi^2(2\alpha).$$

Proof. Recall that the mgf of the gamma random variable X is $(1-\beta t)^{-\alpha}$.

$$\begin{aligned} M_Y(t) &= M_{\frac{2X}{\beta}}(t) = E\left(e^{\frac{2X}{\beta}t}\right) \\ &= E\left(e^{X\left(\frac{2}{\beta}t\right)}\right) = M_X\left(\frac{2}{\beta}t\right) \\ &= (1-2t)^{-\alpha} = (1-2t)^{-\frac{2\alpha}{2}}. \end{aligned}$$

Hence, $Y \sim \chi^2(2\alpha)$. □

The following result states that by squaring a standard normal random variable, we can generate a chi-square random variable, with one degree of freedom.

Theorem 4.2.6 *If X is a standard normal random variable, then X^2 is chi-square random variable with 1 d.f.*

Proof. Because $X \sim N(0, 1)$ the moment-generating function of X^2 is

$$M_{X^2}(t) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = (1-2t)^{-1/2}.$$

This implies that $X^2 \sim \chi^2(1)$. Figure 4.1 gives the probability densities of the random variables X and X^2. □

Theorem 4.2.7 *Let the random sample $X_1, \ldots, X_n$ be from a $N(\mu, \sigma^2)$ distributed. Then $Z_i = (X_i - \mu)/\sigma, i = 1, \ldots, n$ are independent standard normal random variables and*

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

has a χ^2-distribution with n degrees of freedom. In particular, if $X_1, \ldots, X_n$ are independent standard normal random variables, then $Y^2 = \sum_{i=1}^{n} X_i^2$ is chi-square distributed with n degrees of freedom.

Example 14.1. If $X \sim GAM(1,1)$, then what is the probability density function of the random variable $2X$?

Answer: We will use the moment generating method to find the distribution of $2X$. The moment generating function of a gamma random variable is given by

$$M(t) = (1 - \theta\, t)^{-\alpha}, \qquad \text{if} \qquad t < \frac{1}{\theta}.$$

Since $X \sim GAM(1,1)$, the moment generating function of X is given by

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1.$$

Hence, the moment generating function of $2X$ is

$$\begin{aligned} M_{2X}(t) &= M_X(2t) \\ &= \frac{1}{1-2t} \\ &= \frac{1}{(1-2t)^{\frac{2}{2}}} \\ &= \text{MGF of } \chi^2(2). \end{aligned}$$

Hence, if X is $GAM(1,1)$ or is an exponential with parameter 1, then $2X$ is chi-square with 2 degrees of freedom.

Example 14.2. If $X \sim \chi^2(5)$, then what is the probability that X is between 1.145 and 12.83?

Answer: The probability of X between 1.145 and 12.83 can be calculated from the following:

$$
\begin{aligned}
&P(1.145 \leq X \leq 12.83) \\
&\qquad = P(X \leq 12.83) - P(X \leq 1.145) \\
&\qquad = \int_0^{12.83} f(x)\, dx - \int_0^{1.145} f(x)\, dx \\
&\qquad = \int_0^{12.83} \frac{1}{\Gamma\left(\frac{5}{2}\right) 2^{\frac{5}{2}}} x^{\frac{5}{2}-1} e^{-\frac{x}{2}}\, dx - \int_0^{1.145} \frac{1}{\Gamma\left(\frac{5}{2}\right) 2^{\frac{5}{2}}} x^{\frac{5}{2}-1} e^{-\frac{x}{2}}\, dx \\
&\qquad = 0.975 - 0.050 \qquad \text{(from } \chi^2 \text{ table)} \\
&\qquad = 0.925.
\end{aligned}
$$

Example 14.3. If $X \sim \chi^2(7)$, then what are values of the constants a and b such that $P(a < X < b) = 0.95$?

Answer: Since

$$0.95 = P(a < X < b) = P(X < b) - P(X < a),$$

we get

$$P(X < b) = 0.95 + P(X < a).$$

We choose $a = 1.690$, so that

$$P(X < 1.690) = 0.025.$$

From this, we get

$$P(X < b) = 0.95 + 0.025 = 0.975$$

Thus, from chi-square table, we get $b = 16.01$.

2-4 **Student t-Distribution**

Let $W \sim N(0,1)$ and $V \sim \chi^2_{(r)}$ and both W and V are indep. . Let $T = \frac{W}{\sqrt{V/r}}$, we say that T has t-distribution with r degree freedom (df) .

i.e. $T = \frac{W}{\sqrt{V/r}} \sim t(r)$?

Let us find the pdf of T .

$T = \frac{W}{\sqrt{V/r}}$ let U = V

$\because W \sim N(0,1) \Rightarrow f_1(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$, $-\infty < w < \infty$ and $f_2(v) = \frac{1}{2\Gamma\left(\frac{r}{2}\right)}\left(\frac{v}{2}\right)^{\frac{r}{2}-1} e^{\frac{-v}{2}}$, v >0 .

Since W and V are indep.

$$\therefore f(w,v) = f_1(w) f_2(v) = \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} \frac{1}{2\Gamma\left(\frac{r}{2}\right)}\left(\frac{v}{2}\right)^{\frac{r}{2}-1} e^{\frac{-v}{2}}$$

Space of W and V :

$\Omega = \{ (w,v) ; -\infty < w < \infty ,\ 0 < v < \infty \}$

$t = \frac{w}{\sqrt{v/r}} \Rightarrow w = \frac{t\sqrt{u}}{\sqrt{r}} = \frac{t u^{\frac{1}{2}}}{\sqrt{r}} = u_1^{-1}(t,u)$ and $u = v \Rightarrow v = u = u_2^{-1}(t,u)$

Space of T and U :

$\beta = \{ (t,u) ; -\infty < t < \infty ,\ 0 < u < \infty \}$

$$|J| = \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{r}}$$

$\because g(t,u) = f(u_1^{-1}, u_2^{-1}) \, |J|$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\Gamma\left(\frac{r}{2}\right)} \left(\frac{u}{2}\right)^{\frac{r}{2}-1} e^{-\frac{tu}{2r}} e^{\frac{-u}{2}} \frac{\sqrt{u}}{\sqrt{r}} = \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} u^{\frac{r}{2}-\frac{1}{2}} e^{-\frac{u}{2}\left(1+t^2/r\right)}$$

$$\therefore g(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} u^{\frac{r}{2}-\frac{1}{2}} e^{-\frac{u}{2}\left(1+t^2/r\right)} du = \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} \int_0^\infty u^{\frac{r}{2}-\frac{1}{2}} e^{-\frac{u}{2}\left(1+t^2/r\right)} du$$

Let $z = u\left(1+\frac{t^2}{r}\right) \Rightarrow u = \frac{z}{\left(1+\frac{t^2}{r}\right)} \Rightarrow du = \frac{dz}{\left(1+\frac{t^2}{r}\right)}$

$$\Rightarrow g(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} \int_o^\infty \left(\frac{z}{\left(1+\frac{t^2}{r}\right)}\right)^{\frac{r}{2}-\frac{1}{2}} e^{\frac{-z}{2}} \frac{dz}{\left(1+\frac{t^2}{r}\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{r}{2}}\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}\Gamma\left(\frac{r}{2}\right)} \int_o^\infty z^{\frac{r+1}{2}-1} e^{\frac{-z}{2}} dz$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{2^{\frac{r+1}{2}}\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}\Gamma\left(\frac{r}{2}\right)} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)} \int_o^\infty z^{\frac{r+1}{2}-1} e^{\frac{-z}{2}} dz$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}\Gamma\left(\frac{r}{2}\right)} \Gamma\left(\frac{r+1}{2}\right) \overbrace{\int_o^\infty \frac{1}{2^{\frac{r+1}{2}}\Gamma\left(\frac{r+1}{2}\right)} z^{\frac{r+1}{2}-1} e^{\frac{-z}{2}} dz}^{=1}$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}\Gamma\left(\frac{r}{2}\right)} \Gamma\left(\frac{r+1}{2}\right)$$

Hence , $g(t) = \frac{1}{\sqrt{\pi}} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}\Gamma\left(\frac{r}{2}\right)} \Gamma\left(\frac{r+1}{2}\right) \quad , -\infty < t < \infty \ , r > 0$

and denoted by $T = \frac{W}{\sqrt{V/r}} \sim t(r)$

Example 14.6. If $T \sim t(10)$, then what is the probability that T is at least 2.228 ?

Answer: The probability that T is at least 2.228 is given by

$$\begin{aligned} P(T \geq 2.228) &= 1 - P(T < 2.228) \\ &= 1 - 0.975 \qquad \text{(from t} - \text{table)} \\ &= 0.025. \end{aligned}$$

Example 14.7. If $T \sim t(19)$, then what is the value of the constant c such that $P\left(|T| \leq c\right) = 0.95$?

Answer:

$$\begin{aligned} 0.95 &= P\left(|T| \leq c\right) \\ &= P\left(-c \leq T \leq c\right) \\ &= P\left(T \leq c\right) - 1 + P\left(T \leq c\right) \\ &= 2\, P\left(T \leq c\right) - 1. \end{aligned}$$

Hence

$$P\left(T \leq c\right) = 0.975.$$

Thus, using the t-table, we get for 19 degrees of freedom

$$c = 2.093.$$

Theorem 14.7. If $X \sim N(\mu, \sigma^2)$ and $X_1, X_2, ..., X_n$ be a random sample from the population X, then

$$\frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t(n-1).$$

Proof: Since each $X_i \sim N(\mu, \sigma^2)$,

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Thus,

$$\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1).$$

Further, **Know that**

$$n\,\frac{S^2}{\sigma^2} \sim \chi^2(n-1).$$

Hence

$$\frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{(n-1)\,S^2}{(n-1)\,\sigma^2}}} \sim t(n-1)$$

Example 14.8. Let X_1, X_2, X_3, X_4 be a random sample of size 4 from a standard normal distribution. If the statistic W is given by

$$W = \frac{X_1 - X_2 + X_3}{\sqrt{X_1^2 + X_2^2 + X_3^2 + X_4^2}},$$

then what is the expected value of W ?

Answer: Since $X_i \sim N(0,1)$, we get

$$X_1 - X_2 + X_3 \sim N(0,3)$$

and

$$\frac{X_1 - X_2 + X_3}{\sqrt{3}} \sim N(0,1).$$

Further, since $X_i \sim N(0,1)$, we have

$$X_i^2 \sim \chi^2(1)$$

and hence

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 \sim \chi^2(4)$$

Thus,

$$\frac{\frac{X_1 - X_2 + X_3}{\sqrt{3}}}{\sqrt{\frac{X_1^2 + X_2^2 + X_3^2 + X_4^2}{4}}} = \left(\frac{2}{\sqrt{3}}\right) W \sim t(4).$$

Now using the distribution of W, we find the expected value of W.

$$\begin{aligned} E\left[W\right] &= \left(\frac{\sqrt{3}}{2}\right) E\left[\frac{2}{\sqrt{3}} W\right] \\ &= \left(\frac{\sqrt{3}}{2}\right) E\left[t(4)\right] \\ &= \left(\frac{\sqrt{3}}{2}\right) 0 \\ &= 0. \end{aligned}$$

Example 14.9. If $X \sim N(0,1)$ and X_1, X_2 is random sample of size two from the population X, then what is the 75^{th} percentile of the statistic $W = \frac{X_1}{\sqrt{X_2^2}}$?

Answer: Since each $X_i \sim N(0,1)$, we have

$$X_1 \sim N(0,1)$$
$$X_2^2 \sim \chi^2(1).$$

Hence

$$W = \frac{X_1}{\sqrt{X_2^2}} \sim t(1).$$

The 75^{th} percentile a of W is then given by

$$0.75 = P(W \leq a)$$

Hence, from the t-table, we get

$$a = 1.0$$

Hence the 75^{th} percentile of W is 1.0.

2-5 F-Distribution

The *F*-distribution was developed by Fisher to study the behavior of two variances from random samples taken from two independent normal populations. In applied problems we may be interested in knowing whether the population variances are equal or not, based on the response of the random samples. Knowing the answer to such a question is also important in selecting the appropriate statistical methods to study their true means.

Let $U \sim \chi^2_{(n)}$ and $V \sim \chi^2_{(r)}$ and both U and V are indep. . Then

$$F = \frac{U/n}{V/r} \quad \sim f(n,r)$$

Let us find the pdf of F .

$$F = \frac{U/n}{V/r} \quad \text{let} \quad Z = V$$

$$\because U \sim \chi^2_{(n)} \Rightarrow f_1(u) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}(u)^{\frac{n}{2}-1} e^{\frac{-u}{2}} \text{ and } f_2(v) = \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}(v)^{\frac{r}{2}-1} e^{\frac{-v}{2}}, \ v > 0 .$$

Since W and V are indep.

$$\therefore f(u,v) = f_1(u)f_2(v) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}(u)^{\frac{n}{2}-1} e^{\frac{-u}{2}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}(v)^{\frac{r}{2}-1} e^{\frac{-v}{2}}$$

Space of U and V :

$$\Omega = \{ (u,v) ; 0 < u < \infty , \ 0 < v < \infty \}$$

$$f = \frac{u/n}{v/r} \Rightarrow u = f\,z\,\frac{n}{r} = u_1^{-1}(f,z) \quad \text{and} \quad z = v \Rightarrow v = z = u_2^{-1}(f,z)$$

Space of F and Z :

$$\beta = \{ (f,z) ; 0 < f < \infty , \quad 0 < z < \infty \}$$

$$|J| = \begin{vmatrix} z\frac{n}{r} & f\frac{n}{r} \\ 0 & 1 \end{vmatrix} = z\frac{n}{r}$$

$$\because g(t,u) = f\left(u_1^{-1}, u_2^{-1}\right)|J|$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\frac{n}{r}f\,z\right)^{\frac{n}{2}-1} e^{\frac{-\left(\frac{n}{r}f\,z\right)}{2}} \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}(z)^{\frac{r}{2}-1} e^{\frac{-z}{2}}\, z\frac{n}{r}$$

$$= \frac{1}{2^{\frac{n+r}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)}\left(\frac{n}{r}\right)^{\frac{n}{2}} f^{\frac{n}{2}-1} z^{\frac{r+n}{2}-1} e^{\frac{-z}{2}\left(\frac{n}{r}f+1\right)}$$

$$\therefore g(f)=\int_{o}^{\infty} \frac{1}{2^{\frac{n+r}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)}\left(\frac{n}{r}\right)^{\frac{n}{2}} f^{\frac{n}{2}-1}\, z^{\frac{r+n}{2}-1}\; e^{\frac{-z}{2}\left(\frac{n}{r}f+1\right)}\; dz$$

Let $y=z\left(\frac{n}{r}f+1\right) \Rightarrow z=\frac{y}{\left(\frac{n}{r}f+1\right)} \Rightarrow dz = \frac{dy}{\left(\frac{n}{r}f+1\right)}$

$$\therefore g(f)=\int_{o}^{\infty} \frac{1}{2^{\frac{n+r}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)}\left(\frac{n}{r}\right)^{\frac{n}{2}} f^{\frac{n}{2}-1}\left(\frac{y}{\left(\frac{n}{r}f+1\right)}\right)^{\frac{r+n}{2}-1} e^{\frac{-y}{2}}\;\frac{dy}{\left(\frac{n}{r}f+1\right)}$$

$$=\frac{f^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)\left(\frac{n}{r}f+1\right)^{\frac{n+r}{2}}}\left(\frac{n}{r}\right)^{\frac{n}{2}}\Gamma\left(\frac{r+n}{2}\right)\overbrace{\int_{o}^{\infty}\frac{1}{2^{\frac{n+r}{2}}\Gamma\left(\frac{r+n}{2}\right)}y^{\frac{r+n}{2}-1}\; e^{\frac{-y}{2}}\; dy}^{=1\,\left(pdf\ of\ \chi^2_{(r+n)}\right)}$$

$$=\frac{f^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)\left(\frac{n}{r}f+1\right)^{\frac{n+r}{2}}}\left(\frac{n}{r}\right)^{\frac{n}{2}}\Gamma\left(\frac{r+n}{2}\right)$$

$$\therefore g(f)=\begin{cases}\dfrac{f^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{r}{2}\right)\left(\frac{n}{r}f+1\right)^{\frac{n+r}{2}}}\left(\frac{n}{r}\right)^{\frac{n}{2}}\Gamma\left(\frac{r+n}{2}\right) & ;\ f>0\ \ ,n>0,r>0\\ 0 & other\ wise\end{cases}$$

Notes 2-1 :

1- If F ~ $f(n,r)$, then $\frac{1}{F}\sim f(r,n)$.

Proof : $F=\frac{U/n}{V/r} \quad \Rightarrow \frac{1}{F}=\frac{V/r}{U/n} \sim f(r,n)$

2- If T ~ t(r) , then $T^2 \sim f(1,r)$.

Proof : $\because T=\frac{W}{\sqrt{V/r}} \quad \Rightarrow T^2=\frac{W^2}{V/r}$ since $W \sim N(0,1) \Rightarrow W^2 \sim \chi^2_{(1)}$, $V/r \sim \chi^2_{(r)}$

Hence , $T^2 \sim f(1,r)$.

Example 14.11. If $X \sim F(9,10)$, what $P(X \geq 3.02)$? Also, find the mean and variance of X.

Answer:

$$P(X \geq 3.02) = 1 - P(X \leq 3.02)$$

$$= 1 - P(F(9,10) \leq 3.02)$$

$$= 1 - 0.95 \qquad (\text{from F} - \text{table})$$

$$= 0.05.$$

Next, we determine the mean and variance of X using the Theorem 14.8. Hence,

$$E(X) = \frac{\nu_2}{\nu_2 - 2} = \frac{10}{10-2} = \frac{10}{8} = 1.25$$

and

$$Var(X) = \frac{2\,\nu_2^2\,(\nu_1 + \nu_2 - 2)}{\nu_1\,(\nu_2 - 2)^2\,(\nu_2 - 4)}$$

$$= \frac{2\,(10)^2\,(19-2)}{9\,(8)^2\,(6)}$$

$$= \frac{(25)\,(17)}{(27)\,(16)}$$

$$= \frac{425}{432} = 0.9838.$$

Example 14.12. If $X \sim F(6,9)$, what is the probability that X is less than or equal to 0.2439 ?

Answer: We use the above theorem to compute

$$\begin{aligned} P(X \leq 0.2439) &= P\left(\frac{1}{X} \geq \frac{1}{0.2439}\right) \\ &= P\left(F(9,6) \geq \frac{1}{0.2439}\right) \\ &= 1 - P\left(F(9,6) \leq \frac{1}{0.2439}\right) \\ &= 1 - P\left(F(9,6) \leq 4.10\right) \\ &= 1 - 0.95 \\ &= 0.05. \end{aligned}$$

Example 14.13. Let $X_1, X_2, ..., X_4$ and $Y_1, Y_2, ..., Y_5$ be two random samples of size 4 and 5 respectively, from a standard normal population. What is the variance of the statistic $T = \left(\frac{5}{4}\right) \frac{X_1^2+X_2^2+X_3^2+X_4^2}{Y_1^2+Y_2^2+Y_3^2+Y_4^2+Y_5^2}$?

Answer: Since the population is standard normal, we get

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 \sim \chi^2(4).$$

Similarly,

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 \sim \chi^2(5).$$

Thus

$$\begin{aligned} T &= \left(\frac{5}{4}\right) \frac{X_1^2 + X_2^2 + X_3^2 + X_4^2}{Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2} \\ &= \frac{\frac{X_1^2+X_2^2+X_3^2+X_4^2}{4}}{\frac{Y_1^2+Y_2^2+Y_3^2+Y_4^2+Y_5^2}{5}} \\ &= T \sim F(4,5). \end{aligned}$$

Therefore

$$\begin{aligned} Var(T) &= Var[F(4,5)] \\ &= \frac{2(5)^2(7)}{4(3)^2(1)} \\ &= \frac{350}{36} \\ &= 9.72. \end{aligned}$$

2-6 **Order Statistics**

In practice, the random variables of interest may depend on the relative magnitudes of the observed variable. For example, we may be interested in the maximum mileage per gallon of a particular class of cars. In this section, we study the behavior of ordering a random sample from a continuous distribution .

Definition 2-5-1:
Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf $f(x)$.
Let $Y_1, \ldots, Y_n$ be a permutation of $X_1, \ldots, X_n$ such that $Y_1 \le Y_2 \le \cdots \le Y_n$.
Then the ordered random variables $Y_1, \ldots, Y_n$ are called the **order statistics** of the random sample $X_1, \ldots, X_n$.
Here Y_k is called the **kth order statistic**. Because of continuity, the equality sign could be ignored .

Remark 2-5-1 . Although X_i 's are iid random variables, the random variables Yi's are neither independent nor identically distributed.
Thus, the minimum of Xi 's is $Y_1 = \min(X_1, \ldots, X_n)$
and the maximum is $Y_n = \max(X_1, \ldots, X_n)$.
The order statistics of the sample $X_1, \ldots, X_n$ can also be denoted by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$.
Here $X_{(k)}$ is the kth order statistic and is equal to Y_k in Definition One of the most commonly used order statistics is the median, the value in the middle position in the sorted order of the values .

Theorem 2-5-1:
Let $X_1, \dots, X_n$ be a random sample from a population with pdf $f(x)$. Then the joint pdf of order statistics $Y_1, \dots, Y_n$ is

$$f(y_1, y_2, \dots, y_n) = \begin{cases} n!\, f(y_1).f(y_2)\dots f(y_n) & \text{for } y_1 < y_2 < \dots < y_n \\ 0 & \text{other wise} \end{cases}$$

The pdf of the k*th* order statistic is given by the following theorem .

Example 2-5-1 :
Find the distribution of the nth order statistic Y_n of the sample $X_1, \dots, X_n$ from a population with pdf $f(x)$.
Solution :
Let the cdf of Y_n be denoted by $F_n(y)$. Then
$F_n(y) = P(Y_n \le y) = P(\max_{1\le i\le n}(X_i) \le y) = P(X_1 \le y,\ X_2 \le y, \dots, X_n \le y)$

$= P(X_1 \le y) . P(X_2 \le y) \dots P(X_n \le y)$

$= [F(y)]^n$ (by independence).

Hence, the pdf $f_n(y)$ of Y_n is

$$f_n(y) = \frac{d[F(y)]^n}{dy} = n[F(y)]^{n-1}\frac{d}{dy}F(y) = n[F(y)]^{n-1} f(y) .$$

Similarly , For pdf of the order statistic Y_1 :

Let the cdf of Y_1 be denoted by $F_1(y)$. Then
$F_1(y) = P(Y_1 \le y) = P(\min_{1\le i\le n}(X_i) \le y) = 1 - P(\min_{1\le i\le n}(X_i) > y)$

$= 1 - P(X_1 > y,\ X_2 > y, \dots, X_n > y)$

$= [1 - P(X > y)]^n = 1 - [1 - P(X \le y)]^n = 1 - [1 - F(y)]^n$, (by independence).

Hence, the pdf $f_1(y)$ of Y_1 is

$$f_1(y) = \frac{d\{1 - [1 - F(y)]^n\}}{dy} = n[1 - F(y)]^{n-1}\frac{d}{dy}F(y) = n[1 - F(y)]^{n-1} f(y) .$$

Theorem 2-5-2 :

Let $X_1, \ldots, X_n$ be a random sample from a population with pdf $f(x)$. Then the joint pdf of order statistics Y_k, Y_r such that $1 \le k < r \le n$, is

$$f(y_k, y_r) = \begin{cases} \dfrac{n!}{(k-1)!(r-1-k)!(n-k)!}[F(y_k)]^{k-1}[1-F(y_k)]^{r-1-k} \\ \times[1-F(y_r)]^{n-r} f(y_k).f(y_r) \quad \text{for } y_k < y_r \\ 0 \quad \text{other wise} \end{cases}$$

And the pdf of order statistics Y_k such that $1 \le k \le n$, is

$$f(y_k) = \begin{cases} \dfrac{n!}{(k-1)!(n-k)!}[F(y_k)]^{k-1}\,[1-F(y_k)]^{n-k} f(y_k) \quad \text{for } 1 \le k \le n \\ 0 \quad \text{other wise} \end{cases}$$

Example 2-5-2:

Let $X_1, \ldots, X_n$ be a random sample from U [0, 1]. Find the pdf of the *k*th order statistic Y_k.

Solution :

Since the pdf of X_i is $f(x) = 1$, $0 \le x \le 1$, the cdf is $F(x) = x$, $0 \le x \le 1$. Using Theorem the pdf of the *k*th order statistic Y_k reduces to

$$f_k(y) = \frac{n!}{(k-1)!(n-k)!} y^{k-1}\,[1-y]^{n-k} \quad , 0 \le y \le 1$$

which is a beta distribution with $\alpha = k$ and $\beta = n - k + 1$.

Example 2-5-3 :

A string of 10 light bulbs is connected in series, which means that the entire string will not light up if any one of the light bulbs fails . Assume that the lifetimes of the bulbs, $X_1, \ldots, X_{10}$, are independent random variables that are exponentially distributed with mean 2. Find the distribution of the life length of this string of light bulbs.

Solution :

Note that the pdf of X_i is $f(x)=2e^{-2x}$, $0 < x < \infty$, and the cumulative distribution of X_i is $F(x)=\int_0^x 2e^{-2x} = 1-e^{-2x}$.

Let Y represent the lifetime of this string of light bulbs. Then,

$Y = \min(X_1, \ldots, X_{10})$.

Thus,

$F_Y(y)=1-\left(1-1+e^{-2y}\right)^{10}$.

Hence, the density of Y is obtained by differentiating $F_Y(y)$ with respect to y , that is,

$$f_Y(y)=\begin{cases} 10\left(1-1+e^{-2y}\right)^9\left(2e^{-2y}\right)=20e^{-20y} & , 0<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

Example 2-5-4 :

Let $Y_1<Y_2<Y_3<Y_4$ be the order statistics of R.S of size n = 4 from a distribution having a pdf $f(x)$, $f(x)=2x$, $0<x<1$.

a) Find the pdf of Y_3 .

b) Find the joint pdf of Y_1, Y_2, Y_3, Y_4 .

c) Find $P(0.5 < Y_1 < 1)$

Solution :

X_1, X_2, X_3, X_4 be a R.S

$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x) = 2x$$

$$\because Y_1, Y_2, Y_3 \text{ and } Y_4 \in (X_1, X_2, X_3, X_4)$$

$$F(y_i) = \int_0^{y_i} f(y_i)\,dy_i = y_i^2$$

a)

$$\therefore g(y_3) = \frac{4!}{(3-1)!\,(4-3)!}[F(y_3)]^{3-1}[1-F(y_3)]^{4-3} f(y_3)$$

$$= 12\,[y_3^2]^2[1-y_3^2]^1\,(2y_3) = 24y_3^5\,[1-y_3^2] \qquad , 0 < y_3 < 1$$

$$\therefore g(y_3) = 24y_3^5\,[1-y_3^2] \qquad , 0 < y_3 < 1$$

b)

$$\therefore g(y_1, y_2, y_3, y_4) = 4!\,f(y_1)f(y_2)f(y_3)f(y_4) = 4!\,(2y_1)(2y_2)(2y_3)(2y_4)$$

$$= 384\,y_1 y_2 y_3 y_4 \,.$$

c)

$$\therefore g(y_1) = n\,[1-F(y_1)]n^{n-1}\ f(y_1)$$

$$= 4\,[1-y_1^2]^3\,y_1 \qquad , 0 < y_1 < 1$$

$$\therefore P(0.5 < Y_1 < 1) =$$

$$\int_{0.5}^{1} 4\,[1-y_1^2]^3\,y_1\,dy_1 = -2\int_{0.5}^{1} -2\,[1-y_1^2]^3\,y_1\,dy_1 = \left[-2\frac{[1-y_1^2]^4}{4}\right]_{0.5}^{1} = 0 - \left[-2\frac{[1-(0.5)^2]^4}{4}\right]$$

$$= \left[2\frac{[1-(0.5)^2]^4}{4}\right] = \frac{1}{2}\left(\frac{3}{4}\right)^4 = \frac{27}{512} = 0.0527$$

Example 2-5-5 :

Let $X_1, \ldots, X_5$ be a random sample from G(1 ,1)

Let $U = Y_2$, $V = Y_4 - Y_2$, Prove that U *and* V are indep.

Solution :

We find the joint pdf of Y_2 and Y_4

$\because$ X ~ G(1 ,1) $\Rightarrow f(x) = e^{-x}$, $x > 0$

$$\therefore F(x) = \int_0^x e^{-w} dw = 1 - e^{-x}$$

Now , n=5 , $k = 2$, $r = 4$

$$f(y_2, y_4) = \frac{5!}{(2-1)!(4-1-2)!(5-4)!}[1-e^{-y_2}]\left[e^{-y_2} - e^{-y_4}\right]e^{-y_4}e^{-y_2}e^{-y_4}$$

$$= 5!\,[1-e^{-y_2}]\left[e^{-y_2} - e^{-y_4}\right]e^{-2y_4}e^{-y_2} \quad \text{for } 0 < y_2 < y_4 < \infty$$

Since we are with cont. distribution we use the Jacobian trans.

$u = y_2 \Rightarrow y_2 = u = h_1^{-1}(u,v)$, $v = y_4 - y_2 \Rightarrow y_4 = v + u = h_2^{-1}(u,v)$

$$J = \begin{vmatrix} \partial y_2/\partial u & \partial y_2/\partial v \\ \partial y_4/\partial u & \partial y_4/\partial u \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$g(u,v) = f(h_1^{-1}, h_2^{-1})|J| = 5!\,[1-e^{-u}]\,\left[1-e^{-v}\right]e^{-2v}e^{-4u} \quad , 0<u<\infty \ , \ 0<v<\infty$$

$$\therefore g(u) = \int_0^\infty 5!\,[1-e^{-u}]\,\left[1-e^{-v}\right]e^{-2v}e^{-4u}\, dv = 20e^{-4u}\left(1-e^{-u}\right) \quad , \ 0<u<\infty$$

$$\therefore g(v) = \int_0^\infty 5!\,[1-e^{-u}]\,\left[1-e^{-v}\right]e^{-2v}e^{-4u}\, du = 6\,e^{-2v}\left(1-e^{-v}\right) \quad , \ 0<v<\infty$$

$$\because g(u)\,g(v) = \left(20e^{-4u}\left(1-e^{-u}\right)\right).\left(6\,e^{-2v}\left(1-e^{-v}\right)\right) = 120\,[1-e^{-u}]\,\left[1-e^{-v}\right]e^{-2v}e^{-4u}$$
$$= 5!\,[1-e^{-u}]\,\left[1-e^{-v}\right]e^{-2v}e^{-4u} = g(u,v)$$

Thus , that U *and* V are indep. .

EXERCISES :

2-5-1-Let $X_1, \ldots, X_n$ be a random sample from U [0, 1]. Find the joint pdf of Y_2 and Y_5 .

2-5-2-Let $X_1, \ldots, X_n$ be a random sample from exponential distribution with a mean of θ. Show that $Y_1 = \min(X_1, X_2, \ldots, X_n)$ has an exponential distribution with mean θ/n. Also, find the pdf of $Y_n = \max(X_1, X_2, \ldots, X_n)$.

2-5-3-Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution $f(x) = 1/2 \; , 0 \leq x \leq 2$. Find the probability density function for the range $R = (X_{(n)} - X_{(1)})$.

2-5-4-Let $X_1, \ldots, X_n$ be a random sample from a beta distribution with $\alpha = 2$ and $\beta = 3$. Find the joint pdf of Y_1 and Y_n .

2-5-5- Let $X_1, \ldots, X_n$ be a random sample from an exponential population with parameter θ. Let $Y_1, \ldots, Y_n$ be the ordered random variables .

(a) Show that the sampling distributions of Y_1 and Y_n are given by

$$f(x) = \begin{cases} \frac{n}{\theta} e^{\frac{-ny_1}{\theta}} & if \quad y_1 > 0 \\ 0 & other \;\; wise \end{cases}$$

2-5-6- Let $X_1, \ldots, X_n$ be a random sample with $f(x) = 3x^2 , 0 < x < 1$. prove that $U = Y_2 / Y_4$ and $V = Y_4$ are indep. .

2-7 Limiting Distribution

2-7-1 Convergence in Probability

In this section, we formalize a way of saying that a sequence of random variables is getting "close" to another random variable. We will use this concept throughout the lecture .

Definition 2-7-1 *Let* $\{X_n\}$ *be a sequence of random variables and let* X *be a random variable defined on a sample space. We say that* X_n **converges in probability** *to* X *if for all* $\epsilon > 0$

$$\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0,$$

or equivalently,

$$\lim_{n\to\infty} P[|X_n - X| < \epsilon] = 1.$$

If so, we write

$$X_n \xrightarrow{P} X.$$

One way of showing convergence in probability is to use Chebyshev's Theorem

CHEBYSHEV'S THEOREM :

Theorem 3-1-1: Let the random variable X have a mean μ and standard deviation σ . Then for K > 0, a constant,

$$P\{\,|X_n - \mu| < k\sigma\,\} \geq 1 - \frac{1}{k^2}$$

Example 2-7-1 Let $\overline{X}_n$ denoted the mean of a R.S. of size n from distribution having the mean μ and the variance σ_X^2 . Show that $\overline{X}_n$ C.S in probability to μ .

Solution :

$$E(\overline{X})=E(X)=\mu$$

$$\operatorname{var}(\overline{X})=\frac{1}{n}\operatorname{var}(X)=\frac{1}{n}\sigma_X^2$$

For any $\varepsilon>0$ let $\varepsilon=k\frac{\sigma_X}{\sqrt{n}} \Rightarrow k=\frac{\sqrt{n}\,\varepsilon}{\sigma_X}$

$$\lim_{n\to\infty} P\left\{ \left|\overline{X}_n-\mu\right|<\varepsilon \right\}$$

$$=\lim_{n\to\infty} P\left\{ \left|\overline{X}_n-\mu\right|<K\frac{\sigma_X}{\sqrt{n}} \right\} \geq \lim_{n\to\infty}\left(1-\frac{1}{\left(\frac{\sqrt{n}\,\varepsilon}{\sigma_X}\right)^2} \right)=1 \quad \text{(by Chebyshev's)}$$

$$\therefore \lim_{n\to\infty} P\left\{ \left|\overline{X}_n-\mu\right|<\varepsilon \right\}=1 \Rightarrow \overline{X}_n \xrightarrow{C.S} \mu \text{ .}$$

Example 2-7-2 Let $Y_n \sim b(n,p)$. Show that $Y_n/n \xrightarrow{c.s} p$.

Solution :

$$P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\} = P\{ |Y_n - n\,p| < n\varepsilon \}$$

For any $\varepsilon > 0$. Let $n\varepsilon = k\sqrt{np(1-p)} \Rightarrow k = \dfrac{n\varepsilon}{\sqrt{np(1-p)}} = \dfrac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}$

where $\text{var}(Y_n) = np(1-p)$

$$\therefore P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\} = P\{ |Y_n - n\,p| < n\varepsilon \} = P\left\{ |Y_n - n\,p| < k\sqrt{np(1-p)} \right\}$$

$$\geq 1 - \frac{1}{\left(\dfrac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2} \qquad \text{(by Chebyshev's theorem)}$$

$$\therefore P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\} \geq 1 - \frac{1}{\left(\dfrac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2}$$

Take the limit of two sides

$$\lim_{n\to\infty} P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\} \geq \lim_{n\to\infty} \left(1 - \frac{1}{\left(\dfrac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2} \right) = 1$$

$$\therefore \lim_{n\to\infty} P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\} = 1$$

Hence , $Y_n/n \xrightarrow{c.s} p$

Example 2-7-3 : Let $Y_n \sim b(n,p)$. Show that $1 - Y_n/n \xrightarrow{c.s} 1-p$.

Solution :

$$P\left\{ \left|1 - Y_n/n - (1-p)\right| < \varepsilon \right\} = P\left\{ \left|1 - Y_n/n - 1 + p\right| < \varepsilon \right\} = P\left\{ \left| -Y_n/n + p \right| < \varepsilon \right\}$$

$$= P\left\{ \left| (-1)\left(Y_n/n - p \right) \right| < \varepsilon \right\} = P\left\{ |-1| \left| \left(Y_n/n - p \right) \right| < \varepsilon \right\} = P\left\{ \left| Y_n/n - p \right| < \varepsilon \right\}$$

$$= P\left\{ |Y_n - np| < n\varepsilon \right\}$$

By last example we get :

$$\lim_{n\to\infty} P\left\{ \left|1 - Y_n/n - (1-p)\right| < \varepsilon \right\} = \lim_{n\to\infty} P\left\{ \left|Y_n/n - p\right| < \varepsilon \right\} = 1$$

$$\therefore \lim_{n\to\infty} P\left\{ \left|1 - Y_n/n - (1-p)\right| < \varepsilon \right\} = 1$$

Hence , $Y_n/n \xrightarrow{c.s} p$

Example 2-7-4 : Show that : $\dfrac{S^2}{n-1} \xrightarrow{c.s} \sigma^2$?

Solution :

$$\forall \varepsilon > 0 , \ let \ \frac{(n-1)\varepsilon}{\sigma^2} = k\sqrt{2(n-1)} \quad \Rightarrow k = \frac{\frac{(n-1)\varepsilon}{\sigma^2}}{\sqrt{2(n-1)}} \quad ; q = 1-p$$

$$P\left\{ \left| \frac{nS^2}{n-1} - \sigma^2 \right| < \varepsilon \right\} = P\left\{ \left| \frac{nS^2}{\sigma^2} - (n-1) \right| < \frac{(n-1)\varepsilon}{\sigma^2} \right\}$$

apply chebyshev's inequality

$$\Rightarrow P\left\{ \left| \frac{nS^2}{\sigma^2} - (n-1) \right| < \frac{(n-1)\varepsilon}{\sigma^2} \right\} = P\left\{ \left| \frac{nS^2}{\sigma^2} - (n-1) \right| < k\sqrt{2(n-1)} \right\} \geq 1 - \frac{1}{\left(\frac{\frac{(n-1)\varepsilon}{\sigma^2}}{\sqrt{2(n-1)}} \right)^2}$$

$$\therefore \lim_{n\to\infty} P\left\{ \left| \frac{nS^2}{n-1} - \sigma^2 \right| < \varepsilon \right\} \geq \lim_{n\to\infty} \left(1 - \frac{1}{\left(\frac{1}{\sqrt{2(n-1)}} \frac{(n-1)\varepsilon}{\sigma^2} \right)^2} \right) = 1 - 0 = 1$$

$$\Rightarrow \lim_{n\to\infty} P\left\{ \left| \frac{nS^2}{n-1} - \sigma^2 \right| < \varepsilon \right\} = 1$$

$$\therefore \frac{nS^2}{n-1} \xrightarrow{c.s} \sigma^2$$

2-8 Limiting moment generating functions

Definition 2-8-1 Let the r.v. Y_n have the distribution function $f_n(y)$ and the mgf M(t,n) . If there exists a distribution function f(y) with mgf M(t) such that

$$\lim_{n\to\infty} M(t,n) = M(t)$$

Then . Y_n has a limiting distribution with distribution function f(y) .

- **Useful remarks :**

1- $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{an} = e^{ax}$.

2- $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{an} = e^{-ax}$.

3- $\lim_{n\to\infty}\left(1+\frac{x}{n}+\frac{\varphi(n)}{n}\right)^{n} = e^{x} \quad ,\varphi(n)\to 0 \ as\ n\to\infty$.

4- $\lim_{n\to\infty}\left(1-\frac{x}{n}+\frac{\varphi(n)}{n}\right)^{an} = e^{-ax} \quad ,\varphi(n)\to 0 \ as\ n\to\infty$.

Example 2-8-2 Let $Y_n \sim b(n,p)$. Show that the limit of Y_n as $n\to\infty$ becomes a passion .

Solution :

$$M_{Y_n}(t,n) = E\left(e^{tY_n}\right) = \left[(1-p)+pe^{t}\right]^{n} \qquad ,\mu = np \Rightarrow p = \frac{\mu}{n}$$

$$= \left[\left(1-\frac{\mu}{n}\right)+\frac{\mu}{n}e^{t}\right]^{n} = \left[1-\frac{\mu}{n}\left(1-e^{t}\right)\right]^{n}$$

$$\therefore \lim_{n\to\infty}\left[1-\frac{\mu}{n}\left(1-e^{t}\right)\right]^{n} = e^{-\mu\left(1-e^{t}\right)}$$

Hence ,

$$\lim_{n\to\infty} M_{Y_n}(t,n) = e^{-\mu\left(1-e^{t}\right)}$$

The limit of $Y_n \sim b(n,p)$ becomes a passion when $n\to\infty$

That is the r.v which is b(n , p) has a limiting passion distribution with mean μ .

Example 2-8-3 Let $Y_n = \dfrac{Z_n - n}{\sqrt{2n}}$, where $Z_n \sim \chi^2_{(n)}$. Find the limit of Y_n as $n \to \infty$ becomes a N(0,1) .

Solution :

Since $Z_n \sim \chi^2_{(n)} \Rightarrow E(z_n) = n$, $\text{var}(z_n) = 2n$

$$M_{Y_n}(t,n) = E(e^{tY_n}) = E\left(e^{t\left(\frac{z_n - n}{\sqrt{2n}}\right)}\right) = E\left(\exp\left\{\left(\frac{t z_n - t n}{\sqrt{2n}}\right)\right\}\right) = \exp\left(\frac{-t n}{\sqrt{2n}}\right) E\left(\exp\left\{\left(\frac{t z_n}{\sqrt{2n}}\right)\right\}\right)$$

$$= \exp\left(\frac{-t n}{\sqrt{2n}}\right)\left(1 - 2\frac{t}{\sqrt{2n}}\right)^{\frac{-n}{2}} = \left(\exp\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right) - \exp\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right).\left(2\frac{t}{\sqrt{2n}}\right)\right)^{\frac{-n}{2}}$$

$$\Rightarrow M_{Y_n}(t,n) = \left(\exp\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right) - \exp\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right).\left(2\frac{t}{\sqrt{2n}}\right)\right)^{\frac{-n}{2}}$$

Now , by Taylor's formula $\left(e^x = 1 + x + \frac{1}{2!}x^2 + \ldots\ldots \right)$

$$\therefore \exp\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right) = 1 + \frac{\sqrt{2}\,t}{\sqrt{n}} + \frac{1}{2!}\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right)^2 + \frac{1}{3!}\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right)^3 + \ldots$$

Hence ,

$$M_{Y_n}(t,n) = \left(\left(1 + \frac{\sqrt{2}\,t}{\sqrt{n}} + \frac{1}{2!}\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right)^2 + \ldots\right) - t\sqrt{\frac{2}{n}}\left(1 + \frac{\sqrt{2}\,t}{\sqrt{n}} + \frac{1}{2!}\left(\frac{\sqrt{2}\,t}{\sqrt{n}}\right)^2 + \ldots\right)\right)^{\frac{-n}{2}}$$

$$= \left(1 - t^2\frac{1}{n} + \frac{\varphi(n)}{n}\right)^{\frac{-n}{2}}$$

$$\lim_{n\to\infty} M_{Y_n}(t,n) = \lim_{n\to\infty}\left(1 - t^2\frac{1}{n} + \frac{\varphi(n)}{n}\right)^{\frac{-n}{2}} = e^{\frac{t^2}{2}} \qquad ., \varphi(n) \to 0$$

The limit of Y_n becomes a standard normal when $n \to \infty$

EXERCISES :

1 Let $Z_n \sim b(n,p)$. Find the limiting distribution of $Y_n = \dfrac{Z_n - np}{\sqrt{np(1-p)}}$.

2 Let $Z_n \sim P(n)$. Find the limiting distribution of $Y_n = \dfrac{Z_n - n}{\sqrt{n}}$.

3 Let $\overline{X}$ denoted the mean of a R.S. of size n from a passion distribution with mean $\mu = 1$. Show that the mgf of

$$Y_n = \frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma} = \sqrt{n}\left(\overline{X}_n - 1\right) \text{ is given by } Exp\left\{-t\sqrt{n} + n\left(e^{\frac{t}{\sqrt{n}}} - 1\right)\right\}$$

2-9 LARGE SAMPLE APPROXIMATIONS

Theorem 2-9-1 Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mu = E(X_i)$ and $\sigma^2 = Var(X_i) < \infty$ for $i = 1, 2, \ldots, \infty$. Then

$$\lim_{n \to \infty} P(|\overline{S}_n - \mu| \geq \varepsilon) = 0$$

for every ε. Here $\overline{S}_n$ denotes $\frac{X_1 + X_2 + \cdots + X_n}{n}$.

Proof: we have

$$E(\overline{S}_n) = \mu \qquad \text{and} \qquad Var(\overline{S}_n) = \frac{\sigma^2}{n}.$$

By Chebychev's inequality

$$P(|\overline{S}_n - E(\overline{S}_n)| \geq \varepsilon) \leq \frac{Var(\overline{S}_n)}{\varepsilon^2}$$

for $\varepsilon > 0$. Hence

$$P(|\overline{S}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\,\varepsilon^2}.$$

Taking the limit as n tends to infinity, we get

$$\lim_{n\to\infty} P(|\overline{S}_n - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\,\varepsilon^2}$$

which yields

$$\lim_{n\to\infty} P(|\overline{S}_n - \mu| \geq \varepsilon) = 0$$

and the proof of the theorem is now complete.

2-10 Central Limit Theorem

Theorem 2-10-1 Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Define

$$Y_n = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{\sigma}$$

The central limit theorem implies that probability statements about Y_n can be approximated by corresponding probabilities for the standard normal random variable if n is large. (Usually, a value of n greater than 30 will ensure that the distribution of Y_n can be closely approximated by a normal distribution) .

As a matter of convenience, the conclusion of the central limit theorem is often replaced with the simpler statement that Y is *asymptotically normally distributed* with mean μ and variance σ^2/n.

The central limit theorem can be applied to a random sample $X_1, X_2, \ldots, X_n$ from *any* distribution as long as $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ are both finite and the sample size is large.

We will give some examples of the use of the central limit theorem but will defer the proof until the next section (coverage of which is optional). The proof is not needed for an understanding of the applications of the central limit theorem that appear in this text .

Example 2-10-1 Let $X \sim N(6,1)$ and $Y \sim N(7,1)$. Find $P(X > Y)$.

Solution :

Since $P(X > Y) = P(X - Y > 0) = 1 - P(X - Y \le 0)$

$\therefore X - Y \sim N(6-7, 1+1) \Rightarrow X - Y \sim N(-1, 2)$

$$\therefore P(X > Y) = P(X - Y > 0) = 1 - P(X - Y \le 0) = 1 - P\left(\frac{(X-Y)-(-1)}{\sqrt{2}} \le \frac{0-(-1)}{\sqrt{2}}\right)$$

$$= 1 - P\left(Z \le \frac{1}{\sqrt{2}}\right) = 1 - N\left(\frac{1}{\sqrt{2}}\right) = 0.24$$

Example 2-10-2 Let $\overline{X}$ denoted the mean of a R.S. of size n from $N(\mu, 100)$.
Find n . such that $P(\mu - 5 < \overline{X} < \mu + 5) = 0.954$. ; $N(2) = 0.977$

Solution :

$$P(\mu - 5 < \overline{X} < \mu + 5) = 0.977$$

$$\Rightarrow P\left(\frac{((\mu-5)-\mu)\sqrt{n}}{10} < \frac{(\overline{X}-\mu)\sqrt{n}}{10} < \frac{((\mu+5)-\mu)\sqrt{n}}{10}\right) = 0.954$$

$$\Rightarrow P\left(\frac{-5\sqrt{n}}{10} < Z < \frac{5\sqrt{n}}{10}\right) = 0.954$$

$$\Rightarrow P\left(\frac{-\sqrt{n}}{2} < Z < \frac{\sqrt{n}}{2}\right) = 0.954$$

$$\Rightarrow P\left(Z < \frac{\sqrt{n}}{2}\right) - P\left(Z < \frac{-\sqrt{n}}{2}\right) = 0.954$$

$$\Rightarrow P\left(Z < \frac{\sqrt{n}}{2}\right) - \left(1 - P\left(Z < \frac{\sqrt{n}}{2}\right)\right) = 0.954$$

$$\Rightarrow 2P\left(Z < \frac{\sqrt{n}}{2}\right) - 1 = 0.954$$

$$\Rightarrow 2N\left(\frac{\sqrt{n}}{2}\right) - 1 = 0.954 \Rightarrow N\left(\frac{\sqrt{n}}{2}\right) = 0.977$$

$$\therefore \Rightarrow \frac{\sqrt{n}}{2} = 2 \Rightarrow n = 16$$

Example 2-10-3 Let $\overline{X}$ denoted the mean of a R.S. of size 75 from distribution having the pdf $f(x) = 1$, $0 < x < 1$. Find $P(0.45 < \overline{X} < 0.55)$.

Solution :

$$P(0.45 < \overline{X} < 0.55)$$

$$\Rightarrow P\left(\frac{(0.45-\mu)\sqrt{n}}{\sigma_X} < \frac{(\overline{X}-\mu)\sqrt{n}}{\sigma_X} < \frac{(0.55-\mu)\sqrt{n}}{\sigma_X}\right)$$

$$\Rightarrow \mu = E(x) = \int_0^1 x\, f(x)dx = \int_0^1 x\, dx = \frac{1}{2} \ , \ E(x^2) = \int_0^1 x^2\, f(x)dx = \int_0^1 x^2\, dx = \frac{1}{3}$$

$$\Rightarrow \sigma_X^2 = E(x^2) - (E(x))^2 = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

$$\because n = 75 \Rightarrow \sqrt{n} = \sqrt{75} = 5\sqrt{3}$$

$$\Rightarrow P(0.45 < \overline{X} < 0.55) = P\left(\frac{(0.45-0.5)5\sqrt{3}}{1/12} < \frac{(\overline{X}-0.5)5\sqrt{3}}{1/12} < \frac{(0.55-0.5)5\sqrt{3}}{1/12}\right)$$

$$= P(-1.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -1.5)$$

$$= P(Z < 1.5) - (1 - P(Z < 1.5)) = 0.954$$

$$\Rightarrow \ 2P(Z < 1.5) - 1 = 0.866$$

Example 2-10-3 .et $X_1, X_2, \ldots, X_n$ be independent and identically R.S from $b(1,p)$.

Let $Y = X_1 + X_2 + \ldots + X_n$. By using CLT show that

$$\frac{\sqrt{n}\,(\overline{X} - \mu_X)}{\sigma_X} \sim N(0,1)$$

Solution :

Since $Y = X_1 + X_2 + \ldots + X_n$

$E(Y) = E(X_1 + X_2 + \ldots + X_n) = n\,E(x) = n\,(p) = np$

$var(Y) = var(X_1 + X_2 + \ldots + X_n) = n\,var(x) = n\,(p(1-p)) = np(1-p)$

By CLT we get

$$\frac{\sqrt{n}\,(\overline{X} - p)}{\sqrt{(1-p)p}} \sim N(0,1) \quad or \quad \frac{Y - np}{\sqrt{n\,p(1-p)}} \sim N(0,1)$$

Example 2-10-4 Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose density function is

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the approximate value of $P(-0.3 \leq Y \leq 1.5)$ when one uses the central limit theorem?

Answer: First, we find the mean μ and variance σ^2 for the density function $f(x)$. The mean for this distribution is given by

$$\begin{aligned} \mu &= \int_{-1}^{1} \frac{3}{2}x^3\,dx \\ &= \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} \\ &= 0. \end{aligned}$$

Hence the variance of this distribution is given by

$$\begin{aligned} Var(X) &= E(X^2) - [\,E(X)\,]^2 \\ &= \int_{-1}^{1} \frac{3}{2}x^4\,dx \\ &= \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1} \\ &= \frac{3}{5} \\ &= 0.6. \end{aligned}$$

$$P(-0.3 \leq Y \leq 1.5) = P(-0.3 - 0 \leq Y - 0 \leq 1.5 - 0)$$
$$= P\left(\frac{-0.3}{\sqrt{15(0.6)}} \leq \frac{Y-0}{\sqrt{15(0.6)}} \leq \frac{1.5}{\sqrt{15(0.6)}}\right)$$
$$= P(-0.10 \leq Z \leq 0.50)$$
$$= P(Z \leq 0.50) + P(Z \leq 0.10) - 1$$
$$= 0.6915 + 0.5398 - 1$$
$$= 0.2313.$$

Example 2-10-5 Let $X_1, X_2, ..., X_n$ be a random sample of size $n = 25$ from a population that has a mean $\mu = 71.43$ and variance $\sigma^2 = 56.25$. Let $\overline{X}$ be the sample mean. What is the probability that the sample mean is between 68.91 and 71.97?

Answer: The mean of $\overline{X}$ is given by $E\left(\overline{X}\right) = 71.43$. The variance of $\overline{X}$ is given by

$$Var\left(\overline{X}\right) = \frac{\sigma^2}{n} = \frac{56.25}{25} = 2.25.$$

In order to find the probability that the sample mean is between 68.91 and 71.97, we need the distribution of the population. However, the population distribution is unknown. Therefore, we use the central limit theorem. The central limit theorem says that $\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,\, 1)$ as n approaches infinity. Therefore

$$P\left(68.91 \leq \overline{X} \leq 71.97\right)$$
$$= \left(\frac{68.91 - 71.43}{\sqrt{2.25}} \leq \frac{\overline{X} - 71.43}{\sqrt{2.25}} \leq \frac{71.97 - 71.43}{\sqrt{2.25}}\right)$$
$$= P\left(-0.68 \leq W \leq 0.36\right)$$
$$= P\left(W \leq 0.36\right) + P\left(W \leq 0.68\right) - 1$$
$$= 0.5941.$$

EXERCISES :

1- -Compute an approximate probability that the mean of R.S of size 15 from a distribution having pdf $f(x)=3x^2$,$0<x<1$, is between 3/5 and 4/5 .

2- If $Y\sim b(100,0.5)$. approximate P(Y=50),[**Note P(Y=50) = P(49.5<Y<50.5)]** .

3- Let $\overline{X}$ denoted the mean of a R.S. of size n=100 from $\chi^2_{(50)}$. Compute an approximate value of P(49 < $\overline{X}$ < 51) .

4- Let $\overline{X}$ denoted the mean of a R.S. of size n=128 from $G(2,4)$. Compute an approximate value of P(7 < $\overline{X}$ < 9) .

5- Let $X_1, X_2, \ldots, X_{25}$ and $Y_1, Y_2, \ldots, Y_{25}$ be independent and identically R.S from N(0,16) and N(1 , 9) respectively , Let $\overline{X}$, $\overline{Y}$ be the mean of them resp. , Find P($\overline{X} > \overline{Y}$) .

2-12 Theorems on limiting distribution

Theorem 2-12-1 If $X_n \xrightarrow{C.S} c$, then $X_n/c \xrightarrow{C.S} 1$.

Proof :

$$\lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = \lim_{n\to\infty}\left\{\frac{|X_n - c|}{|c|} \geq \frac{\varepsilon}{|c|}\right\} = \lim_{n\to\infty}\left\{\left|\frac{X_n - c}{c}\right| \geq \frac{\varepsilon}{|c|}\right\} = \lim_{n\to\infty}\left\{\left|\frac{X_n}{c} - 1\right| \geq \frac{\varepsilon}{|c|}\right\}$$

For any $\varepsilon' > 0$ let $\varepsilon' = \frac{\varepsilon}{|c|}$

$$\therefore \lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = \lim_{n\to\infty}\left\{\left|\frac{X_n}{c} - 1\right| \geq \varepsilon'\right\}$$

Since $X_n \xrightarrow{C.S} c$ then $\lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = 0$

Thus ,

$$\lim_{n\to\infty}\left\{\left|\frac{X_n}{c} - 1\right| \geq \varepsilon'\right\} = 0$$

Hence ,

$$X_n/c \xrightarrow{C.S} 1 \ .$$

Theorem 2-12-2 If $X_n \xrightarrow{C.S} c$, then $\sqrt{X_n} \xrightarrow{C.S} \sqrt{c}$; $c > 0$.

Proof :

$$\lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = \lim_{n\to\infty}\left\{\left|\left(\sqrt{X_n} - \sqrt{c}\right)\left(\sqrt{X_n} + \sqrt{c}\right)\right| \geq \varepsilon\right\} = \lim_{n\to\infty}\left\{\left|\left(\sqrt{X_n} - \sqrt{c}\right)\right|\left|\left(\sqrt{X_n} + \sqrt{c}\right)\right| \geq \varepsilon\right\}$$

$$= \lim_{n\to\infty}\left\{\left|\left(\sqrt{X_n} - \sqrt{c}\right)\right|\left(\sqrt{X_n} + \sqrt{c}\right) \geq \varepsilon\right\} = \lim_{n\to\infty}\left\{\left|\left(\sqrt{X_n} - \sqrt{c}\right)\right| \geq \frac{\varepsilon}{\sqrt{X_n} + \sqrt{c}}\right\}$$

$$\geq \lim_{n\to\infty}\left\{\left|\left(\sqrt{X_n} - \sqrt{c}\right)\right| \geq \frac{\varepsilon}{\sqrt{c}}\right\}$$

For any $\varepsilon' > 0$ let $\varepsilon' = \frac{\varepsilon}{\sqrt{c}}$

$$\therefore \lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} \geq \lim_{n\to\infty}\left\{\left|\sqrt{X_n} - \sqrt{c}\right| \geq \varepsilon'\right\}$$

Since $X_n \xrightarrow{C.S} c$ then $\lim_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = 0$

Thus ,

$$\lim_{n\to\infty}\left\{\left|\sqrt{X_n} - \sqrt{c}\right| \geq \varepsilon'\right\} = 0$$

Hence ,

$$\sqrt{X_n} \xrightarrow{C.S} \sqrt{c} \quad ; c > 0 \ .$$

Theorem 2-12-3 If $X_n \xrightarrow{C.S} X$, then $X_n - X \xrightarrow{C.S} 0$.

Proof :

$\lim\limits_{n\to\infty}\{|X_n - c| \geq \varepsilon\} = \lim\limits_{n\to\infty}\{|(X_n - X) - 0| \geq \varepsilon\}$

Since $X_n \xrightarrow{C.S} X$ then $\lim\limits_{n\to\infty}\{|X_n - X| \geq \varepsilon\} = 0$

Thus ,

$\lim\limits_{n\to\infty}\{|(X_n - X) - 0| \geq \varepsilon\} = 0$

Hence ,

$X_n - X \xrightarrow{C.S} 0$.

Theorem 2-12-4 If $X_n \xrightarrow{C.S} X$ and $Y_n \xrightarrow{C.S} Y$, then

$X_n + Y_n \xrightarrow{C.S} X + Y$.

Proof :

$\lim\limits_{n\to\infty}\{|X_n + Y_n - (X+Y)| \geq 2\varepsilon\} = \lim\limits_{n\to\infty}\{|X_n + Y_n - X - Y| \geq 2\varepsilon\} = \lim\limits_{n\to\infty}\{|X_n - X + Y_n - Y| \geq 2\varepsilon\}$

$\leq \lim\limits_{n\to\infty}\{|X_n - X| + |Y_n - Y| \geq 2\varepsilon\} = \lim\limits_{n\to\infty}\{|X_n - X| \geq \varepsilon\} + \lim\limits_{n\to\infty}\{|Y_n - Y| \geq \varepsilon\}$

Since $X_n \xrightarrow{C.S} X$ then $\lim\limits_{n\to\infty}\{|X_n - X| \geq \varepsilon\} = 0$

Also , Since $Y_n \xrightarrow{C.S} Y$ then $\lim\limits_{n\to\infty}\{|Y_n - Y| \geq \varepsilon\} = 0$

For any $\varepsilon' > 0$ let $\varepsilon' = 2\varepsilon$

Thus ,

$\lim\limits_{n\to\infty}\{|X_n + Y_n - (X+Y)| \geq \varepsilon'\} = 0$

Hence ,

$X_n + Y_n \xrightarrow{C.S} X + Y$.

Theorem 2-12-4 If $X_n \xrightarrow{C.S} X$ and if k is constant , then $X_n k \xrightarrow{C.S} X k$.

Theorem 2-12-5 If $X_n \xrightarrow{C.S} 0$, then $X_n^2 \xrightarrow{C.S} 0$.

Theorem 2-12-6 If $X_n \xrightarrow{C.S} a$ and $Y_n \xrightarrow{C.S} b$; a , b are constants , then $X_n Y_n \xrightarrow{C.S} ab$.

Theorem 2-12-7 If $X_n \xrightarrow{C.S} a$ and $Y_n \xrightarrow{C.S} b$; a , b$\neq 0$ are constants ,

then $X_n / Y_n \xrightarrow{C.S} a/b$.

Example 2-12-1 If $Y_n/n \xrightarrow{C.S} p$ and $1-Y_n/n \xrightarrow{C.S} 1-p$, then

$$\left(Y_n/n\right)\left(1-Y_n/n\right) \xrightarrow{C.S} p(1-p)$$

Solution :

By theorem (2-12-6) the proof is complete .

Example 2-12-2 If $Y_n/n \xrightarrow{C.S} p$ and $1-Y_n/n \xrightarrow{C.S} 1-p$, then

$$\left(Y_n/n\right)\left(1-Y_n/n\right) \Big/ p(1-p) \xrightarrow{C.S} 1$$

Solution :

By theorem (2-12-6) and theorem (2-12-7) the proof is complete .

CHAPTER 3

Point Estimation

INTRODUCTION

The field of statistical inference consists of those methods used to make decisions or to draw conclusions about a **population.** These methods utilize the information contained in a **sample** from the population in drawing conclusions. This chapter begins our study of the statistical methods used for inference and decision making.

Statistical inference may be divided into two major areas: **parameter estimation** and **hypothesis testing .** As an example of a parameter estimation problem, suppose that a structural engineer is analyzing the tensile strength of a component used in an automobile chassis. Since variability in tensile strength is naturally present between the individual components because of differences in raw material batches, manufacturing processes, and measurement procedures (for example), the engineer is interested in estimating the mean tensile strength of the components. In practice, the engineer will use sample data to compute a number that is in some sense a reasonable value (or guess) of the true mean. This number is called a **point estimate .**We will see that it is possible to establish the precision of the estimate.

Now consider a situation in which two different reaction temperatures can be used in a chemical process, say and . The engineer conjectures that results in higher yields than does . Statistical hypothesis testing is a framework for solving problems of this type. In this case, the hypothesis would be that the mean yield using temperature is greater than the mean yield using temperature Notice that there is no emphasis on estimating yields; instead, the focus is on drawing conclusions about a stated hypothesis .

Suppose that we want to obtain a point estimate of a population parameter . We know that before the data is collected, the observations are considered to be random variables , say Therefore, any function of the observation, or any **statistic,** is also a random variable . For example, the sample mean and the sample variance are statistics and they are also random variables.

Since a statistic is a random variable, it has a probability distribution. We call the probability distribution of a statistic a **sampling distribution .** The notion of a sampling distribution is very important and will be discussed and illustrated later in the chapter.

When discussing inference problems, it is convenient to have a general symbol to represent the parameter of interest. We will use the Greek symbol (theta) to represent the parameter. The objective of point estimation is to select a single number, based on sample data, that is the most plausible value for . A numerical value of a sample statistic will be used as the point estimate.

In general, if X is a random variable with probability distribution , characterized by the unknown parameter , and if is a random sample of size n from X, the statistic is called a **point estimator** of . Note that is a random variable because it is a function of random variables. After the sample has been selected, takes on a particular numerical value called the **point estimate** of .

Definition 3-1 Let $X \sim f(x;\theta)$ and $X_1, X_2, ..., X_n$ be a random sample from the population X. Any statistic that can be used to guess the parameter θ is called an estimator of θ. The numerical value of this statistic is called an estimate of θ. The estimator of the parameter θ is denoted by $\widehat{\theta}$.

One of the basic problems is how to find an estimator of population parameter θ. There are several methods for finding an estimator of θ. Some of these methods are:

(1) Moment Method

(2) Maximum Likelihood Method

(3) Bayes Method

(4) Least Squares Method

(5) Minimum Chi-Squares Method

(6) Minimum Distance Method