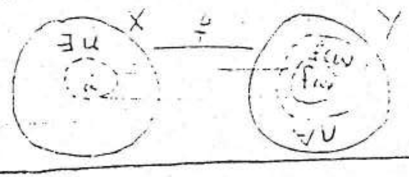
Chapter (5)

Continuouse functions + Product Top + Product spa[illegible]

1 Continuouse functions.

Dfn Let $f: X \longrightarrow Y$ be a function. Let $a \in X$ then f is $\boxed{\text{continuous at } a}$ iff $\forall V$ (open) of Y that containing $f(a)$ $\exists$ an U (open) of X containing a + $f(U) \subseteq V$



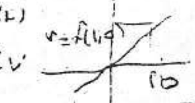
* $\boxed{f \text{ is discontinuouse at } x=a}$ $\Longleftrightarrow \exists V$ (open) in Y containing $f(a)$ $\exists \forall U$ (open) in X containing a we have $f(U) \not\subseteq V$

Dfn $\boxed{\text{global def}}$
$f: (X, T_X) \longrightarrow (Y, T_Y)$ is continuouse $\boxed{\text{iff}}$ f is cts at each $x \in X$.

examples

*① $f: (R, T_S) \longrightarrow (R, T_{dis})$ define $\boxed{f(x)=x}$ Is f cts at 10? [illegible]

NO because $\exists V = \{10\}$ open in T_{dis} containing $f(x)$ + $\forall U$ open T_u containing 10 [illegible] $f(U) = U \not\subseteq V$



② $g: (R, T_{dis}) \longrightarrow (R, T_S)$ + $g(x) = x$ is g continuous at 10? [illegible]

Yes: because let V being open set in T_S containing $g(10) = 10$ then take $U = \{10\}$ we have [illegible] U: U open in T_{dis} + $g(U) = g(\{10\}) = \{10\} \subseteq V$

an example (show) that f is discontinue But open & closed

$(R, T_u) \longrightarrow (R, T_{dis})$ & $f(x) = x$

an example (show) that f is cts & (not) open & (not) closed

$(R, T_{dis}) \longrightarrow (R, T_u)$

characterization function of $A \subseteq X$ is function

$\chi_A : X \longrightarrow R$

$\ni \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

cts $\Longleftrightarrow$ A is open & closed in X

χ_A is cts

A closed since $\{1\} \underset{closed}{\subseteq} R \Longrightarrow A$ is closed in X

A closed since $\{0\} \underset{closed}{\subseteq} R \Longrightarrow A$ is open in X

if $0, 1 \in V \Longrightarrow f^{-1}(V) = X$ open

$\in$ & $1 \notin V \Longrightarrow f^{-1}(V) = X - A$ open (since A is closed)

$\in V$ & $0 \notin V \Longrightarrow f^{-1}(V) = A$ (open)

$\notin V$ & $1 \notin V \Longrightarrow f^{-1}(V) = \varphi$ (open)

$\therefore \chi_A$ is cts.

Q.E.D

(5) Let $f : (R, T_s) \longrightarrow (R, T_s)$ be cts at c then

(a) $\Leftrightarrow$ Given any $\varepsilon > 0$ $\exists \delta > 0 \ni |f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$

(b) $\Leftrightarrow$ $\forall$ open $V \ni f(c)$ $\exists$ open $U \ni c$ & $f(U) \subseteq V$

Proof

$(a \Rightarrow b)$ Let V be open & $f(c) \in V$ then $\exists \varepsilon > 0 \ni$ $f(c) \in (f(c) - \varepsilon, f(c) + \varepsilon) \subseteq V$

then $\exists \delta > 0 \ni f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$, $\forall x \in (c - \delta, c + \delta)$

Take $U = (c - \delta, c + \delta)$ then U is open $c \in U$ & $f(U) \subseteq (f(c) - \varepsilon, f(c) + \varepsilon) \subseteq V$

$(b \Rightarrow a)$ Let $\varepsilon > 0$ Take $V = (f(c) - \varepsilon, f(c) + \varepsilon)$ then V is open & $f(c) \in V$ then $\exists U \ni c$ (open) & $f(U) \subseteq V$

then $\exists \delta > 0 \ni (c - \delta, c + \delta) \subseteq U$

so $f(x) \in V = (f(c) - \varepsilon, f(c) + \varepsilon)$ $\forall x \in U \supseteq (c - \delta, c + \delta)$

Dfn

(1) $f : X \longrightarrow Y$ is called an (open function) iff $\forall U$ open in X then $f(U)$ ~~is~~ open in Y

(2) $f : X \longrightarrow Y$ is called (a closed function) iff $\forall F$ closed in X then $f(F)$ is closed in Y

(3) $f : X \longrightarrow Y$ is called (a homeomorphism) iff

① f is cts

② f is (1-1 & onto) or bijection

③ $f^{-1} : Y \longrightarrow X$ is cts or f is open $\therefore X \overset{f}{\cong} Y$

(4) two Top-spaces (X, T_X), (Y, T_Y) are said to be $\boxed{\text{homeomorphism}}$ iff $\exists$ a homomorphism from (X, T_X) to (Y, T_Y) & denoted $(X, T_X) \cong (Y, T_Y)$

Theorms + Ex

Let $f:(X,\tau_X) \longrightarrow (Y,\tau_Y)$ then the following statement are equivalent

f is cts

inverse image of each open subset is open

($\forall U \subseteq Y$ (open) $\Longrightarrow f^{-1}(U)$ is open in X)

inverse image of each basic open set is open

inverse image of each subbasic open set is open

inverse image of each closed subset is closed

for each $A \subseteq X$ then $f(\bar{A}) \subseteq \overline{f(A)}$

for each $B \subseteq Y$ then $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$

f is cts let $V \underset{open}{\subseteq} Y$ $\boxed{Try}$ $f^{-1}(V)$ is open in X

$\Rightarrow f(x) \in V$ (open) but f is cts at x

$\exists U_x$ (open) in X containing x $\ni f(U_x) \subseteq V \Rightarrow U_x \subset f^{-1}(V)$

then $f^{-1}(V)$ is open.

B basic open set then B is open + so $f^{-1}(B)$ is open.

S subbasic open set then S is a basic open set so $f^{-1}(S)$ is open.

Let S be subbase of Y then $\boxed{Y - C = \bigcup_{\alpha\in\Delta} B_\alpha}$ (open)

finite $\sqcap$ of element of S $\forall\alpha\in\Delta$ }

$B_\alpha = \sqcap S_i$ where Δ_α is finite + $S_i \in S$ $\forall i \in \Delta_\alpha$

$f^{-1}(\sqcap S_i) = \prod_{i\in\Delta_\alpha} f^{-1}(S_i)$ is open $\forall \alpha \in \Delta$

$f^{-1}(Y-C) = f^{-1}(\bigcup B_\alpha) = \bigcup f^{-1}(B_\alpha)$ is open

$f^{-1}(Y-C) = X - f^{-1}(C)$ so $f^{-1}(C)$ is closed

(5-6) Let $A \subseteq X$ Take $C = \overline{f(A)}$ which is closed.

But $A \subseteq f^{-1}(C)$ so $\bar{A} \subseteq \overline{f^{-1}(C)} = f^{-1}(C)$

Thus $f(\bar{A}) \subseteq f f^{-1}(C) \subseteq C$ Hence $f(\bar{A}) \subseteq \overline{f(A)}$

(6-1): Let $p \in X$ + $V \underset{open}{\subseteq} Y$ $\ni f(p) \in V$

Put $A = f^{-1}(Y-V)$ Let $U = X - \bar{A}$ open

$\underline{SO}$ $p \in U$ [suppose $p \notin U \Rightarrow p \in \bar{A} \Rightarrow f(p) \in f(\bar{A}) \subseteq \overline{f(A)}$

$\Rightarrow f(p) \in \overline{f(A)} \subseteq Y - V \Rightarrow f(p) \notin V$) #

+ $f(U) \subseteq V$ [if not Let $t \in U \Rightarrow t \notin \bar{A}$

suppose $f(t) \notin V \Rightarrow f(t) \in Y - V \Rightarrow t \in f^{-1}(Y-V) = A \subseteq \bar{A}$ #

(2) Composition of two continuous function is also cts

(i.e $f: X \to Y$ + $g: Y \to Z$ be cts fn's $\Longrightarrow g\circ f: X \to Z$ is cts)

(Proof)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Let U be open in Z $\boxed{Try}$ $(g\circ f)^{-1}(U)$ is open in X

$\therefore (g\circ f)^{-1}(U) = f^{-1}\circ g^{-1}(U) = f^{-1}(g^{-1}(U))$

$U \underset{open}{\subseteq} Z$, g cts $\Rightarrow$ $\therefore g^{-1}(U)$ is open in Y

f is cts $\Rightarrow$ $f^{-1}(g^{-1}(U))$ is open in X

$\Longrightarrow (g\circ f)^{-1}(U)$ is open in X

$\therefore g\circ f$ is a cts fn

Let $f: X \to Y$ be function & $A \subseteq X$ then
$f/_A : A \to Y \ni (f/A)(x) = f(x) \quad \forall x \in A$
is called restruction function

[Restruction of any cts fn is cts].

($f: X \to Y$ & $A \subseteq X$ $\Rightarrow$ $f/_A : A \to Y$ is cts)

$U \underset{open}{\subseteq} Y$ Try $(f/A)^{-1}(U)$ is open

$(f/A)^{-1}(U) = f^{-1}(U) \sqcap A$ where $f^{-1}(U)$ is open in X

$f^{-1}(U) \sqcap A$ is open in A so $(f/A)^{-1}(U)$ is open

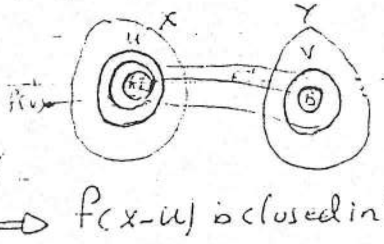
$\therefore f/A$ is cts.

Q.E.D

A mapping $f: X \to Y$ is closed iff $\forall B \subseteq Y, \forall U \underset{open}{\subseteq} X$

$f^{-1}(B) \subseteq U$

$\exists$ (open) in Y $\ni B \subseteq V$ & $f^{-1}(V) \subseteq U$



Suppose f be closed

$B \subseteq Y$ & $f^{-1}(B) \subseteq U \underset{open}{\subseteq} X$

$(X-U)$ is closed in X $\Rightarrow$ $f(X-U)$ is closed in Y

$V = Y - f(X-U)$ is open in Y

$B \subseteq V$ [if not $y \in B - V$ $\Rightarrow$ $y \in f(X-U)$
$\Rightarrow y \notin V \Rightarrow y \notin Y - f(X-U)$
so $\exists x \in X - U \ni f(x) = y \in B$
& $x \in f^{-1}(B) \subseteq U$ #)

$f^{-1}(V) \subseteq U$ [if $x \in f^{-1}(V)$ then $f(x) \in V = Y - f(X-U)$
$\Rightarrow f(x) \notin f(X-U) \Rightarrow x \notin X - U$
$\Rightarrow x \in U$

Let $C \underset{closed}{\subseteq} X$ Try $f(C)$ is closed in Y!
Suffision to (show) $Y - f(C)$ is open in Y

assume f is given & satisfies the conduction stat in Theorm

Take $y \in Y - f(C)$ $\Rightarrow y \notin f(C) \Rightarrow f^{-1}(y) \notin C$

$\therefore f^{-1}(y) \sqcap C = \phi$ i.e. $f^{-1}(y) \subseteq X - C$ (open)

by assumption $\exists V \underset{open}{\subseteq} Y \ni y \in V$ & $f^{-1}(V) \subseteq X - C$

thus $V \cap f(C) = \phi$ [if not $t \in V \cap f(C) \Rightarrow \exists x \in C$
$\ni f(x) = t \Rightarrow x \in f^{-1}(t) \subseteq f^{-1}(V) \cap C \neq$ #

$\Rightarrow y \in V \subseteq Y - f(C)$

$\therefore Y - f(C)$ is open & hence $f(C)$ is closed $\therefore f$ is closed

(5) a mapping $f: X \to Y$ is open $\Leftrightarrow$ $\forall B \subseteq Y \ \forall C \underset{closed}{\subseteq} X$
with $f^{-1}(B) \subseteq C$ $\exists D$ in Y $\ni B \subseteq D$ & $f^{-1}(D) \subseteq C$

Proof Ex

(6) Prove that $f: X \to Y$ is closed iff $\forall A \underset{closed}{\subseteq} X$ then
$\{y \in Y : f^{-1}(y) \sqcap A \neq \phi\}$
is closed in Y

Proof

claim $f(A) = \{y \in Y : f^{-1}(y) \sqcap A \neq \phi\} = B$ [f(A) نريد أن نثبت]

$(\subseteq)$
let $t \in f(A) \Rightarrow (t = f(x)$ for some $x \in A) \Rightarrow x \in f^{-1}(t)$ & $x \in A$
then $x \in f^{-1}(t) \sqcap A \Rightarrow t \in B$

(2) let $t \in B \Rightarrow f^{-1}(t) \sqcap A \neq \phi$ say $x \in f^{-1}(t) \sqcap A$
$\Rightarrow f(x) = t$ & $x \in A \Rightarrow t \in f(A)$

$\therefore f(A) = B = \{y : Y : f^{-1}(y) \sqcap A \neq \phi\}$ is closed in Y

Q.E.D

Let X, Y be toplogical spaces $\boxed{\text{then}}$ the Projection

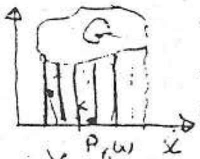
$P_{r_1}: X\times Y \longrightarrow X$; $P_{r_1}(x,y)=x$

① conts ② onto ③ open

$U \subseteq X$ ∴ $P_{r_1}^{-1}(U) = U\times Y$ (since Y is open & $U \underset{open}{\subseteq} X$)

$U\times Y \underset{open}{\subseteq} X\times Y$ ∴ P_{r_1} is cts

$x \in X$ then P_{r_1} is onto (cleare).



$G \subseteq X\times Y$ $\boxed{\text{Try}}$ $P_{r_1}(G)$ is open in X

$\rightarrow \exists (x,t) \in G \ni P_{r_1}(x,t)=x$

open in (Tproduct) $\boxed{\text{then}}$ U open in X & V open in Y

$U\times V \subseteq G$ take P_{r_1} for every side, we have

$U \subseteq P_{r_1}(G)$ then $P_{r_1}(G)$ is open in X ∴ P_{r_1} is open

---

Let $\prod X_i$ be the product space of $X_1, \dots, X_n$ then $\forall k = 1, \dots, n$

the projection $P_k: \prod_{i=1}^{n} X_i \longrightarrow X_k$ is ① cts ② open ③ onto

---

Let $U_k \subseteq X_k$ then $P_k^{-1}(U_k)$ is an element of a subbase of the product topology & therefor is open

$P_k^{-1}(U_k) = X_1 \times X_2 - \times U_k \times - \times X_n$

it is enough to show that the **image** of any basic open set is open

(since every open set is union of basic open sets & the image of union is union of images)

Let $\prod U_i$ be basic open set then $P_k\left(\prod_{i=1}^{n} U_i\right) = U$ which is open in X_k

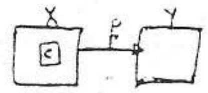
Let $t \in X_k$ ∃ $x \in \prod X_i$ whose k-th coordinate $x_k = t$

then it is clear that $P_k(x) = x_k = t$

⑨ For a bijective function $h: X \longrightarrow Y$ the following statement are equivalent ① h is homeomorphism

② h is cts & closed

③ both h & h^{-1} are cts



Proof. $\boxed{1 \to 2}$ Let $C \subseteq X$, closed, $\boxed{\text{then}}$ $X - C$ is open. & $h(X-C)$ is open

But $h(X-C) = h(X) - h(C) = Y - h(C)$ open so $h(C)$ is closed

$(2 \to 3)$ f $C \underset{closed}{\subseteq} X$ then $(h^{-1})^{-1}(C) = h(C)$ which is closed in Y

so h^{-1} is cont

$(3 \to 1)$. Let $U \underset{open}{\subseteq} X$, since h^{-1} is cont then $(h^{-1})^{-1}(U)$ is open

But $(h^{-1})^{-1}(U) = h(U)$ so h is open

---

⑩ Let $X \xrightarrow{f} Y$ be bijection then the following statement are equivalents ① f is homeomorphism

② $f: X \to Y$ is cts & open

③ f is cts & closed

④ $f(\overline{A}) = \overline{f(A)}$ for each subset $A \subseteq X$.

Proof

$(1-2)$ It is cleare f is cts let G be an open in X $\underline{\underline{\text{Try}}}$ $f(G)$ is open

$f(G) = (f^{-1})^{-1}(G)$ open in Y (becase $f^{-1}: Y \to X$ is conts).

$(2-3)$ Let A be a closed set in X $\underline{\underline{\text{Try}}}$ $f(A)$ is closed in Y

$f(A^c) = f(X-A)$ is open in Y (f is open function & A^c open in X)

$(f(A^c))^c = Y - f(X-A)$ is closed set in Y (since f is bijection)

$f(A)$

$(3-4)$ Since f is conts then $(\subseteq)$ $f(\overline{A}) \subseteq \overline{f(A)}$

(2) we know $A \subseteq \overline{A} \Rightarrow f(A) \subseteq$

Since f is bijects But $\overline{A}$ closed in X then $f(\overline{A})$ is closed in Y

Since f is closed function then $\overline{f(A)} \subseteq \overline{f(\overline{A})} \subseteq f(\overline{A})$

constant

... X_1, X_2 be a top-spaces $f: X \longrightarrow X_1 \times X_2$ is function

f is cts $\Longleftrightarrow$ $Pr_i \circ f$ is conts for each $i = 1, 2$.

(composition of two cts fns is cts)

$Pr_1 \circ f$, $Pr_2 \circ f$ are cts (Req) $f: X \longrightarrow X_1 \times X_2$ is cont

Let G be any basic open set in $X_1 \times X_2$ - Try $f^{-1}(G)$ is open in X

$G = U \times V$ for some U is open in X_1 & V is open in X_2

(since G is basic in $X \times Y$)

$f^{-1}(U \times V) = \underbrace{(Pr_1 \circ f)^{-1}(U)}_{\text{open in } X} \cap \underbrace{(Pr_2 \circ f)^{-1}(V)}_{\text{open in } X}$ (open in X)

$f^{-1}(G)$ is open in X then f is cts.

... be a space & R have the standard topology

$f: X \to R$ is cts $\Longleftrightarrow$ $\forall a \in R$ both of the sets

$\{x \in X : f(x) > a\}$ & $\{x \in X : f(x) < a\}$

are open

$\{x \in X : f(x) > a\} = f^{-1}(a, \infty)$

$\{x \in X : f(x) < a\} = f^{-1}(-\infty, a)$

Suppose f cts then $f^{-1}(a, \infty)$ & $f^{-1}(-\infty, a)$ are

$\forall a \in R$ so $\{x \in X : f(x) > a\}$ & $\{x \in X : f(x) < a\}$ are open.

$f^{-1}(a, \infty) \underset{\text{open}}{\subseteq} X$ & $f^{-1}(-\infty, a) \underset{\text{open}}{\subseteq} X$ but $f^{-1}(R) = X$

$R = A \cup B \cup \{a\}$ where $A \cap B = \phi$

$f^{-1}(a) = X - \{f^{-1}(A) \cup f^{-1}(B)\}$ where $f^{-1}(A) \cap f^{-1}(B) = \phi$

is closed.

(13) Prove or disprove

(1) all constant fns are cts

(2) if the domain of the fn has the discrete top then the fn is continuous

(3) All injective fns are cts

Proof

(1) $X \xrightarrow{f} Y$, $f(x) = a \quad \forall x \in X$ let $p \in X$ & let $V \underset{\text{open}}{\subseteq} Y$ with $a = f(p)$

Take $\boxed{U = X}$ then $x \in U$ & $f(U) = \{a\} \subseteq V$

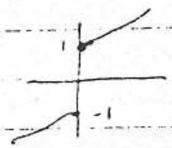
Note that: if $V \subseteq Y$ (open) then $f^{-1}(V) = \begin{cases} X & \text{if } a \in V \\ \phi & \text{if } a \notin V \end{cases}$

-2- X - (discrete space) $\xrightarrow{f} Y$

& $V \subseteq Y$ (open) then $f^{-1}(V) \subseteq X$ or $f^{-1}(V) \in P(X) \Rightarrow f^{-1}(V)$ open

-3- all injective fns are cont X

ex $f(x) = \begin{cases} x - 1, & x \leq 0 \\ x + 1 & x > 0 \end{cases}$



or $(R, T_{L.r}) \xrightarrow{i} (R, T_s)$ $V = (a, b)$ then $i^{-1}(V) = (a, b) \in T_L$

(14) Let $X = \{a, b, c\}$, $T_X = \{\phi, X, \{a\}, \{b, c\}\}$

& $Y = \{x, y, z\}$, $T_Y = \{\phi, Y, \{y\}, \{z\}, \{y, z\}\}$

(a) Let $f = \{(a, x), (b, z), (c, z)\}$ prove or disprove f is cts

(b) find all cts fn from X to Y

$f^{-1}(\phi) = \phi$, $f^{-1}(Y) = X$, $f^{-1}(\{y\}) = \phi$

$f^{-1}(\{z\}) = \{b, c\}$ & $f^{-1}(\{y, z\}) = \{b, c\}$

Since the inverse image of each open set in Y is open in X then f is conts.

Hereditary :- A property P is called hereditary $\longleftrightarrow$ whenever X has P then every nonempty subspace of X has P

Topological property :- A property of a space X is called a top-property if whenever $Y \cong X$ then Y has also this property.

Let $X \times Y$ be a product space & $(x_0, y_0) \in X \times Y$ then $X \cong X \times \{y_0\}$ & $\boxed{Y \cong \{x_0\} \times Y}$

$h : X \times \{y_0\} \longrightarrow X$ by $h(x, y_0) = x$ $(h = P_1 | X \times \{y_0\})$

– h is onto : Let $p \in X$ then $(p, y_0) \in X \times \{y_0\}$ & $h(p, y_0) = p$

Let $p, q \in X \times \{y_0\}$ & $p \neq q \Rightarrow p = (x_1, y_0)$ & $q = (x_2, y_0)$ where $x_1 \neq x_2$ then $h(p) = x_1 \neq x_2 = h(q)$

$\mathcal{B} = \{U \times V : U \underset{open}{\subseteq} X$ & $V \underset{open}{\subseteq} Y\}$ is a base for $X \times Y$

& $\mathcal{B}_{(X \times \{y_0\})} = \{U \times V \cap (X \times \{y_0\}) = U \times \{y_0\} ; U \underset{open}{\subseteq} X\}$ for the relative topology

to prove that the image of each basis is open

open set B union of basis open sets)

(16) IF $f : X \longrightarrow Y$ is a homomorphism & $A \subseteq X$ then $A \cong f(A) \subset Y$

Proof $h : X \to Y$, $(h|A) : A \to Y$

to show homomorphism

1- 1-1 Let $a_1, a_2 \in A$, $a_1 \neq a_2$ then $(h/A)(a_1) = h(a_1) \neq h(a_2) = (h/A)(a_2)$

2- onto : obvious

3- Cont : obvious

4- open : Let $V \subseteq A$ (open) then $V = U \cap A$ where U is open in X then $(h/A)(V) = h(V) = h(U \cap A) = \underset{open}{h(U)} \cap h(A)$ open in $h(A)$

(17) For any two spaces X & Y prove $X \times Y \cong Y \times X$

Proof : Define $h : X \times Y \longrightarrow Y \times X$ by $h(x, y) = (y, x)$

– h onto : $h(X \times Y) = Y \times X$

– h 1-1 : $(x_1, y_1) \neq (x_2, y_2) \Rightarrow (y_1, x_1) \neq (y_2, x_2)$

– open / cts : $h(U \times V) = V \times U$ & $h^{-1}(V \times U) = U \times V$

(18) Let $f : X \longrightarrow Y$ be a function. Let $g : X \longrightarrow f \subseteq X \times Y$ be given by $g(x) = (x, f(x))$

Prove g is homeomorphism iff f is cts

Proof

$\Leftarrow$ ① g is onto

① g is 1-1 Let $x_1 \neq x_2 \Rightarrow g(x_1) = (x_1, f(x_1)) \neq (x_2, f(x_2)) = g(x_2)$

② g is conts : let $g(X) \cap (U \times V)$ be a basic open subset of $g(X)$

$W = g(X) \cap (U \times V) = \{(x, f(x)) : x \in U$ & $f(x) \in V\} = \{x \in f^{-1}(V) \cap U\}$

$\therefore g^{-1}(W) = U \cap f^{-1}(V)$ open in X

④ g open $U \subseteq X$ $\therefore g(U) = \{(x, f(x)) : x \in U\} = g(X) \cap (U \times Y)$

... the property of a space having an isolated point ... topological property

... isolated point of $X \Leftrightarrow U \cap X - \{x\} = \phi$ for some open U containing x_0.

... $U - \{x_0\} = \phi \longrightarrow U = \{x_0\}$

... an isolated pt of a space $X \Leftrightarrow \{x_0\}$ is open

... $X \cong Y$ & x_0 is an isolated point of X

... an isolated point of Y. $\{h(x_0)\} = h(\{x_0\})$ (h hom, open)

... example of a bijective fn $f: X \to Y \ni f$ is cts ... $f^{-1}: Y \to X$ is not conts

... that $f: X \to Y$ is homeo $\Leftrightarrow f^{-1}: Y \to X$ is homeomorph

... from $f = i\ (R, T_s) \longrightarrow (R, T_{l.r})$

... & $V \in T_{l.r} \Rightarrow i^{-1}(V) = V \in T_{l.r} \subseteq T_s$ ① cts

... $f^{-1} = i^{-1}$ is not cts $(c,1) \in T_s$ but $i(c,1) = (c,1) \notin T_{l.r}$

... $\Leftrightarrow f^{-1}: Y \xrightarrow{hom} X$

... bijection $\Leftrightarrow f^{-1}$ bijection & f open $\Rightarrow f^{-1}$ open.

... cts $\Leftrightarrow f^{-1}$ open

... Top for R which make the two ... space non-homeomorphic

(R, T_s) & $(R, T_{l.r})$ suppose $\exists$ a hom $h: (R, T_s) \to R, T_{l.r}$

... $U = (0,1)$ & $V = (2,3)$

... $h(U) \overset{h\ 1\text{-}1}{=} h(U \cap V) = h(\phi) = \phi$ #

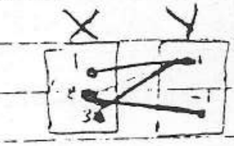
(متناظر)

② The identification Toplogy

Def Let (X, T_X) be a top-space, Y any set then $f: X \xrightarrow{onto} Y$ & the topology induced on Y by f & (X, T_X) is called the identification Top on Y

Def ② $(X, T_X) \xrightarrow[onto]{f\ cts} (Y, T_Y)$ then f is called an identification function
f is cont. & onto

Examples



① $X = \{1,2,3\}$ & $T_X = \{\phi, X, \{1\}, \{3\}, \{1,3\}\}$

$Y = \{1, -1\}$ & $T_Y = \{\phi, Y, \{1\}, \{-1\}\}$

$f = \{(1,1), (2,-1), (3,1)\}$ Is f an identification function?

Solution: NO since f not cts because $f^{-1}(\{-1\}) = \{2\}$ not open in X

---

② $X = \{1,2,3\}$ & $T_X = \{\phi, X, \{1,3\}, \{2\}\}$, $Y = \{1,-1\}$ & $T_Y = \{\phi, Y, \{1\}, \{-1\}\}$

$f = \{(1,1), (2,-1), (3,1)\}$ Is f an identification function

Yes f is onto & cts Tidentification $= \{\phi, Y, \{1\}, \{-1\}\} = T_Y$

$\therefore f$ is an identification function.

---

③ $X = \{1,2,3\}$, $T_X = \{\phi, X, \{1,2\}, \{2\}\}$ & $Y = \{1,-1\}$, $T_Y = \{\phi, Y, ...$

$f = \{(1,1), (2,-1), (3,-1)\}$ is function identification function?

NO Since Tidentification $= \{\phi, Y, \{1\}, \{-1\}\} \neq T_Y$

---

④ The identification function f is continuouse if $V \in T_Y$ then $V \in T_f$ then $f^{-1}(V)$ open in domean $\therefore f$ is cts

But converse is **not** True.

Example $(R, T_s) \xrightarrow{i} (R, T_{l.r})$ is onto & cts

but $T_i = \{U \subseteq R : i^{-1}(U) \in T_s\} = \{U \subseteq R : U \in T_s\}$

$(0,1) \in T_i$ but $(0,1) \notin T_{l.r}$

Theorms & Ex

(1) Let (X,T_X), (Y,T_Y) be two spaces $\ni f: X \to Y$ if f is cts + open (closed) then f is an identif f_n

Proof: $T_f = T_Y$

Let $V \in T_Y \Rightarrow V \subseteq Y$ + $f^{-1}(U) \in T_X$ (since f cts)

Hence $V \in T_f$

Let $V \in T_f \Rightarrow V \subseteq Y$ + $f^{-1}(U) \in T_X$ (since f open)

$f(f^{-1}(U))$ is open in T_Y. But $f f^{-1}(U) = V$ (onto)

so $V \in T_Y$

If closed then $X - f^{-1}(U)$ is closed

$X - f^{-1}(U)$ is closed in (X, T_X). But $f(X - f^{-1}(U)) = Y - V$

is open in T_Y.

(2) The composition of two identification fn's is an identification

$(X,T_X) \xrightarrow{f} (Y, T_Y) \xrightarrow{g} (W, T_W)$, $\underbrace{\qquad\qquad}_{g \circ f}$

Let f, g be identification fn's. Let $G \subseteq W$

$(g\circ f)^{-1}(G)$ is open in X

$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$

$g^{-1}(G) \in T_Y$ + Thus $G \in T_W$

Hence $g\circ f$ is an identification function

$(X,T_X) \xrightarrow{f} (Y,T_Y)$, f is identification fn

$\ni$ ① $V \subseteq Y$ open in $Y \Leftrightarrow f^{-1}(V)$ open in X

② $C \subseteq Y$ closed in $Y \Leftrightarrow f^{-1}(C)$ closed in X

③ Let $(X,T_X) \xrightarrow{f} (Y,T_Y)$ be cts if $\exists g: Y \to X$ a cts $\ni f\circ g = i_Y$ $\boxed{\text{then}}$ f is an identification function

Proof $f\circ g = i_Y$

$Y \xrightarrow{g} X \xrightarrow{f} Y$ لما شرطان onto + cts

f is onto: if $y_0 \in Y$ then $(f\circ g)(y_0) = i_Y(y_0) = y_0$

So $f(g(y_0)) = y_0$ But $g(y_0) = x_0 \Rightarrow f(x_0) = y_0$

f is identif f_n $\boxed{\text{Try}}$ $T_f = T_Y$

first $T_Y \subseteq T_f$ (because f is conts)

Moreover if $V \in T_f \Rightarrow f^{-1}(U) \in T_X$ so $g^{-1}(f^{-1}(U)) \in T_Y$

(because g is also cts)

But $g^{-1}(f^{-1}(U)) = (f\circ g)^{-1}(U) = i_Y^{-1}(U) = U$ Hence $V \in T_Y$

④ Let (X,T_X), (Y,T_Y) be spaces $f: X \to Y$ an identification function then the identification topology is the Largest topology on Y which makes f continuose.

Proof

Let τ^* be a topology on Y $\ni f: (X,T_X) \to (Y, T^*)$ is cts Try $T_Y \supseteq \tau^*$

Let $V \in \tau^* \Rightarrow f^{-1}(U) \in T_X$ (f cts)

Now $V \in Y$, $f^{-1}(V) \in T_X \Rightarrow V \in T_{identif}$

But $T_{identif} = T_Y$ since f identification fn

then $V \in T_Y$.

⑤ Let $(X,T_X) \xrightarrow[\text{inde}]{f} (Y,T_Y) \xrightarrow[f_n]{g} (Z,T_Z)$ then g is cts $\Leftrightarrow$ $g\circ f$ cts

Proof $\Rightarrow$ obvious (f cts + the composition of two cts is cts)

$\Leftarrow$ Let $g\circ f$ be cts Let $W \underset{\text{open}}{\subseteq} T_Z \Rightarrow (g\circ f)^{-1}(W)$ is open in X

+ f is an identif fn so $g^{-1}(W) \in T_Y$ Hence g is cts

Produce

... not necessary for an identification function to be open
... ted.
... example: $X=\{1,2,3,4\}$ $\quad T_X=\{\phi, X, \{1\}, \{2,3\}, \{1,2,3\}\}$

$Y=\{a,b,c\}$ $\quad f: X \to Y \ni$

$f=\{(1,a),(2,b),(3,c),(4,c)\}$

$T_f=\{\phi, Y, \{a\}\}$ where f identification

$f(\{2,3\})=\{b,c\} \notin T_f$ i.e. f is not open. also $f(\{4\})=\{c\}$ not closed

... $f: X \to Y$ be an identification fn. if Z is any set &
$g: Y \to Z$ is a surjection
... that $g \circ f$ is an identification function iff
g is identification function.

$$X \xrightarrow{f} Y \xrightarrow[\text{onto}]{g} Z \qquad \overbrace{\quad}^{g\circ f}$$

$\Rightarrow$ suppose $g\circ f$ is an identification function $T_{g\circ f}=T_Z=T_g$

$U \in T_Z \Rightarrow (g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ open $\Rightarrow g^{-1}(U)$ is open in Y

then $V \in T_g$

$U \in T_g$ then $g^{-1}(U)$ open in Y then $f^{-1}g^{-1}(U)$ open in X

then $(f\circ g)^{-1}(U)$ open in Y $\Rightarrow U \in T_Z$

$\therefore T_Z = T_g$.

... Composition of identification function is an identification.

... A function $f: X \to A \ni f/_A = i_A$ is called $\boxed{\text{retraction}}$

i.e. $f(x)=x \quad \forall x \in A$ & A is called retract of X

... (A, T_A) — subspace of (X,T)

(8) Let X be a space & $A \subset X$ a subspace if $\exists$ a $f: X \to A \ni f/_A = i_A$ then f is called a retraction of X onto A & A is called a retract of X under f.

$\boxed{\text{Proof}}$ that every retract is an identification

Solve

every retraction is an identification function

$f: X \to A$ $\quad f$ is cts onto

Let $U \subseteq A$ $\Rightarrow f^{-1}(U)$ is open in X $\Rightarrow f^{-1}(U)\cap A$ is open in ...

Try $V = f^{-1}(U)\cap A$

$(\subseteq)$ if $x \in V$ then $f(x)=x \quad \forall x\in A \Rightarrow x \in f^{-1}(V)$ & $x \in A \Rightarrow x \in ...$

$(\supseteq)$ if $x \in f^{-1}(U)\cap A$ then $x\in A$ & $x \in f^{-1}(U)$ $\Rightarrow x \in A$ & $f(x) \in U \Rightarrow f(x) = ...$

(9) Let $f: X \to Y$ be cts surjection $\Rightarrow f$ is iden fn $\Leftrightarrow$ the following condition holds, $\forall$ space Z & each function $g: Y \to Z$ the continuity of $g\circ f$ implies the cts function of g

$\Leftarrow$

$$X \xrightarrow[\text{cts-onto}]{f} (Y, T_Y) \xrightarrow{i} (Y, T_f) \qquad \overbrace{\quad}^{i\circ f}$$

Note that i is cts (if $U \in T_f$ then $(i\circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(U)$ open in ...

By assumption, i cont. Thus if $V \in T_f$ then $V = i^{-1}(U) \in T_Y$

then $T_f \subseteq T_Y$

General products & Product topology & product space

Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of set then

$\{f : f : \Delta \longrightarrow \bigcup_{\alpha\in\Delta} X_\alpha : f(\alpha) \in X_\alpha, \forall \alpha \in \Delta\}$

the α-th coordinate of the point f

is called α-th coordinate of X

$X_2 = \{3, -3\}$ $\Delta = \{1, 2\}$

$\{f : f : \{1, 2\} \xrightarrow{f_n} \{10, 20, 3, -3\}, \ni$

$\ldots$ & $f(2) \in \{-3, 3\}$

$\{(1,10), (2,-3)\}, \{(1,10), (2,3)\}$

$\{(1,20), (2,-3)\}, \{(1,20), (2,3)\}$

$(10,-3), (10,3), (20,-3), (20,3)$

indexset لا تختلف ايش لا تغير ملحوظة

$\prod_{\alpha\in A} A_\alpha \cap \prod_{\alpha\in A} B_\alpha = \prod_{\alpha\in A} A_\alpha \cap B_\alpha$

Projection

$\pi_\beta : \prod_{\alpha\in\Delta} X_\alpha \longrightarrow X_\beta$ defined by

$\pi_\beta(x) = x_\beta$ is called the projection of $\prod_{\alpha\in\Delta} X_\alpha$

Let be the product of a family of sets $\{X_\alpha : \alpha \in \Delta\}$

$P_\alpha : \prod X_\alpha \longrightarrow X_\alpha$ by

$P_\alpha(f) = f(\alpha)$

projection function

بعض النظريات الهامة

(1) for the family of sets $\{X_\alpha : \alpha\in\Delta\}$, $\prod_{\alpha\in\Delta} X_\alpha \neq \phi \Leftrightarrow \forall \alpha\in\Delta, X_\alpha \neq \phi$

[axiom of choice]

Proof: $\rightarrow$ Given $\prod X_\alpha \neq \phi$ to show $X_\alpha \neq \phi\ \forall \alpha\in\Delta$

Since $\prod X_\alpha \neq \phi$ therefor $\exists f \in \prod_{\alpha\in\Delta} X_\alpha \ni f(\alpha) \in X_\alpha\ \forall \alpha \in \Delta$

$X_\alpha \neq \phi\ \forall \alpha\in\Delta$

$\leftarrow$ cleare

(2) **axiom of choise** $\rightarrow$ if $\{A_\alpha : \alpha\in\Delta\}$ is family of non-empty pairwise disjont sets There is exist $B \subset \bigcup A_\lambda \ni B \cap A_\lambda$ has exactly one element for each $\lambda \in \Delta$

(3) **choice functin** $\rightarrow$ If $\{A_\lambda : \lambda\in\Delta\}$ is an indexed family of nonempty pair wise disjont sets there is a functin $f : \Delta \to \bigcup A_\lambda \ni f(\lambda) \in A_\lambda$ for each $\lambda \in \Delta$ (f is called choice function)

Dfn (3): Suppose $\{(X_\alpha, T_\alpha), \alpha\in\Delta\}$ a collection of Top-space $P_\alpha : \prod_{\alpha\in\Delta} X_\alpha \longrightarrow X_{\alpha_0}$ the α_0-projection map $P_{\alpha_0}(f) \ldots$

* **Define**

$S = \{P_0^{-1}(H_{\alpha_0}) : H_{\alpha_0} \in T_{\alpha_0}, \alpha_0 \in \Delta\}$ then S is subbase for some top on $\prod X_\alpha$

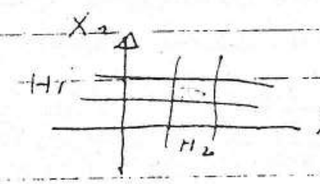
Now $\beta[S] = \{\bigcap_{i=1}^{n} P_{\alpha_i}^{-1}(H_{\alpha_i}) : \alpha_i \in \Delta, H_{\alpha_i} \in T_{\alpha_i}, n \in N\} \cup \{\prod_{\alpha\in\Delta} X_\alpha\}$

$T = T[B]$

example to show this

$P_1 : X_1 \times X_2 \longrightarrow X_1$ $\quad P_1(f) = f(1) \Leftrightarrow P_1(a,b) = a$

$P_2 : X_1 \times X_2 \longrightarrow X_2$ $\quad P_2(f) = f(2) \Leftrightarrow P_2(a,b) = b$



$H_1 \times X_2$

$X_1 \times H_2$

$= H_1 \times X_2 \cap X_1 \times H_2$

$= H_1 \times H_2 \times X_1 \times X_2$

$= H_{\alpha_1} \times H_{\alpha_2} \times \dots \times H_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} X_\alpha \quad i = 1, 2, \dots, n$

$T(B)$) the product topology also $\boxed{\text{the Tychonoff}}$ product topology

$U_\beta \subseteq X_\beta$ we can take about

$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in \Delta} G_\alpha$ where $G_\beta = U_\beta$ + $G_\alpha = X_\alpha \;\forall \alpha \neq \beta$

since $\pi_\beta^{-1}(U_\beta) = \{ x \in \prod X_\alpha : x_\beta = \pi_\beta(x) \in U_\beta \} \subseteq \prod_{\alpha \in \Delta} G_\alpha$

$y \in \pi_\beta^{-1}(U_\beta)$ so

$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in \Delta} G_\alpha$ ☐

(Theorm st Ex)

be the product space of spaces (X_α, T_α)

π_β is ① onto (or) surjection

② open

$\longrightarrow X_\beta$

~~by choice function~~

for $\alpha \in \Delta$, $\alpha \neq \beta$ choose a point $P_\alpha \in X_\alpha$

Let x be point of $\prod X_\alpha$ where $x_\beta = s$ + $x_\alpha = P_\alpha$ when $\alpha \neq \beta$

then $\pi_\beta(x) = x_\beta = s$

② $\boxed{\text{to show } \pi_\beta \text{ is open}}$ الجهة

It is enough to show that the image of any basic open set is open

such basic open set is of the form $\prod_{\alpha \in \Delta} U_\alpha$ when $U_\alpha \underset{open}{\subseteq} X_\alpha$ $\forall \alpha \in \Delta$, + $U_\alpha = X_\alpha$ for all finitely many $\alpha \in i$

then $\Longrightarrow$

$\pi_\beta(\prod_{\alpha \in \Delta} U_\alpha) = U_\beta$ which is open in X_β

$\Rightarrow \pi_\beta$ is an open function.

(2) Let $\prod_{\alpha \in \Delta} X_\alpha$ be product space of space X_α, Let $B_\alpha \subseteq X_\alpha$ اثبات

① Show $\overline{\prod_{\alpha \in \Delta} B_\alpha} = \prod_{\alpha \in \Delta} \overline{B_\alpha}$

② Let $D_\alpha \subseteq X_\alpha$ Proof $\prod_{\alpha \in \Delta} D_\alpha$ is dence in $\prod_{\alpha \in \Delta} X_\alpha$ iff D_α is dence in X_α $\forall \alpha \in \Delta$.

Proof ①

to show $\overline{\prod_{\alpha \in \Delta} B_\alpha} = \prod \overline{B_\alpha}$

($\subseteq$) since $B_\alpha \subseteq \overline{B_\alpha}$ $\forall \alpha \in \Delta$ then $\prod_{\alpha \in \Delta} B_\alpha \subseteq \prod \overline{B_\alpha}$

+ $\overline{\prod_{\alpha \in \Delta} B_\alpha} \subseteq \prod_{\alpha \in \Delta} \overline{B_\alpha}$ مغلق

(2) Suppose $x \notin \overline{\prod_{\alpha \in \Delta} B_\alpha}$ i.e [$\exists$ basic open set U

$\ni x \in U = \prod_{\alpha \in \Delta} U_\alpha$, $U_\alpha = X_\alpha$, $\forall \alpha \neq \alpha_1, \alpha_2, \dots, \alpha_n$]

+ $U_{\alpha_1} \times U_{\alpha_2} \times \dots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \dots, \alpha_n} X_\alpha \cap \prod B_\alpha = \phi$] تعريف closure

$\rightarrow \exists\, U_{\alpha_i} \cap B_{\alpha_i} = \phi \Rightarrow x_{\alpha_i} \in U_{\alpha_i}$ so $x_{\alpha_i} \notin \overline{B_{\alpha_i}}$

$\rightarrow x \notin \prod_{\alpha \in \Delta} \overline{B_{\alpha_i}}$ $\quad \therefore \overline{\prod B_\alpha} \subseteq \prod \overline{B_\alpha}$

$\rightarrow \overline{\prod B_\alpha} = \prod \overline{B_\alpha}$

$\Rightarrow$ suppose $\prod_{\alpha\in\Delta} D_\alpha$ is dence in $\prod_{\alpha\in\Delta} X_\alpha$ (i.e.)

$= \overline{\prod_{\alpha\in\Delta} D_\alpha} \overset{\text{by }①}{=} \prod \overline{D}_\alpha \Rightarrow \overline{D_\alpha} = X_\alpha \ \forall \alpha\in\Delta$

$\therefore D_\alpha$ is dence in $X_\alpha \ \forall\alpha$

$\overline{D}_\alpha = X_\alpha \ \forall\alpha\in A \Rightarrow \prod_{\alpha\in\Delta} \overline{D}_\alpha = \prod_{\alpha\in\Delta} X_\alpha \overset{\text{by }①}{=} \overline{\prod_{\alpha\in\Delta} D_\alpha}$

$(\prod A_\alpha)^\circ \neq \prod A_\alpha^\circ$ ingeneral where $A_\alpha \subseteq X_\alpha$

$A_\alpha \subseteq X_\alpha$ (open) $\Rightarrow \prod_{\alpha\in\Delta} A_\alpha^\circ = \prod_{\alpha\in\Delta} A_\alpha$

$(\prod A_\alpha) \neq (\prod_{\alpha\in\Delta} A_\alpha)^\circ$

... be family of discrete spaces show

$\prod X_\alpha$ is discrete $\Leftrightarrow$ A is finite

ⓔ

suppose $\prod_{\alpha\in\Delta} X_\alpha$ has discrete by contraduction

suppose A is infinite

... then $\{f\}$ is open (we can written as arbitrary union of finite intersection element)

$\Rightarrow \prod U_\alpha = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha\neq\{\alpha_1,\ldots,\alpha_n\}} X_\alpha$

$f \in \prod_{\alpha\in\Delta} U_\alpha \subseteq \{f\}$ # A is finite

... finite Let $x \in \prod_{\alpha\in A} X_\alpha \Rightarrow x_\alpha \in X_\alpha$

$\ldots = \overset{\text{open}}{\{x_1\}} \times \overset{\text{open}}{\{x_2\}} \times \cdots \times \{x_n\} \Rightarrow X = \{(x_1, \ldots, x_n)\}$ is open in $\prod_{\alpha\in\Delta} X_\alpha$

... has the discrete Topology

$\ldots = \prod f_\alpha^{-1}(x_\alpha)$ open so $\prod X_\alpha$ has discrete topology.

Q.E.D

③ For each $\alpha\in\Delta$, Let $f_\alpha : X_\alpha \to Y_\alpha$ be a cts fn we define a fn $f : \prod X_\alpha \to \prod Y_\alpha$ as follows for $x = (x_\alpha) \in \prod X_\alpha$ we let $f(x) \in \prod_{\alpha\in\Delta}$

$\ni (f(x))_\alpha = f_\alpha(x_\alpha) \ \forall \alpha\in A$

ⓐ Prove f is cts

ⓑ if $\prod W_\alpha \subseteq \prod X_\alpha$ show $f(\prod W_\alpha) = \prod_{\alpha\in\Delta} f_\alpha(W_\alpha)$

Proof

(a) $\prod X_\alpha \xrightarrow{\Pi_\alpha} X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{\Pi_\alpha^{-1}} \prod Y_\alpha$, f

Now Π_α is cts $\forall\alpha\in\Delta$ + f_α is cts $\forall\alpha\in\Delta$ (since Π_α is open), Π_α^{-1} is

Now $f = \Pi_\alpha^{-1} \circ f_\alpha \circ \Pi_\alpha \ \forall\alpha\in A \Rightarrow f$ is cts.

(Let U be a sub baise open $\prod Y_\alpha$ then $U = \Pi_\alpha^{-1}(U_\alpha) \ni U_\alpha \underset{\text{open}}{\subseteq} Y_\alpha$

then $f_\alpha^{-1}(U) = f_\alpha^{-1}(\Pi_\alpha^{-1}(U_\alpha))$ open in X_α $\therefore P_\alpha^{-1}(f_\alpha^{-1}(U))$ open in $\prod X_\alpha$.

(b) ($\subseteq$) Let $x \in \prod_{\alpha\in\Delta} W_\alpha$ then $x_\alpha \in W_\alpha \subseteq X_\alpha \ \forall\alpha\in\Delta$ then $f_\alpha(x_\alpha) \in f_\alpha(W_\alpha)$

$(f(x))_\alpha \in f_\alpha(W_\alpha)$

then $f(x) \in \prod_{\alpha\in\Delta} f_\alpha(W_\alpha)$ then $f(\prod_{\alpha\in\Delta} W_\alpha) \subseteq \prod f_\alpha(W_\alpha)$ — (1)

(2)

Let $f(x) \in \prod_{\alpha\in\Delta} f_\alpha(W_\alpha)$ then $(f(x))_\alpha \in f_\alpha(W_\alpha) \ \forall\alpha\in A \Rightarrow f_\alpha(x_\alpha) \in f_\alpha(\ldots)$

$\exists\, \varepsilon_\alpha \in W_\alpha \ \forall\alpha\in\Delta \ (\ni f_\alpha(\varepsilon_\alpha) = f_\alpha(X_\alpha))$ but f_α is 1-1

then $\varepsilon_\alpha = X_\alpha$ so $X_\alpha \in W_\alpha \ \forall\alpha\in A \Rightarrow x \in \prod_{\alpha\in\Delta} W_\alpha$

then $f(x) \in f(\prod W_\alpha)$ then $\prod_{\alpha\in\Delta} f_\alpha(W_\alpha) \subseteq f(\prod_{\alpha\in\Delta} W_\alpha)$ — (2)

$\therefore$ by (1) + (2) (get $f(\prod W_\alpha) = \prod f_\alpha(W_\alpha)$)

$\alpha \doteq \prod_{\alpha\in D}$

جاد احمد السواح

$\prod X_\alpha$ be product space of space X_α, we choose a fixed point $b_\alpha \in X_\alpha$. Show subspace $Y_\beta = \{x \in \prod X_\alpha : x_\alpha = b_\alpha, \forall \alpha, \alpha \neq \beta\}$ is homorphic to the factors space X_β $\forall \beta \in \Delta$

... a fixed point $q_\alpha \in X_\alpha$, $\forall \alpha \in \Delta$ ... the subset $D = \{x \in \prod X_\alpha : x_\alpha = q_\alpha$ $\forall$ but finite many $\alpha \in \Delta\}$ ... dence in $\prod_{\alpha \in \Delta} X_\alpha$

$h : Y_\beta \longrightarrow X_\beta$ defined $h(x) = \pi_\beta(x)$ $\forall x \in Y_\beta$

since $h = \pi_\beta / Y_\beta$

Let $x_1, x_2 \in Y_\beta \ni x_1 \neq x_2$ then $\pi_\beta(x_1) \neq \pi_\beta(x_2) \Rightarrow h(x_1) \neq h(x_2)$

Let $y_\beta \in X_\beta \Rightarrow x \in \prod X_\alpha \ni x_\alpha = b_\alpha$ $\forall \alpha \in \Delta$, $\alpha \neq \beta$

$X_\beta = Y_\beta$ is open of Y_β & $h(x) = y_\beta$

Let $\prod U_\alpha$ be basic open set in $\prod X_\alpha \ni U_\alpha \in X_\alpha$ for all but finite many of them) U_α is proper open subset of X_α then $\prod U_\alpha \cap Y_\beta$ is basic open set in Y_β

$h(\prod U_\alpha \cap Y_\beta) = \underline{h(\prod U_\alpha)} \cap \underline{h(Y_\beta)} = U_\beta \cap X_\beta = U_\beta$ is open X_β

$\prod U_\alpha$ be non-empty basic open set in $\prod_{\alpha \in \Delta} X_\alpha$ then $U_\alpha = X_\alpha$ but finitly many of them are proper open subset of X_α

$\in U_\alpha$ for all but {except possibly finite many of U_α is not]

$\prod U_\alpha$ contains at least one point of D

Thus

$\prod_{\alpha \in \Delta} U_\alpha \neq \phi$ so D is dence in $\prod X_\alpha$

Q.E.D

(7) Let $\{X_\alpha ; \alpha \in \Delta\}$ & $\{Y_\beta, \beta \in B\}$ be two family of spac

Let $\varphi : \Delta \longrightarrow B$ bijection $\ni X_\alpha \cong Y_{\varphi(\alpha)}$ $\forall \alpha \in \Delta$

Show

$$\prod_{\alpha \in \Delta} X_\alpha \cong \prod_{\beta \in B} Y_\beta$$

Proof: suppose $\Delta \xrightarrow{\varphi} B$, $f_\alpha : X_\alpha \longrightarrow Y_{\varphi(\alpha)}$ homomorphic

defind $h : \prod_{\alpha \in \Delta} X_\alpha \longrightarrow \prod_{\beta \in B} Y_\beta$ [by $[h(x)]_{\varphi(\alpha)} = f_\alpha(x_\alpha)$] $\ni \varphi(\alpha)$

Claim h is hom

(1) (1-1) Let $x, y \in \prod_{\alpha \in \Delta} X_\alpha \ni x \neq y$ then $\exists \alpha_0 \in \Delta \ni x_{\alpha_0} \neq y_{\alpha_0}$ & since f_α is hom then $f_{\alpha_0}(x_{\alpha_0}) \neq f_{\alpha_0}(y_{\alpha_0})$ so

$[h(x)]_{\varphi(x_0)} \neq [h(y)]_{\varphi(x_0)}$ then $h(x) \neq h(y)$

(2) h is onto. Let $y \in \prod_{\beta \in B} Y_\beta$ Try $\exists x \in \prod X_\alpha \ni h(x) = y$

Let $\beta_0 \in B$ then $\exists \alpha_0 \in \Delta \ni \varphi(\alpha_0) = \beta_0$ but f_{α_0} is hom $\exists x_{\alpha_0} \in X$

$\ni f_{\alpha_0}(x_{\alpha_0}) = y_{\beta_0}$ then $[h(x)]_{\beta = \varphi(x_0)} = f_{\alpha_0}(x_{\alpha_0}) = y_{\beta_0} \Rightarrow h(x) \ldots$

(3) h is cts

$f_\alpha \circ \pi_\alpha = \pi_\beta \circ h \Rightarrow h$ is cts

$$\begin{tikzcd} \prod X_\alpha \arrow[r, "h"] \arrow[d, "\pi_\alpha"] & \prod Y_\beta \arrow[d, "\pi_\beta"] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\beta \end{tikzcd}$$

(4) h is open. Let $U = \prod_{\alpha \in \Delta} U_\alpha$ be a basic open set in $\prod_{\alpha \in \Delta} X_\alpha$ where U_α in X_α & $U_\alpha = X_\alpha$ for all but finitly many α, say $\{\alpha_1, \ldots, \alpha_n\}$

then $h(\prod_{\alpha \in \Delta} U_\alpha) = \prod_{\beta \in B} W_\beta$ where for $\beta \in B$ with $\beta = \varphi(\alpha)$ we have $W_\beta = f_\alpha(U_\alpha)$ then W_β is open & $W_\beta = Y_\beta$ for all But finitly many $\beta \in B$

Q.E.D

...lly finite: a family of subsets of Top-space is called a locally finite iff each point of the space has nbd meeting only finitly many elements of the family.

... union of any sub family from Locall finite ... of closed sets is closed.

... locally finite family of closed subset of X ... union in X a function on X is cts iff ... restriction of each A_λ is consts.

... family of $\mathcal{A}$ (locally finite closed)

$\boxed{\text{Try}}$ $X - C$ open

... $\exists U$ nbd of $x \ni U \cap$ (finite may elements of $\mathcal{A}$)

... B_n then $U \cap B_i \neq \phi$ $i = 1, \ldots, n$

... $\alpha \neq 1, 2, \ldots n$ then $U \subseteq X - B_\alpha$ $(\alpha \neq 1, 2, \ldots, n)$

... open $\ni x \in V \subseteq U \subseteq X - B_\alpha$

... open set $\exists V_i$ open $\ni x \in V_i \subseteq X - B_i$ $i = 1, \ldots, n$

... $\subseteq \bigcap_{i=1}^{n} (X - B_i)$ & $x \in V' \subseteq \bigcap_{\alpha \in D} (X - B_\alpha)$

... then $x \in G \subseteq \bigcap_{\alpha \in D} (X - B_\alpha) = X - \bigcup_{\alpha \in D} B_\alpha = X - C$ open

... C is closed.

... be locally finite family of closed subset of X

... $= X$ Let $f: X \xrightarrow{cts} Y \Leftrightarrow f/_{A_\lambda}: X \xrightarrow{cts} Y$ $\forall \lambda \in \Delta$

... cts Let $C \subseteq_{closed} Y \Rightarrow f^{-1}(C)$ closed in X

... $= f^{-1}(C) \cap A_\lambda$ closed (since A_λ closed) $\Rightarrow f/_{A_\lambda}$ is cts.

... $f/_{A_\lambda}$ is cts Let $C \subseteq_{closed} Y$ then $f/_{A_\lambda}^{-1}(C) \subseteq_{closed} A_\lambda$

... $\bigcup_{\lambda \in \Delta} f/_{A_\lambda}^{-1}(C)$ since $\cup A_\lambda = X$ which is closed f is cts.

(4) The weak Topology

Let X be arbitrary set, $\forall \alpha \in A$, Let $f_\alpha: X \longrightarrow (X_\alpha, \tau_\alpha)$ the weak Topology (Initial topology) on X induced by the family $\{f_\alpha : \alpha \in A\}$ of functions is the smallest topol. on X making each f_α continuous.

(*) The weak Topology on X induced by the family $\{f_\alpha : \alpha \in \Delta\}$ is the topology generated by the sub base $S = \{f_\alpha^{-1}(V_\alpha) : \alpha \in \Delta, V_\alpha \in \tau_\alpha\}$

& Base of the weak Topology is of the form

(*) $\{\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(V_{\alpha_i}) : n \in \mathbb{N}, \alpha_i \in A, V_{\alpha_i} \in \tau_{\alpha_i}\} \cup \{\prod_{\alpha \in P} X_\alpha\}$

Proposition of Weak Topology

(1) The weak Topology on A set X induced by $f_\alpha: X \to (X_\alpha, \tau_\alpha)$, is the smallest topology on X that makes each f_α continuous

Proof

(i) f_α is continuous? $\forall \alpha \in A$, $f_\alpha: X \to X_\alpha$ is cts $\forall \alpha \in A$ if V_α is subset X_α then $f_\alpha^{-1}(V_\alpha) \in S \subseteq \tau_w$

$f_\alpha^{-1}(V_\alpha)$ is open in X so $f_\alpha: X \to X_\alpha$ is cts

(ii) suppose τ' is a Top on $X \ni f_\alpha: (X, \tau') \longrightarrow X_\alpha$ is continuous function $\forall \alpha \in A$

It is enough to show that the sub base

$S = \{f_\alpha^{-1}(V_\alpha) : \alpha \in \Delta, V_\alpha \subseteq_{open} X_\alpha\} \subseteq \tau'$

but if $V_\alpha \subseteq_{open} X_\alpha$ then f_α being cts we have $f_\alpha^{-1}(V_\alpha) \in \tau'$

Thus τ is smalles than τ'

& τ' is arbitrary then τ is the smallest.

... The weak topology T_w induced by the fns ... $\to X_\alpha$ then $g: Y \to X$ is cts $\Leftrightarrow$ $f_\alpha \circ g$ is cts $\forall \alpha \in A$

$$Y \xrightarrow{g} X \xrightarrow{f_\alpha} X_\alpha, \quad f_\alpha \circ g$$

... cts then $\forall \alpha \in A$, $f_\alpha \circ g$ is composition of two cts fn ... fore it is cts

... $f_\alpha \circ g$ is cts (Tw) (g is cts) It is enough to show ... a sub base S for X we have $g^{-1}(S)$ is open in Y ... $s \in S$

... be a sub basic open subset of X ... $f_\alpha^{-1}(V_\alpha)$ for some $\alpha \in A$ & $V_\alpha \underset{opn}{\subseteq} X_\alpha$ } then

$= g^{-1}(f_\alpha^{-1}(V_\alpha)) = (f_\alpha \circ g)^{-1}(V_\alpha)$

... open in Y because $f_\alpha \circ g$ is cts $\therefore g$ is cts.

... X arbitrary set, $\forall \alpha \in A$, We let T_α be a top on X ... the top on X

... supremam topology generated by the (subbase $\bigcup_{\alpha \in A} T_\alpha$)

... called $T = \sup\{T_\alpha : \alpha \in A\} = \bigcup_{\alpha \in A} T_\alpha$

$\Rightarrow (X, T_\alpha)$ is weak Top is called super mam Topology

⑤، ④ بين تبين

... $\{T_\alpha : \alpha \in \Delta\}$ be a family of Top on a fixed set X & X_α the space ... of X with Top T_α, The identity function from $X \xrightarrow{i_\alpha} X_\alpha$ ... $W(\Phi) = \{i_\alpha : \alpha \in \Delta\}$ is supremam Topologyes on T_X

... $\cong$ to Δ in $\prod X_\alpha$ subset of $\prod X_\alpha$

... $\Delta = \{x \in \prod X_\alpha : X_\alpha = X_\beta \ \forall \alpha, \beta\}$

Proof

① T is Top generated by subbase $\bigcup_{\alpha \in \Delta} T_\alpha$ Define $i_\alpha : (X, T) \to (X_\alpha, T_\alpha)$

$\therefore T_w = T = \{i_\alpha^{-1}(V_\alpha) : \alpha \in A : V_\alpha \underset{cpn}{\subseteq} X_\alpha\} = \{V_\alpha : \alpha \in A, V_\alpha \underset{cpn}{\subseteq} X_\alpha\}$

to show $T_w = T$

$\Rightarrow$ let $V_\alpha \in T \Rightarrow V_\alpha \in T_{w\alpha} \Rightarrow T_\alpha \subseteq T_{w\alpha} \ \forall \alpha \in A$

the $\bigcup_{\alpha \in A} T_\alpha \subseteq T_w$ then $S(\bigcup T_\alpha) \subseteq T_w \Rightarrow B(S(\bigcup T_\alpha)) \subseteq T_w$

$\therefore T(B(S(\bigcup T_\alpha))) \subseteq T_w$ —— ①

but we know that weak Top is smallest Topology on X

i.e $T_w \subseteq T$ —— ②

$\therefore$ by ① & ② get $T_w = T$.

② Defin $f : X \to \prod X_\alpha$ by $f(x) = (X_\alpha)_{\alpha \in A} \ni X_\alpha = X \ \forall \alpha$

to show homomorphism

it sufficiant to show f is embedding

① Since the Top in X is weake topology induced $\{i_\alpha : \alpha \in \Delta\}$

② The final $\{i_\alpha : \alpha \in \Delta\}$ seperited point for X

$\therefore$ Let $x \neq y$

$i_\alpha(x) = (X_\alpha)_{\alpha \in \Delta} \ni X_\alpha = X \quad \forall \alpha \in A$

& $i_\alpha(y) = (Y_\alpha)_{\alpha \in \Delta} \Rightarrow Y_\alpha = y \quad \forall \alpha \in A$

if $i_\alpha(x) = i_\alpha(y)$ then $(X_\alpha)_{\alpha \in A} = (Y_\alpha)_{\alpha \in A} \Rightarrow X_\alpha = Y_\alpha \ \forall \alpha$

$\Rightarrow x = y$ #

$i_\alpha(x) \neq i_\alpha(y)$

$\therefore$ separated point form X

$\therefore f(x) \cong \Delta \quad \therefore f$ is embedded by ① & ②

E. E. D

47

(1) embedding of X in Y

X, Y be two space & $f: X \to Y$ is embedding of X in Y

$X \cong f(X)$ subset of Y $\quad (X) \approx (f(X))$

(2) evaluation map:

$\{f_\alpha : \alpha \in \Delta\}$ be a family of functions from $X \to X_\alpha$

evaluation map $e: X \to \prod_{\alpha \in \Delta} X_\alpha$ is defined as follows

any $x \in X$ we let $e(x)$ to be that point of the product space given as $(e(x))_\alpha = f_\alpha(x) \quad \forall \alpha \in \Delta$

called the evaluation function

(3) Separate point of X

$\Phi = \{f_\alpha : \alpha \in \Delta\}$ be a family of functions $: X \to X_\alpha$

given $x, y \in X$ with $x \neq y$ $\exists \alpha \in \Delta$

$\ni f_\alpha(x) \neq f_\alpha(y)$ $\quad$ يعني 1-1

(4) Separates points from closed set in X

we say that a family $\{f_\alpha : X \to X_\alpha, \alpha \in \Delta\}$ Separates pt from closed set in X **if**

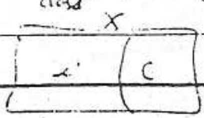
Given any $C \subseteq_{\text{closd}} X$ & any point $x \notin C$

$\exists \alpha \in A$

$\ni f_\alpha(x) \notin \overline{f_\alpha(C)}$

$f_\alpha : X \to X_\alpha \quad \forall \alpha \in \Delta$

pt from closed set in X if Given any $C \subseteq_{\text{closd}} X$ & $\forall x \notin C$



$\exists f_\alpha(x) \notin \overline{f_\alpha(C)}$

بدون برهان

(4) Let X be a Top-space & for each $\alpha \in A$, let $f_\alpha : X \to X_\alpha$ then evaluation map $e: X \to \prod_{\alpha \in \Delta} X_\alpha$ is an embedding iff (1) X has weak Top induced by (2) $\{f_\alpha : \alpha \in \Delta\}$ separates in X.

$X \xrightarrow{f_\alpha} X_\alpha$, $e \searrow \uparrow \pi_\alpha$, $\prod X_\alpha$ $\Rightarrow f_\alpha = \pi_\alpha \circ e$

Proof $\Rightarrow$

suppose e is an embedding i.e. $X \cong e(X)$

show (1) since $\prod_{\alpha \in \Delta} X_\alpha$ has the weak Top induced by $\{\pi_\alpha : \alpha \in \Delta\}$ & $X \xrightarrow{e} e(X) \subseteq \prod_{\alpha \in \Delta} X_\alpha$, The subspace $e(X)$ has weak Top induced by $\{\pi_\alpha / e(X) : \alpha \in \Delta\}$ so X has weak Top induced by $\{\pi_\alpha \circ e, \alpha \in A\} = \{f_\alpha, \alpha \in A\}$

show (2) since e is an embedding $\therefore$ 1-1

if $x, y \in X$, $x \neq y$ then $e(x) \neq e(y)$ $\exists \alpha \in A \ni [e(x)]_\alpha \neq [e(y)]_\alpha$

$\Rightarrow f_\alpha(x) \neq f_\alpha(y)$

this means that $\{f_\alpha : \alpha \in A\}$ separates point in X.

$\Leftarrow$ بالعكس

assume that (1) & (2) are satisfy to show e is embedding

(1) e is 1-1 $\quad$ if $x, y \in X$, $x \neq y$ $\exists \alpha \in A \ni f_\alpha(x) \neq f_\alpha(y)$ so $(e(x))_\alpha \neq$ ... & hence $e(x) \neq e(y)$ $\therefore e$ is 1-1

(2) e is cts $\quad$ e is cts $\Leftrightarrow$ $\pi_\alpha \circ e$ is cts $\Leftrightarrow$ f_α is cts

but f_α is cts $\forall \alpha \in A$ (because X has weak Top induced by $\{f_\alpha : \alpha \in \Delta\}$)

$\therefore e$ is cts.

(3) e is open

It is enough to show the image of any sub basic open subset of X is open

let $f_\alpha^{-1}(V_\alpha)$ be subbasic open set in X, when ever V_α is open in X_α for some

then $e(f_\alpha^{-1}(V_\alpha)) = e((\pi_\alpha \circ e)^{-1}(V_\alpha)) = e(e^{-1} \circ \pi_\alpha^{-1})(V_\alpha)$

$= e(e^{-1}(\pi_\alpha^{-1}(V_\alpha)))$

$= \pi_\alpha^{-1}(V_\alpha) \cap e(X)$

is open in $e(X)$

$$\begin{tikzcd} X \arrow[r, "f_\alpha"] \arrow[d, "e"'] & X_\alpha \\ \prod X_\alpha \arrow[ur, "\pi_\alpha"'] & \end{tikzcd}$$

Q.E.D

$f_\alpha^{-1} = (\pi_\alpha \circ e)^{-1}$

-8

... be a family of cts fn on a top space X then ... the collection ... separates point from closed set in X $\Longleftrightarrow$ the collection $\{f_\alpha^{-1}(V_\alpha) : V_\alpha \underset{open}{\subseteq} X_\alpha : \alpha \in A\}$ is base for the Top on X

... separates point from closed sets in X ... by assumption

... $\underset{open}{\subseteq} X$ then $x \notin X/U$ (closed) $f_\alpha(x) \notin \overline{f_\alpha(X-U)}$ (i.e) $f_\alpha(x) \in X_\alpha - \overline{f_\alpha(X-U)} = V_\alpha$ open in X_α

... $\ni f_\alpha(x) \in V_\alpha$ i.e. $x \in f_\alpha^{-1}(V_\alpha)$

... $f_\alpha^{-1}(V_\alpha) \subseteq U$

... show that $y \notin f_\alpha^{-1}(V_\alpha)$, when ever $y \notin U$))

... will show that $y \notin f_\alpha^{-1}(V_\alpha)$ ... $f_\alpha(y) \in f_\alpha(X-U) \subseteq \overline{f_\alpha(X-U)}$

... $\Rightarrow y \in X-U \Rightarrow$ ... $\Rightarrow y \notin f_\alpha^{-1}(V_\alpha)$ ... $x \in f_\alpha^{-1}(V_\alpha) \subseteq U$

... $\notin X_\alpha - \overline{f_\alpha(X-U)} = V_\alpha$

... $\beta = \{f_\alpha^{-1}(U_\alpha) : U_\alpha \underset{open}{\subseteq} X_\alpha, \alpha \in A\}$ is base of X

... $C \underset{open}{\subseteq} X$ & $x \in X-C$ $[x \notin C]$

... $\exists \alpha : f_\alpha(x) \notin \overline{f_\alpha(C)}$

... show $\exists \alpha$ ... $X-C$ is open contains x. Since B is a base

... closed then $X-C$ is open ... $\exists \alpha \in A \ni U_{\alpha_1} \underset{open}{\subseteq} X_{\alpha_1} \ni x \in f_{\alpha_1}^{-1}(U_{\alpha_1}) \subseteq X-C$

we will show that $f_{\alpha_1}(x) \notin \overline{f_\alpha(C)}$ in fact we show $f_\alpha(x) \in U_{\alpha_1}$, $U_{\alpha_1} \cap f_{\alpha_1}(C) = \phi$

to show $U_{\alpha_1} \cap f_{\alpha_1}(C) = \phi$

... $t \in U_{\alpha_1} \cap f_{\alpha_1}(C) \Rightarrow t \in U_{\alpha_1}$ & $t \in f_{\alpha_1}(C)$
then $f_\alpha^{-1}(t) \in f_{\alpha_1}^{-1}(U_{\alpha_1})$ & $\exists s \in C \ni f_\alpha(s) = t$

$s \in f_{\alpha_1}^{-1}(t) \subseteq f_{\alpha_1}^{-1}(U_{\alpha_1})$

then $s \in f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap C$ #

then $U_{\alpha_1} \cap f_{\alpha_1}(C) = \phi$

Q. E. D

---

(6) IF $\{f_\alpha : \alpha \in \Delta\}$ is family of cts fn from sp... $\{f_\alpha : \alpha \in \Delta\}$ separated point from closed set in $X \Rightarrow X$ has weak Top induced by Φ

Proof by (5) has base $\Rightarrow$ has $w(\Phi)$

(7) IF $\{f_\alpha : \alpha \in A\} = \Phi$ is a family of cts fn from T_1 space into sp X_α that separates points from closed sets in X

then The evelation map is an embedding

Proof

by (6) X has weak Top induced by Φ

Since X is T_1-space $\Rightarrow \Phi$ separate point then by ...

The evelation map is embedding.

(8) IF X is T_1-space then when every a family $\{f_\alpha : \alpha \in \Delta\}$ sep point from closed sets $\boxed{\text{then}}$ it is also separates point

Proof

$\forall X \neq y$ & $x, y \in X$ Since $\{y\}$ closed in T_1 then $\exists \alpha \in A \ni$ then $f_\alpha(x) \notin \overline{f_\alpha(\{y\})}$

$x \notin \{y\}$ closed

but $f_\alpha(y) \in \overline{f_\alpha(\{y\})}$ for some $\alpha \in A$

then $f_\alpha(x) \neq f_\alpha(y)$

(Notes) separates points from closed set + T_1 then separated p...

5 Quotient space

Let (X,T) be a top-space, ρ is an equivalence relation on X. Define $f: X \xrightarrow{\text{onto}} X/\rho$ by $f(x)=[x]$ (equivalence class for x); X/ρ is called quotient set $=\{[x] : x \in X\}$.

The quotient topology on X/ρ is the identification topology induced by f & (X,T). The set X/ρ together with the quotient topology is called The quotient space.

Ex: $X=\{1,2,3,4\}$, $T_X=\{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2,3\}, \{1,2\}, \{1,2\}, \{2,3\}\}$

Define $a \rho b \Leftrightarrow a+b$ is even, then ρ is an equivalent relation in X.

$[1]=[3]$, $[2]=[2,4]=[4]$

$X/\rho=\{[1]=\{1,3\}, [2]=\{2,4\}\}$, $f: X \to X/\rho$, $f(x)=[x]$

f is cts $\Rightarrow \{\phi, \{1,3\}, \{2,4\}, X/\rho\} \neq T_\rho$

quotient topology: $f^{-1}(\{1\})=\{1,3\}$ open in X & $f^{-1}(\{2\})=\{2,4\}$ not open in X.

$(X/\rho, T_{q\rho})$ is called the quotient space.

ملاحظة حالتنا هذه حالة معاكسة لـ product top
حالة مسماة strong topology

If $f_\alpha: X_\alpha \to Y$, $\alpha \in \Delta$, The strong Topology on Y is $\tau=\{U \subseteq Y \ni f^{-1}(U) \underset{opn}{\subseteq} X_\alpha, \forall \alpha \in \Delta\}$

Quotient Topology on Y

$f: X \to Y$, take $T_f=\{V \subseteq Y : f^{-1}(V) \underset{opn}{\subseteq} X\}$

then T_f is a Topology called Quotient topology on Y induced by f.

Theorems + Ex

1) T_f is the largest Top on Y that makes f continuous.

Proof

$f: X \xrightarrow{cts} Y$, let $V \underset{opn}{\subseteq} Y$ then $f^{-1}(V)$ open in $X \Rightarrow f$ is cts

to show largest Top: Let τ' be another Top on $Y \ni f: (X) \to (Y,\tau')$

Now if $W \in \tau'$ then $W \subseteq Y$ then $f^{-1}(W) \underset{opn}{\subseteq} X \Rightarrow W \in T_f$ i.e. $\tau' \subseteq T_f$

2) Let $f: X \to Y$ be a Quotient map then $C \underset{closed}{\subseteq} Y \Leftrightarrow f^{-1}(C)$ is closed

Proof: let $C \underset{closed}{\subseteq} Y$

$\Rightarrow$ since f is cts $\Rightarrow f^{-1}(C)$ is closed in X

$\Leftarrow$ suppose $f^{-1}(C)$ is closed in X, take $(Y-C)$

Since $f^{-1}(Y-C)=X-f^{-1}(C) \Rightarrow f^{-1}(Y-C)$ open, so $Y-C$ is open $\Rightarrow C$ closed

3) Let $f: (X,\tau) \xrightarrow{\text{onto}} (Y,\tau')$ be cts fn & f is open (closed) fn then the top τ' on Y is the quotient top induced by f (τ)

i.e. $\tau'=T_f$

Proof

Case 1: If f is open, to show $\tau'=T_f$

($\subseteq$) $\tau' \subseteq T_f$ (always because T_f is largest top.)

($\supseteq$) if $V \in T_f \Rightarrow f^{-1}(V)$ open in $X \Rightarrow f(f^{-1}(V)) \overset{\text{onto}}{=} V$ is τ'-open

i.e. $V \in \tau' \Rightarrow T_f \subseteq \tau'$ $\therefore \tau'=T_f$

Case 2: if f is closed, to show $\tau'=T_f$

($\subseteq$) $\tau' \subseteq T_f$ (because f is cts + T_f is largest topology)

($\supseteq$) let $V \in T_f \Rightarrow f^{-1}(V)$ is open in $X \Rightarrow X-f^{-1}(V)$ is closed in X

then $f(X-f^{-1}(V))=Y-V$ is τ-closed $\Rightarrow V \in \tau$ (open)

i.e. $V \in \tau \Rightarrow T_f \subseteq \tau'$ then $\tau'=T_f$

$X \to Y$ be Quotient map then f_n $g: Y \to Z$ is cts

$\boxed{}$ $g \circ f$ is cts

$\Rightarrow$ since f is cts & g is cts $\Rightarrow g \circ f$ is cts

$\Leftarrow$ $\underbrace{X \xrightarrow{f} Y \xrightarrow{g} Z}_{g \circ f}$

$g \circ f$ is cts to show g is cts

$U \underset{open}{\subseteq} Z$ Try $\boxed{g^{-1}(U) \underset{open}{\subseteq} Y}$ (It is enough to show that $f^{-1}(g^{-1}(U))$ is open in X)

$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ & (since f is cts) $V \subseteq Z$ open

$\Rightarrow f^{-1}(g^{-1}(U))$ is open thus $g^{-1}(U)$ is open in Y & this ... es that g is cts

---

$(R, \mathfrak{I})$, $A = \{x : x \le 0\} \sqcup \{x : x \ge 1\}$, $A \subseteq R$. Describe the ... space topology on R/A obtained by collapsing A to a point

ينا / نبيت بيت

$\leftarrow \overset{0}{]} \ominus \overset{1}{[} \rightarrow$

$U \in \mathfrak{I}_p \Leftrightarrow p^{-1}(U) = U \in \mathfrak{I}_S$

V open $\Leftrightarrow p^{-1}(a) = (-\infty, a) \cup (b, \infty)$ where $0 < a$ & $b < 1$

... contain the element prescribed

above

... an equivalent relation R on R^1 as follows, For each $x, y \in R^1$ ... $\Leftrightarrow x - y$ is an integer. What is the quotient space Top on R^1/R

... equivalence classes are of the form $(n + r)$ where $0 \le r < 1$, $r \in R$ & $n \in N$ $\forall \epsilon(q)$

... $\in R_1/R$

... R^1/R open if $p^{-1}(U) \underset{open}{\subseteq} R$ then $V = \cup \{E : F(p) \cup (n + a, n + b)\}$ where $(a, b) \subseteq R$.

---

(7) Prove that if X/R_1 is a quotient space of X & $(X/R_1)/R_2$ a quotient space of X/R_1 then $(X/R_1)/R_2$ is homeomorphic to a quotient space of X.

Proof suppose X/R_1 is a quotient of X &

$X \xrightarrow{P_1} X/R_1 \xrightarrow{P_2} (X/R_1)/R_2$

$P = P_2 \circ P_1$

claim $X/R_P \cong (X/R_1)/R_2$

$h(R_P(x)) = P(x)$

$R_{P(x)} \in V$

open $V \underset{open}{\subseteq} X/R_P \Rightarrow h(V)$ open

however $h(U)$ is open $\Leftrightarrow p^{-1}(h(U))$ is open. But $p^{-1}(h(U)) = \cup \{R_{P(x)}\} = P_0^{-1}(U)$ open in X

where $X \xrightarrow{P_0} X/R_P$

400

~~(8)~~ Let $f: X \to A$ be a retraction of the space X onto the subset $A \subseteq X$

Prove $A \cong X/R(f)$

Proof: $f: X \to A$ $\therefore f/A = i_A$ f is identification f_n

$X \xrightarrow{f} A$, $P \searrow X/R(f)$, $\nearrow h$ $= \{f^{-1}(y) : y \in ...\}$

A has the quotient top induced by the identification function $X/R(f)$ has the Decomposition $\{f^{-1}(y) : y \in ...\}$

So h is homeomorphism & $A \cong X/R(f)$

---

(9) Define an equivalent relation R on R^2 as follows, $\forall (x, y)$ & $(z, w) \in R^2$

$(x, y) R (z, w) \Leftrightarrow x = z$. Describe the quotient space Top on R^2/R & Prove that the quotient space R^2/R is homeomorphic ...

Proof $(x, y) \sim (x', y') \Leftrightarrow x = x'$ the equ-class straight lines parallel to the Y ... passing through X. Let $R^2 \xrightarrow{\pi_1} R$, $P \searrow R^2/R \nearrow h$ ... π_1 is identif f_n

$\pi_1(x, y) = x \Rightarrow h(x) = \pi_1^{-1}(x)$ of ... (Line parallel to Y-axis passing through ...)

π_1 is onto, cts, open so π_1 induced the quotient topology R

& $R^2/\sim$ is a decomposition space of R^2 with $h(x) = \pi^{-1}(x)$

So $R^2/\sim \cong R$

ملاحظة : quotient topology لا نقول علاقة

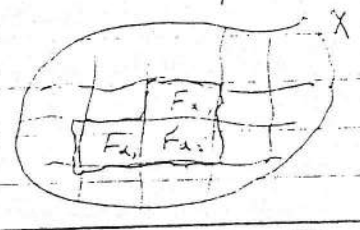
Def: Partition (Decomposition)

Let X be Top-space. A decomposition $\mathcal{D}$ of X is a non-empty collection of disjoint subsets of X whose union is X

Let X be a Top-space & $\mathcal{D}$ a decomposition of X. A subset of $\mathcal{F} \subseteq \mathcal{D}$ is called open $\Leftrightarrow$ $\bigcup\{F : F \in \mathcal{F}\}$ is open in X

$\mathcal{D} = \{F_1, F_2, \dots, F_n\}$

$\mathcal{F} = \{F_{\alpha_1}, F_{\alpha_2}, \dots\}$



Show that this does give a topology on $\mathcal{D}$

φ is open because $\varphi \subseteq \mathcal{D}$ & $\varphi \underset{open}{\subseteq} X$

$\mathcal{D} = \{$is open union for all $F\}$ is X & $X \in T_\alpha$ $\therefore \mathcal{D} = \bigcup_{F \in \mathcal{D}} F$

F_1, F_2 open. $\mathcal{F} = F_1 \cap F_2$ is open

$\mathcal{F} = \bigcup_{\alpha \in \Delta} F_\alpha$

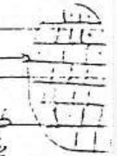
$\bigcup\{F : F \in \mathcal{F}\} = \bigcup_\alpha (\bigcup F_i : F_i \in T_\alpha\})$ We call this Top Decompotion topology & $(\mathcal{D}, T_\mathcal{D})$ is the decompotion space.

Notes Let X be a space & let $\mathcal{D}$ be composition of X

Let $\exists\ P : X \to \mathcal{D}$ is natural map

$\ni p(x) \to \mathcal{D}_x$

$T_p = T_\mathcal{D}$, T_p is the quotient Top induced by natural map حل تمرين في الامتحان



then every decomposition space $\subseteq$ quotient space

(10) The toplogy on a decomposition space $\mathcal{D}$ of X is the quotient topology induced by the natural map $P : X \to \mathcal{D}$

مهم

Proof T_p: Quotient topology induced on $\mathcal{D}$ by p

$T_\mathcal{D} \subseteq T_p$ (quotient topology strongest) من التعريف

Let $V \subseteq \mathcal{D}$ & let $V \in T_p$ then $p^{-1}(V) \underset{open}{\subseteq} X$

but $p^{-1}(V) = \bigcup\{F : F \in V\}$ then $V \in T_\mathcal{D}$ & $T_p \subseteq T_\mathcal{D}$

then $T_p = T_\mathcal{D}$

To show that the Decomposition space $\mathcal{D}$ of X is the quotient top induced by the natural map p

$X \xrightarrow{p} \mathcal{D}$

We have to show that

① p is onto. Let $F \in \mathcal{D}$, $F \neq \varphi$. $\exists x \in F \ni p(x) = F$

② p is cts. $F \underset{open}{\subseteq} \mathcal{D}$ $p^{-1}(F) = \{\bigcup\{F : F \in \mathcal{F}\}$ open in X

$F \underset{open}{\subseteq} \mathcal{D} \Leftrightarrow \mathcal{F} = \bigcup\{F : F \in \mathcal{F}\} \underset{open}{\subseteq} X$

$X \xrightarrow{p} \mathcal{D}$ $p(x) = F$, $F \in \mathcal{D}$, $x \in F$

of $f: X \to Y$ if Y has the quotent top induced by f then $Y \cong \mathcal{D}_f$ where the elements of $\mathcal{D}$ $= \{f^{-1}(y) : y \in Y\}$

under a homeomorphism $h: Y \to \mathcal{D} \Rightarrow f \circ h$ is the natural map p of X onto $\mathcal{D}$

خنا دالة كيف يكمل جرخطوة

$h(y) = f^{-1}(y)$

$$\begin{array}{ccc} X & \xrightarrow{f \text{ quotient}} & Y \\ & \searrow^{p} & \downarrow h? \\ & & \mathcal{D}_f \end{array}$$

$\Rightarrow h$ to hom ?

Let us define a function $h: Y \to \mathcal{D}_f$ by $h(y) = f^{-1}(y)$ then $h \circ f = p$

1-1: if $y_1 \neq y_2 \Rightarrow f^{-1}(y_1) \cap f^{-1}(y_2) = \phi$ by def of a fn

$h(y_1) \cap h(y_2) = \phi$

$h(y_1) \neq h(y_2)$ — is 1-1

onto: if $D \in \mathcal{D}_f$ then $D = f^{-1}(y)$ for some $y \in Y$ Thus $h(y) = f^{-1}(y) = D$ then h is onto

cts: h is cts $\Leftrightarrow h \circ f$ is cts + f cts but $h \circ f = p$ which is cts $\therefore h$ is cts

is open: Let $V \subseteq Y$ be open want to show $h(V) \subseteq \mathcal{D}_f$ be open

to show $p^{-1}(h(V))$ open in X

$p^{-1}(h(V)) = (h \circ f)^{-1} h(V) = f^{-1}(h^{-1} h(V)) = f^{-1}(V)$ open because f is cts

Then $h(V)$ is open in $\mathcal{D}_f$.

بسم الله الرحمن الرحيم
بكر احمد بكر العدلي
نينوى - تلعفر
مرتم ....
" ....

Dfn: $P: X \to \mathcal{D}$, let $\mathcal{D}$ be a decomposition of X a subset A op in X is called $\boxed{\text{Saturated}}$ with respect to $\mathcal{D}$ iff whenever $D \in \mathcal{D}$, $D \cap A \neq \phi$ then $D \subseteq A$ or $A = p^{-1}(G)$ for some $G \underset{open}{\subseteq} \mathcal{D}$

المنزلق

Dfn 9.8: Let $\mathcal{D}$ be a decomposition of a space X, $\mathcal{D}$ is called upper semi-cts if $\forall D \in \mathcal{D}$ $\forall$ open set U in X with $D \subseteq U$ $\exists$ an open saturated subset $V \ni D \subseteq V \subseteq U$

(12) $\mathcal{D}$ is upper semi cts $\Leftrightarrow$ $\forall U \underset{open}{\subseteq} X$ the set $U^* = \cup\{D \in \mathcal{D} : D \subseteq U\}$ is open in X

Proof $\Rightarrow$ let $\mathcal{D}$ be upper semi-cts, let $U \underset{open}{\subseteq} X$ & $U^* = \cup\{D \in \mathcal{D} : D \subseteq U\}$ if $x \in U^*$ then $x \in D \subseteq U$ for some $D \in \mathcal{D}$ by the def, $\exists$ an open saturated set $V \ni x \in D \subseteq V \subseteq U$ Thus $x \in V \subseteq U^*$

$\Leftarrow$

Let $D \in \mathcal{D}$ & $D \subseteq U$ where U is open in X, Put $V = U^*$ then $D \subseteq V \subseteq U$ where V is open & saturated then $\mathcal{D}$ is upper semi cts.

(13) The natural map $P: X \to \mathcal{D}$ is closed $\Leftrightarrow$ $\mathcal{D}$ is upper semi-

Proof $\Rightarrow$ let $D \in \mathcal{D}$, $U \underset{open}{\subseteq} X$ with $D \subseteq U$ then $X \setminus U$ is closed in X & since p is cls take $V = p^{-1}(\underbrace{\mathcal{D} - p(X-U)}_{open \quad closed}) = X - p^{-1}(p(X-U))$ then V is open & saturated also $D \subseteq V \subseteq$

$D \cap (X - U) = \phi$ then $D \notin p(X-U)$ then $D \in \mathcal{D} - p(X-U)$

P takes each point to the set that contains this point $\Rightarrow D \subseteq p^{-1}[\mathcal{D} - p(X-U)$

& the same way we can show that $V \subseteq U$

Let $D_1 \subseteq V = X - p^{-1}p(X-U)$ then $A \cap p^{-1}p(X-U) = \phi$ but $X-U \subseteq p^{-1}p(X-U)$ $\Rightarrow D_1 \cap X - U =$

$\Leftarrow$ $P: X \to \mathcal{D}$ let $C \underset{closed}{\subseteq} X$ & Try $p(C)$ closed or $\mathcal{D} - p(C)$ open

$p^{-1}[\mathcal{D} - p(C)]$ is saturated so it is contained & complete

Let $D \in \mathcal{D}$, $\ni D \subseteq p^{-1}(\mathcal{D} - p(C)) \subseteq X - C$ & $X - C$ is open $\exists$ an open saturated set $\Rightarrow D \subseteq V \subseteq X - C$ then $D \subseteq V \subseteq p^{-1}(\mathcal{D} - p(C)]$ then $D \in p(V) \in \mathcal{D} - p$ where $p(V)$ open

$\mathcal{D} - p(C)$ open

$p(C)$ closed $\square$

"ا لتمارين"

Let $\{f_\alpha : \alpha \in A\}$ be a family of fns $X \to X_\alpha$, $\alpha \in A$

(ⅱ) assume X_α is T_2, $\forall \alpha \in A$. Show $(X, W(\Phi))$ is $T_2 \iff \Phi$ separates point in X

assume S_α is a subbase of X_α, $\alpha \in A$. Show

$$S = \{f_\alpha^{-1}(S_\alpha),\ S_\alpha \in S_\alpha,\ \alpha \in A\} \text{ is subbase for } W(\Phi).$$

by contraduction suppose Φ is not separte point in X

$\exists x, y \in X \ni x \neq y$ & $f_\alpha(x) = f_\alpha(y)$ $\forall \alpha \in A\}$

X is T_2-space $\exists$ two open $C, D \ni x \in C$ & $y \in D$ & $C \cap D = \phi$

X has weak Top $\exists$ two a subbasic openset

$f_{\alpha_2}^{-1}(U_{\alpha_2}) \ni x \in f_{\alpha_1}^{-1}(U_{\alpha_1}) \subseteq C$

& $y \in f_{\alpha_2}^{-1}(U_{\alpha_2}) \subseteq D$

$f_{\alpha_1}(x) \in U_{\alpha_1}$ & $f_{\alpha_2}(y) \in U_{\alpha_2}$ Now

$f_{\alpha_1}(x) = f_{\alpha_2}(y) \longrightarrow C \cap D \neq \phi$ #

$\therefore \Phi$ seperats the point of X

Φ separats point of X, let $x, y \in X \ni x \neq y \Rightarrow \exists \alpha \in A \ni f_{\alpha}(x) \neq f_{\alpha}(y)$

T_2-space $\forall \alpha \in A$ $\exists V, U$ two disjoint openset $\ni f_\alpha(x) \in U$ & $f_\alpha(y) \in V$ & $U \cap V$

$f_{\alpha}^{-1}(U)$ open in X & $y \in f_{\alpha}^{-1}(V)$ open X, $f_{\alpha}^{-1}(U) \cap f_{\alpha}^{-1}(V) = f_{\alpha}^{-1}(U \cap V) = f_{\alpha}^{-1}(\phi) = \phi$

$\therefore (X, W(\Phi))$ is T_2-space

$\in X$, $x \in U$

$\Rightarrow x \in B \subseteq U$ (since B is basic openset) $\Rightarrow B$ is finite $\cap$ of sub

openset $W(\Phi)$, $B = \{\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$ where $U_{\alpha_i} \underset{\text{open}}{\subseteq} X_{\alpha_i}$ $i = 1, \dots, n\}$

$x \in f_{\alpha_i}^{-1}(U_{\alpha_i})$ then $f_{\alpha_i}(x) \in U_{\alpha_i} \underset{\text{open}}{\subseteq} X_\alpha$ & S_α is subbase of X_α

can be written as finite $\cap$ of elements of S_{α_i}, $U_{\alpha_i} = \bigcap_{j=1}^{m} S_{\alpha_{ij}}$

$f_{\alpha_i}(x) \in S_{\alpha_{ij}}$ $\forall j = 1, 2, \dots, n$ $\longrightarrow x \in f_{\alpha_i}^{-1}(S_{\alpha_{ij}})$ $\forall j = 1, \dots, n$

$f_{\alpha_i}^{-1}(S_{\alpha_{ij}}) = f^{-1}(U_{\alpha_i})$ then $x \in \bigcap_{i=1}^{n}(\bigcap_{j=1}^{m} f_{\alpha_i}^{-1}(S_{\alpha_{ij}})) \subseteq U$

$f_{\alpha_i}^{-1}(S_{\alpha_{ij}}) \in S$ $\forall i = 1, \dots, n$ $\Rightarrow x \in \bigcap_{i=1}^{n}(\bigcap_{j=1}^{m} f_{\alpha_i}^{-1}(S_{\alpha_{ij}})) \subseteq U$

$\therefore S$ is subbase of $W(\Phi)$ Q.E.D

~~Chapter (4)~~

~~Separation & Countability axioms~~

(i) T_1, T_2, T_0 space

Dfn

(1) T_0-space: (X, T) is called T_0-space $\iff$ for any $x, y \in X$, $x \neq y$ $\exists$ open set containing only one of them

(2) T_1-space: a (X, T) is called T_1-space $\iff \forall x, y \in X$ & $x \neq y$ $\exists U_x, U_y$ (opensets) $\ni$ $x \in U_x$, $y \notin U_x$, $y \in U_y$, $x \notin U_y$

(3) T_2-space: a (X, T) is called T_2-space $\iff \forall x, y \in X$, $x \neq y$ $\exists U_x, U_y$ opensets $\ni x \in U_x$ & $y \in U_y$ & $U_x \cap U_y = \phi$

Type	T_0	T_1	T_2
① (X, T_{ind})	⊗ not since $T_{ind} = \{\phi, X\}$ $\forall x \neq y$ then X contains both x & y	⊗	⊗
2 $(R, T_{l.r})$	✓ Let $x, y \in R$ & $x \neq y$ say $x < y$ then $U = L_y = (-\infty, y)$ contains only x not y	⊗ Not Let $x \neq y \in R$ (say $x < y$) if V open & $y \in V$ then $x \in V$	⊗
③ (X, T_{dis})	✓	✓	✓

$(X, T_{cof}) \cong (X, T_{coc})$ ② $(\mathbb{R}, T_{l.r}) \cong (\mathbb{R}, T_{r.r})$

" اسئلة عامة "

$\{f_\alpha, \alpha \in A\}$ be a family of fns $X \to X_\alpha, \alpha \in A$ ⓘ

assume X_α is T_2, $\forall \alpha \in A$ Show $(X, W(\Phi))$ is $T_2 \Leftrightarrow \Phi$ separates point in X

assume S_α is a sub base of X_α, $\alpha \in A$ Show

$S = \{f_\alpha^{-1}(S_\alpha), S_\alpha \in S_\alpha, \alpha \in A\}$ is sub base for $W(\Phi)$

by contradiction suppose Φ is not separate point in X

$\exists x, y \in X \ni x \neq y$ & $f_\alpha(x) = f_\alpha(y)$ $\forall \alpha \in A$

X is T_2-space $\exists$ two open $C, D \ni x \in C$ & $y \in D$ & $C \cap D = \phi$

X has weak Top $\exists$ two a sub basic opensets

$f_{\alpha_2}^{-1}(U_{\alpha_2})$ $\ni x \in f_{\alpha_1}^{-1}(U_{\alpha_1}) \subseteq C$

& $y \in f_{\alpha_2}^{-1}(U_{\alpha_2}) \subseteq D$

$f_{\alpha_1}(x) \in U_{\alpha_1}$ & $f_{\alpha_2}(y) \in U_{\alpha_2}$ Now

$f_{\alpha}(x) = f_{\alpha_2}(y) \longrightarrow C \cap D \neq \phi$ #

$\therefore \Phi$ seperats the point of X

Φ separats point of X, let $x, y \in X \ni x \neq y \Rightarrow \exists \alpha \in A \ni f_{\alpha}(x) \neq f_{\alpha}(y)$

T_2-space $\forall \alpha \in A$ $\exists V, U$ two disjoint open set $\ni f_{\alpha}(x) \in U$ & $f_\alpha(y) \in V$ & $U \cap V$

$f_\alpha^{-1}(U)$ open in X & $y \in f_\alpha^{-1}(V)$ open X, $f_{\alpha}^{-1}(U) \cap f_{\alpha}^{-1}(V) = f_\alpha^{-1}(U \cap V) = f_\alpha^{-1}(\phi) = \phi$

$\therefore (X, W(\phi))$ is T_2-space

$U \subseteq X$, $x \in U$

$\ni x \in B \subseteq U$ (since B is basic open set) $\Rightarrow B$ is finite $\cap$ of sub

open set $W(\phi)$, $B = \{\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$ where $U_{\alpha_i} \underset{open}{\subseteq} X_{\alpha_i}$ $\forall i = 1, \ldots, n\}$

$f_{\alpha_i}^{-1}(U_{\alpha_i})$ then $f_{\alpha_i}(x) \in U_{\alpha_i} \underset{open}{\subseteq} X_\alpha$ & S_α is sub base of X_α

can be written as finite $\cap$ of elements of S_{α_i}, $U_{\alpha_i} = \bigcap_{j=1}^{m} S_{\alpha_{ij}}$

$f_{\alpha_i}(x) \in S_{\alpha ij}$ $\forall j = 1, 2, \ldots, n \longrightarrow x \in f_{\alpha_i}^{-1}(S_{\alpha_{ij}})$ $\forall j = 1, \ldots, n$

$f_{\alpha_i}^{-1}(S_{\alpha ij}) = f^{-1}(U_{\alpha_i})$ then $x \in \bigcap_{i=1}^{n}(\bigcap_{j=1}^{m} f_{\alpha_i}^{-1}(S_{\alpha_{ij}})) \subseteq U$

$f_{\alpha_i}^{-1}(S_{\alpha_{ij}}) \in S$ $\forall i = 1, \ldots, n$ $\Rightarrow x \in \bigcap_{i=1}^{n}(\bigcap_{j=1}^{m} f_{\alpha_i}^{-1}(S_{\alpha_{ij}})) \subseteq U$

$\therefore S$ is subbase of $W(\Phi)$ Q.E.D

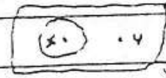
~~Chapter (6)~~

Separation & countability axioms

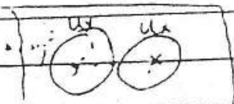
~~Separation axioms~~ ⓘ: T_1 T_2 T_0 space

Dfn

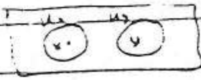
① T_0-space (X, T) is called T_0-space $\Leftrightarrow$ for any $x, y \in X$, $x \neq y$ $\exists$ open set containing only one of them



② T_1 space : a (X, T) is called T_1-space $\Leftrightarrow$ $\forall x, y \in X$ & $x \neq y$ $\exists$ V_x, V_y (open sets) $\ni$ $x \in U_x$, $y \notin U_x$, $y \in U_y$, $x \notin U_y$



③ T_2-space a (X, T) is called T_2-space $\Leftrightarrow$ $\forall x, y \in X$, $x \neq y$ $\exists U_x, U_y$ open sets $\ni x \in U_x$ & $y \in U_y$ & $U_x \cap U_y = \phi$



Type	T_0 التوبولوجي	T_1	T_2
① (X, T_{ind})	⊗ not since $T_{ind} = \{\phi, X\}$ $\forall x \neq y$ then X contains both x & y 1988	⊗	⊗
2- $(R, T_{l.r})$	✓ Let $x, y \in R$ & $x \neq y$ say $x < y$ then $L_y = (-\infty, y)$ contains only x not y	⊗ Not Let $x \neq y \in R$ (say $x<y$) if V is open & $y \in V$ then $x \in V$	⊗
③ (X, T_{dis})	✓	✓	✓

$(X, T_{cof}) \cong (X, T_{coc})$ ② $(\mathbb{R}, T_{l.r}) \cong (\mathbb{R}, T_{r.r})$ ①:

54

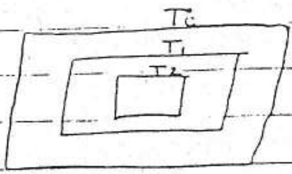
	T_0	T_1	T_2
$(\mathbb{R}, \tau_u)$	✓ Let $x, y \in \mathbb{R}, x \neq y$ assume $x < y$ $(-\infty, \frac{x+y}{2})$ is open containg x but not y $(\frac{x+y}{2}, y+1)$ is open containg y not x	✓ similary	✓
$(\mathbb{R}, \tau_{cof})$		✓ Let $x, y \in X, x \neq y$ $x \in \{y\}^c$, $y \notin \{y\}^c$, $\{y\}^c$ open $y \in \{x\}^c$, $x \notin \{x\}^c$ + $\{x\}^c$ open	⊗ Let $U, V \subseteq_{open} \mathbb{R}$ + $U \neq \phi \neq V$ Try $\boxed{U \cap V \neq \phi}$ supose $U \cap V = \phi$ $R - U\cap V = R - \phi = R$ $\underbrace{R-U \cup R-V}_{finite} = R$ (finite, finite, infinite) #

Sorgenfrey Line $= E$ is T_2-space E

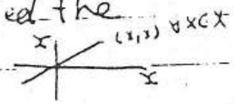
Let $x, y \in E$ + $x \neq y$ (say $x<y$) $\exists\ a, b \ni$

$x < a \leq y < b$ Then $U = [x, a)$ + $V = [y, b)$

$U \cap V = \phi$ So E is T_2-space



$X \times X$ then $\Delta = \{(x,x) : x \in X\}$ is called the diagonal



(Theorems + ~~Ex~~) ~~T_1, T_2~~ T_0

① Every $\boxed{T_2 \text{ is } T_1}$ and $\boxed{T_1 \text{ is } T_0}$ but converse is not tr

(i.e.) $T_2 \longrightarrow T_1 \longrightarrow T_0$

$\not\longleftarrow$ ② cof $\not\longleftarrow$ ① $(\mathbb{R}, \tau_{lr})$

② Let τ_1 + τ_2 be two Topologies on $X \ni \tau_1 \subseteq \tau_2$

if (X, τ_1) is T_i-space then (X, τ_2) is T_i-space

$i = 0, 1, 2$

Proof

$i = 0$

Let $x, y \in X$ + $x \neq y$ $\exists\ U \in \tau_1 \ni U$ contains only one of them.

But $U \in \tau_2$ So (X, τ_2) is T_0-space

For $i=2$

Let $x, y \in X$ + $x \neq y$ $\exists\ \tau_1$ opensets $U, V \ni x \in U$ + $y \in V$ + $U \cap V$

But $\tau_1 \subseteq \tau_2$ so U + V are τ_2-openset Hence (X, τ_2) is T_2-space

③ Being T_i-space is a Topological property:

Proof

$i=0$ Let $X \xrightarrow{h} Y$ (X is T_0-space)

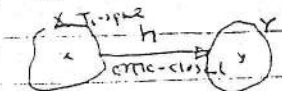
to show h is 1-1 Let $y_1, y_2 \in Y$ + $y_1 \neq y_2$ then since h is onto

$\exists\ x_1, x_2 \in X \ni h(x_1) = y_1$ + $h(x_2) = y_2$ + $x_1 \neq x_2$ h is 1-1

But X is T_0-space So $\exists\ U \subseteq_{open} X \ni U$ containe only one of x_1 + x_2

Say x_1 So $h(U)$ is open in Y + containe only $h(x_1) = y_1$

$i=1$

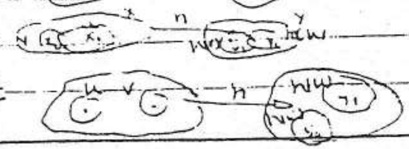


$\forall\ \in Y \Rightarrow \exists x \in X \ni h(x) = y$

$h(\{x\}) = \{y\}$

or

$i=2$



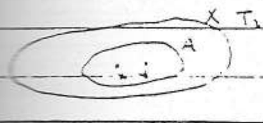
$h(U) \cap h(V) = h(U \cap V) = h(\phi) = \phi$

i.e. every subspace of T_i-space is T_i-space

T_i-space is a hereditary property for $i=0,1,2$

Let $(X,\mathcal{T})$ is T_0-space & Let $(A,\mathcal{T}_A)$ be subspace of $(X,\mathcal{T})$
Let $x,y \in A$ & $x \neq y$ But $A \subseteq X$ & T_0-space (X)
$\exists$ U open in X which contains only one of x,y say y
Let $V = U \cap A \Rightarrow V$ is open in A & contains only y $\square$

Let $(X,\mathcal{T})$ is T_1-space & $A \subseteq X$ & Try $(A,\mathcal{T}_A)$ is T_1-space
Let $x,y \in A$ & $x \neq y$ But $A \subseteq X$ so $x,y \in X$ But X is T_1-space
$\exists$ U_x, V_y open in X $\ni x \in U_x$ & $y \in U_y$ & $x \notin U_y$ & $y \notin U_x$
$\Rightarrow x \in U_x \cap A$ & $y \notin U_x \cap A$
& $y \in U_y \cap A$ & $x \notin U_y \cap A$
But $U_x \cap A$ & $U_y \cap A$ is open in A since $U_y, U_x \in \mathcal{T}$
then $x \in U_x \cap A \underset{opn}{\subseteq} A$ & $y \in U_y \cap A \underset{opn}{\subseteq} A$ & $x \neq y$
then A is T_1-space.



$U, V \underset{opn}{\subseteq} X$ & $U \cap V = \phi$
$U_1 = U \cap A$ & $V_1 = V \cap A$

The product space $X \times Y$ is T_i-space $\Leftrightarrow$ X & Y are T_i-space where $i=0,1,2$

$\Rightarrow$ let $X \times Y$ be T_i-space $X \cong X' \subseteq X \times Y$ (But X' is T_i-space because T_i-space is hereditary property) so X is T_i-space (because T_i is Top-property) $\therefore X$ is T_i-space & Y is T_i-space

$\Leftarrow$ Let X & Y be T_0-space Let (x_1,y_1) & $(x_2,y_2) \in X \times Y$
$(x_1,y_1) \neq (x_2,y_2)$ (Assume $(x_1 \neq y_1)$)
$x_1, x_2 \in X$ & $x_1 \neq x_2$ since X is T_0-space $\exists$ $U \underset{opn}{\subseteq} X$
& contains only one of them say x_1

Take $V = U \times Y$ then V is open in $X \times Y$ & contains only (x_1,y_1) &

$i=1$
Let X, Y be T_1-space & Let $(x_1,y_1), (x_2,y_2) \in X \times Y$ with $(x_1,y_1) \neq (x_2,y_2)$
say $(x_1 \neq x_2)$ Since X is T_1-space $\exists$ U, V opens in X $\ni x_1 \in U$ & $x_2 \notin U$
& $x_2 \in V$ & $x_1 \notin V$

Take $U \times Y = G$ & $H = V \times Y$ $\Rightarrow$ G, H are open in $X \times Y$ with
$(x_1,y_1) \in G$, $(x_2,y_2) \notin G$ & $(x_2,y_2) \in H$, $(x_1,y_1) \notin H$

$i=2$
similarly but $U \cap V = \phi$ $\Longrightarrow$ $G \cap H = \phi$

⑥ In general $\prod_{i=1} X_k$ is T_i-space $\Leftrightarrow$ X_k is T_i-space $i=0,1,2$

Proof
$\Rightarrow$ similar to ⑤
$\Leftarrow$ by induction ① $n=2$ ② assume is true for $n \geq 2$
③ For the case $n+1$ $\prod_{k=1}^{n+1} X_k \cong \left(\prod_{n=1}^{n} X_k\right) \times X_{n+1}$

⑦ The product space $\prod_{\alpha \in \Lambda} X_\alpha$ is T_i-space $\Leftrightarrow$ X_α is T_i space $i=0,1,2$

Proof $\Rightarrow$ $X \cong Y \subseteq \prod_{\alpha \in \Lambda} X_\alpha$ (clear)
$\Leftarrow$
Let $i=2$ Let $x, y \in \prod X_\alpha$ $x \neq y$ $\Rightarrow \exists \alpha_0 \in \Lambda \ni x_{\alpha_0} \neq y_{\alpha_0}$
where $x_{\alpha_0}, y_{\alpha_0} \in X_{\alpha_0}$ $\exists$ two disjoint open subsets $U_{\alpha_0}, V_{\alpha_0}$
$\ni x_{\alpha_0} \in U_{\alpha_0} \ni y_{\alpha_0} \in V_{\alpha_0}$ Put $U = \pi_{\alpha}^{-1}(U_{\alpha_0})$ & $V = \pi_{\alpha_0}^{-1}(V_{\alpha_0})$
الى دراسة

نظريات رالدسلة مرح

... X we define equ-relation $\sim$ by $x \sim y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}$ ...
... that identification space $X/\sim$ is T_0-space.

... $x, y \in X/\sim$ $\boxed{\text{then}}$ $[x], [y] \in X/\sim$ take $y \in [y]$ & $x \in [x]$
... then $x \neq y \rightarrow \overline{\{x\}} \neq \overline{\{y\}}$
... $t \in X/\sim \ni t \in \overline{\{x\}}$ & $t \notin \overline{\{y\}}$
$\exists\, \mathcal{U} \ni t$ & $\mathcal{U} \cap \{y\} = \phi$, ($y \notin \mathcal{U}$) & $\mathcal{U} \cap \{x\} \neq \phi$ $x \in \mathcal{U}$)
$\therefore X/\sim$ is T_0-space

Q.E.D

... T_0-space $\Leftrightarrow$ given any $x, y \in X$ with $x \neq y$ then $\overline{\{x\}} \neq \overline{\{y\}}$

... $\rightarrow$
... X be T_0-space Let $x, y \in X$ with $x \neq y \Rightarrow \exists\, \mathcal{U}$ open
... exactly one of them say $x \in \mathcal{U}$, $\mathcal{U} \cap \{y\} = \phi$
$\Rightarrow x \notin \overline{\{y\}}$ But $x \in \overline{\{x\}}$ so $\overline{\{x\}} \neq \overline{\{y\}}$

... $\overline{\{x\}} \neq \overline{\{y\}}$ whenever $x, y \in X$ with $x \neq y$
... $\in X \ni t \in \overline{\{x\}}$ but $t \notin \overline{\{y\}}$ then $\exists$ open $\mathcal{U} \ni t \in \mathcal{U}$
... $\mathcal{U} \cap \{y\} = \phi$ (i.e. $y \notin \mathcal{U}$) But $t \in \overline{\{x\}}$ therefor $\mathcal{U} \cap \{x\} \neq \phi$
... $x \in \mathcal{U}$ so X is T_0-space

... $T_1 \longleftrightarrow \forall\, x \neq y$ in X, $\overline{\{x\}} \cap \overline{\{y\}} = \phi$

$\rightarrow$ Let X be T_1-space & Let $x \neq y$ in X by contradiction
... $\overline{\{x\}} \cap \overline{\{y\}} \neq \phi \Rightarrow t \in \overline{\{x\}} \cap \overline{\{y\}}$ then $x \in \overline{\{x\}}$
... (open) & $t \in \mathcal{U} \Rightarrow \mathcal{U} \cap \{x\} \neq \phi \Rightarrow x \in \mathcal{U}$
... $\overline{\{y\}}$ so $\forall\, \mathcal{U}$ (open) & $t \in \mathcal{U}$, $\mathcal{U} \cap \{y\} \neq \phi \Rightarrow y \in \mathcal{U}$
... is T_1-space the $\overline{\{x\}} \cap \overline{\{y\}} = \phi$ #

$\leftarrow$ Let $x \neq y$ & $\overline{\{x\}} \cap \overline{\{y\}} = \phi$ then $x \in \overline{\{x\}}$ & $x \notin \overline{\{y\}}$
So $\exists$ open $\mathcal{U}$ where $x \in \mathcal{U}$ & $\mathcal{U} \cap \{y\} = \phi$ so $y \notin \mathcal{U}$
then X is T_2-space

(11) For any $x, a \in X$, $x \neq a$ $\exists\, \mathcal{U}$ open $a \in \mathcal{U}$ & $x \notin \overline{\mathcal{U}}$ $= T_2$

Then The diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$
(or) X is $T_2 \Leftrightarrow$ the diagonal Δ is closed in $X \times X$

Proof
$\Rightarrow$ Let X be T_2-space to show Δ is closed (Try $X \times X - \Delta$ open)
Let $(a,b) \in \Delta^c$
Now $(a,b) \notin \Delta \rightarrow a \neq b$ using sumption then $\exists\, \mathcal{U}(a)$ open
in $X \ni a \in \mathcal{U}(a)$ & $b \notin \overline{\mathcal{U}_a} \Rightarrow b \in (\overline{\mathcal{U}(a)})^c$ open
$(a,b) \in \mathcal{U}(a) \times (\overline{\mathcal{U}_a})^c$ open in $X \times X$
& $\mathcal{U}_a \times (\overline{\mathcal{U}_a})^c \subseteq \Delta^c$ to show this Let $(x,y) \in \mathcal{U}_a \times (\overline{\mathcal{U}_a})^c$
$\Rightarrow x \in \mathcal{U}_a$ & $y \in (\overline{\mathcal{U}_a})^c$ but $\mathcal{U}_a \subseteq \overline{\mathcal{U}_a}$
$\Rightarrow \mathcal{U}_a \cap (\overline{\mathcal{U}_a})^c = \phi \Rightarrow x \neq y$
$\Rightarrow (x,y) \in \Delta^c$

(12) X is $T_2 \Leftrightarrow$ diagonal Δ is closed in $X \times X$ يكا (cr) سرحال برهان

Proof
$\Rightarrow$ Let $(x,y) \in X \times X - \Delta$ then $x \neq y$ Since X is T_2-space $\exists\, \mathcal{U}, V$
open $\ni x \in \mathcal{U}$ & $y \in V$ & $\mathcal{U} \cap V = \phi$ So $(x,y) \in \mathcal{U} \times V \subseteq X \times X - \Delta$
[$\mathcal{U} \times V \cap \Delta = \phi$ because if not $\exists (t,t) \in \mathcal{U} \times V \Rightarrow t \in \mathcal{U}$ & $t \in V$
then $t \in \mathcal{U} \cap V = \phi$ #) $\therefore X \times X - \Delta$ is open so Δ is closed.
$\Leftarrow$ Let Δ closed in $X \times X$, Let $x, y \in X$ & $x \neq y$ then $(x,y) \notin \Delta$
i.e $(x,y) \in X \times X - \Delta$ & so $\exists$ a basic open set $\mathcal{U} \times V \ni (x,y) \in \mathcal{U} \times V \subseteq X \times X - \Delta$
& $\mathcal{U}, V$ opens $\ni x \in \mathcal{U}$ & $y \in V$ & $\mathcal{U} \cap V = \phi$ (if not $\Rightarrow t \in \mathcal{U} \cap V$
$\Rightarrow (t,t) \in \mathcal{U} \times V \subseteq X \times X - \Delta$ #)
$\therefore X$ is T_2-space

X is T_2-space $\Longleftrightarrow$ Given any $x, y \in X$ with $x \neq y$ $\exists$ open set U $\ni x \in U$ & $y \notin \overline{U}$ $\Rightarrow y \in V$ & $U \cap V = \phi$; $y \notin \overline{U}$

Proof

$\Rightarrow$ Let X is T_2-space if $x, y \in X$ & $x \neq y$ then $\exists$ open U & V $\ni x \in U$, $y \in V$ & $U \cap V = \phi$ thus $y \notin \overline{U}$

$\Leftarrow$ take $x, y \in X$ with $x \neq y$ By assumption $\exists$ open U $\ni x \in U$ & $y \notin \overline{U}$ Put $V = X - \overline{U}$ then U, V opens & $x \in U$, $y \in V$ & $U \cap V = \phi$

Q.E.D

(X,T) is T_1-space $\Longleftrightarrow$ all singeltons are closed set or ($\{x\}$ is closed $\forall x \in X$)

Proof $\Rightarrow$ suppose (X,T) is T_1-space ; let $x \in X$ to show $\{x\}$ is closed

Try $\boxed{X - \{x\}}$ is open , if $y \in X - \{x\}$ then $x \neq y$ so $\exists$ U, V opens $\ni x \in U$ & $y \in V$

$y \notin U$ & $x \notin V$ Thus $\{x\} \cap V = \phi$ hence $y \in V \subseteq X - \{x\}$

$\therefore X - \{x\}$ is open & therefor $\{x\}$ is closed .

$\Leftarrow$

Let $\{x\}$ be closed $\forall x \in X$ if $x \neq y \in X$ Put $U = X - \{y\}$ & $V = X - \{x\}$

so It is cleare that U, V are open $x \in U$ & $y \notin U$ & $y \in V$ & $x \notin V$

Thus X is T_1-space .

Q.E.D

The cofinite topology on a set X is the smallest topology that given T_1-space.

Proof

Let (X, T_{cof}) , Let T be a topology on X $\ni (X,T)$ is T_1-space Try $T_{cof} \subseteq T$

Let $U \in T_{cof}$ if $U = \phi, X$ then $U \in T$

other wise $\phi \neq X - U$ is finite $\therefore X - U = \{x_1, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$

But $\{x\}$ is closed in X $\forall x \in X$ So $X - U = U^c = \bigcup_{i=1}^{n} \{x_i\}$ is closed in T $\Longrightarrow U \in T$

(16) X is T_1-space $\Longleftrightarrow$ all finite sets in X are closed

Proof $\Rightarrow$ Let X is T_1-space & Let A is any finite subset of X

Try A is closed set , $A = \{a_1, a_2, \ldots, a_n\}$

Since T_1-space so $\{a_1\}, \ldots, \{a_n\}$ are closed so $\{a_1\} \cup \ldots \cup \{a_n\} = A$ is closed

$\Leftarrow$

Given all finite subset of X is closed to show X is T_1-space

all finite subset of X are closed then all singelton set of X are closed since the X is T_1 by ⊕

(17) Let $f: X \to Y$ & $g: X \to Y$ be continuse functions into a T_2-space then (a) The set $A = \{x \in X : f(x) = g(x)\}$ is closed

(b) if D is dense subset of X & $f|_D = g|_D$ then $f = g$

Proof

(a) Try ($X - A$ is open) Let $t \in X - A$

(1) then $t \notin A$ i.e $f(t) \neq g(t)$ Since Y is T_2-space $\exists$ G_1, G_2 open $\ni f(t) \in G_1$ & $g(t) \in G_2$ & $G_1 \cap G_2 = \phi$

(2) Since f, g are cts at t $\exists$ U, V opens containing t $\ni f(U) \subseteq G_1$ & $g(V) \subseteq G_2$

Take $W = U \cap V \Longrightarrow W$ is open $t \in W$, $W \cap A = \phi$

i.e $W \subseteq X - A$ [if $W \cap A \neq \phi \Rightarrow p \in W \cap A \Rightarrow p \in W$ & $p \in A$ then $p \in V$ & $p \in U$ & $p \in A$ $f(p) \in G_1$ & $g(p) \in G_2$ & $f(p) = g(p)$ then $G_1 \cap G_2 \neq \phi$ ≠

$\therefore X - A$ is open $\Rightarrow A$ is closed

Let $A = \{x \in X : f(x) = g(x)\}$ closed. Note that $D \subseteq A$

so $X = \overline{D} \subseteq \overline{A} = A \Rightarrow A = X$ then $f = g$

example) تمرين

Let $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$ be define $f(t) = t^2$ $\forall t \in \mathbb{Q}$ find $f(\sqrt{2})$

solution

define $g : \mathbb{R} \xrightarrow{cts} \mathbb{R}$, $g(x) = x^2$. Since $\mathbb{Q}$ dence in $\mathbb{R}$

$\mathbb{R}$ is T_2-space $+$ $f(x) = g(x)$ $\forall x \in \mathbb{Q} \Rightarrow f(x) = g(x)$ $\forall x \in \mathbb{R}$

$f(\sqrt{2}) = g(\sqrt{2}) = (\sqrt{2})^2 = 2$

Let X is T_1-space, $A \subseteq X$ then $x \in A' \Leftrightarrow U \cap A$ is infinite $\forall$ open $U \ni x$

Proof $\Rightarrow$ to show $U \cap A$ is infinite $\forall$ open U containing x (suppose $x \in \overline{A}$)

Let $\exists$ open $U \ni x \in U$ $+$ $U \cap A$ is finite بالتناقض

$U \cap A = \{x_1, x_2, \ldots, x_n\} \Rightarrow U \cap A - \{x\}$ is finite

$U \cap A - \{x\} = \{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$ closed $\subseteq U$

Take $V = U - \{x_1, \ldots, x_n\} = U \cap \{X - \{x_1, \ldots, x_n\}\} \Rightarrow$

is open $+$ contains x. More over $V \cap A - \{x\} \subseteq U \cap (A) - \{x\}$ $\{x_1, \ldots, x_n\}$

so $U \cap A - \{x\} = \phi$ #

$\therefore U \cap A$ is infinit $\forall$ open $U \ni x$

$\Leftarrow$ suppose

$U \cap A$ is infinite $\forall$ open contains x then $U \cap A - \{x\}$ is infinite $+$ hence $U \cap A - \{x\} \neq \phi$

$\forall$ open contains $x \Rightarrow x \in A'$

---

(19) $f : X \to Y$ is 1-1, $f \subseteq X \times Y$ (closed) show X is T_1-space

Proof

Let $x, w \in X$ $+$ $x \neq w$ then $\exists! y \in Y$ $\ni f(x) = y$, $(x, y) \in f$ $+ (w, y) \notin f$

$(w, y) \in X \times Y - f$ (open), $\exists$ $U \times V$ open in $X \times Y$ where $V \subseteq_{open} Y$, $U \subseteq_{open} X$

$+ (w, y) \in U \times V \subseteq X \times Y - f \Rightarrow U \times V \cap f = \phi$

then $w \in U$ $+$ $x \notin U$ [otherwise $(x,y) \in U \times V + (x, y) \in f$ #

So X is T_1-space

---

(20) Let $f : X \to Y$ (open $+$ onto), $f \subseteq_{closed} X \times Y$

Show (Y is (T_2) (hausdorff))

Proof

Let $w \neq y \in Y$ $\exists x \in X$ $\ni f(x) = y$, $(x, y) \in f$ but $(x, w) \notin f$

we have $X \times Y - f$ (open) since f is closed $\exists$ $U \times V$ open in $X \times Y - f$

$(x, w) \in U \times V \subseteq X \times Y - f \Rightarrow x \in U$ $+$ $w \in V$ $+$ $y \notin V$ $+$ since f ...

otherwise $(x, y) \in U \times V$ but $U \times V \cap A = \phi$ #

$+ x \in U$ (open) $\xrightarrow{f\ open} f(U) \subseteq_{open} Y$ $+$ $f(x) = y \in f(U)$

So $V \cap f(U) = \phi \Rightarrow Y$ is T_2

---

(21) Let $f : X \to Y$ be cts where Y is T_1-space $+$ X be arbitrary Top-space for any $y \in Y$

Prove $f^{-1}(y) \subseteq_{closed} X$

Proof

Let $y \in Y$ (T_1) then $\{y\}$ is closed (because T_1)

So $Y - \{y\}$ is open $+$ $f^{-1}(Y - \{y\}) = f^{-1}(Y) - f^{-1}(y) \overset{onto}{=} X - f^{-1}(y)$

is open in X so $f^{-1}(y)$ is closed in X

Q.E.D

Regular

(X, T) be a T_1-space & $A \subset X$. Prove that the derived set A' [is a] closed subset of X.

We want to show ($\overline{A'} = A'$). But $\overline{A'} = A' \cup A''$

So Try $A'' \subset A'$

[Let] $p \in A''$, let $U_{(p)} \in X$ (open) + try $U_{(p)} \cap A - \{p\} \neq \phi$

since $p \in A''$ then $U_p \cap A' - \{p\} \neq \phi$, $\exists x \in U_p$, $x \in A'$

[$x \neq$] p then $x \in U_p - \{p\} \subseteq X$ (open) (X is T_1)

$U_p - \{p\} \cap A - \{x\} \neq \phi$ then $U_p \cap A - \{p\} \neq \phi$

then $p \in A'$

$\therefore$ A' is closed

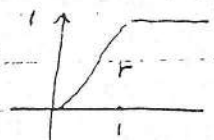
$\boxtimes$

[Def] (a retract of X): (X, T) be Top-space, $A \subseteq X$, A is retract of X iff $\exists r: X \to A$ cts $\ni r/_A = 1_A$ ($r(x) = x$ $\forall x \in A$)

[Example] ① Is $[0,1]$ a retract of R? Yes

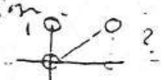
because $\exists r: R \xrightarrow{cts} [0,1]$, $r(x) = x$ $\forall x \in [0,1]$

$$r(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \\ 0 & \text{if } x < 0 \end{cases}$$



cts & it is called retraction function

[Is] $(0,1)$ retraction of R? NO



[Th] (X is T_2 space), A is a retract of X $\Rightarrow$ A must be closed set

[Proof] (Try A^c (open)), Let $t \in A^c$, since A is a retract so $\exists f: X \xrightarrow{cts} A$

$\ni f/_A = 1_A$, since $f(t) \in A$, $t \notin A$, $f(t) \neq t$ but X is T_2

$\exists U, V$ open $\ni f(t) \in U$ & $t \in V$ & $U \cap V = \phi$, since f cts at t

[$\exists H$] (open), $H \ni t$ & $f(H) \subseteq U$ claim $t \in \underline{H \cap V} \subseteq A^c$

[Let x] $\in H \cap V \Rightarrow x \in H$ & $x \in V \Rightarrow f(x) \in f(H) \subseteq U$, $f(x) \in U$

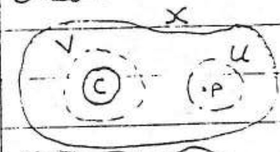
[but] $U \cap V = \phi$ so $x \neq f(x) \Rightarrow x \notin A \Rightarrow x \in A^c$

$\therefore$ A^c is open $\Rightarrow$ A is closed.

(iii) Regular, T_3-space & Complete Regular

Dfns:

① regular space: A Top-space X is Called regular space if Given any closed subset C of X & any $p \in X - C$ $\exists$ U, V opens $\ni p \in U$, $C \subseteq V$ & $U \cap V = \phi$



-2- T_3-space: (X, T) is Called T_3-space $\Leftrightarrow$ ① T_1-space ② Regular

-3- Completely Hausdorff ($T_{2\frac{1}{2}}$): X is called C.H ($T_{2\frac{1}{2}}$) if $\forall x, y \in X$, $x \neq y$ $\exists U, U$ opens $\ni x \in U_x$, $y \in U_y$ & $\overline{U_x} \cap \overline{U_y} = \phi$

④ Completely regular: a topological space is call Completely regular iff for any F closed set & $p \in X - F$ $\exists g: X \xrightarrow{cts} R[0,1]$ $\ni g(F) = 1$ & $g(p) = 0$

⑤ Tychonoff space ($T_{3\frac{1}{2}}$) = Completely regular + T_1

Notes ① The regular $\not\Rightarrow$ T_0

because (R, T_{ind}) is regular since $X \in T$ (clopen)

$\forall x \in X$ $X \subseteq X$, $\phi \in \Phi$ $\phi \subseteq \phi$ $\phi \cap X = \phi$

but not T_0

60

Theorms + Ex

The $T_3 \Rightarrow T_{2\frac{1}{2}}$

Let $x,y \in X$, $x \neq y$. Since X is T_1-space so $\{x\}$ is closed, $y \notin \{x\}$ by X is Regular $\exists\, U_x, U_y$ open sets $x \in U_x$ & $y \in U_y$ & $U_x \cap U_y = \phi$ by regularty so $\exists\, V_x$ open $x \in V_x \subseteq \overline{V_x} \subseteq U_x$

$\exists\, V_y$ open $y \in V_y \subseteq \overline{V_y} \subseteq U_y$

$\Rightarrow x \in V_x$ (open), $y \in V_y$ & $\overline{V_x} \cap \overline{V_y} = \phi$

$\therefore X$ is $T_{2\frac{1}{2}}$-space.

Characterization of regular space.

A space X is regular $\Leftrightarrow$ Given any $x \in X$ & U open contains x $\exists\, V$ open $\ni x \in V \subseteq \overline{V} \subseteq U$

Proof

$\Rightarrow$ suppose X be regular, Let $x \in X$ & U (open) with $x \in U$ then $X-U$ closed with $x \notin X-U$ by reglarity $\exists\, V, W$ (opens) $\ni x \in V$ & $X-U \subseteq W$ & $V \cap W = \phi$ so $V \subseteq X-W \subseteq U$ & take $\overline{V} \subseteq \overline{X-W} = X-W \subseteq U \Rightarrow x \in V \subseteq \overline{V} \subseteq U$

$\Leftarrow$ $C \subseteq X$ & $x \in X-C$ where $(X-C)$ is open by assumption, $\exists\, V$ (open) $\ni x \in V \subseteq \overline{V} \subseteq (X-C)$ take $W = X-\overline{V}$ then V & W are open $\ni x \in V$, $C \subseteq W$ & $V \cap W = \phi$ $\therefore X$ is regular ☒

(3) The T_3-space is T_2-space but converse is not True

Proof

Let X be a T_3-space & Let $x,y \in X$, $x \neq y$. Since X is T_3 & hence T_1-space then $\{y\}$ is closed & $x \notin \{y\}$

By regularity of X $\exists\, U, V$ (opens) $\ni x \in U$, $\{y\} \subseteq V$ & $U \cap V = \phi$

(ii) To show $T_2 \not\Rightarrow T_3$-space

Counter example: Let R, $S = \{U \subseteq R : U$ (open interval)$\}$ $\cup$ ... $S \subseteq \beta \subseteq \tau$, (R,τ) is T_2-space

because $x,y \in R$ with $x \neq y$ choose $a,b,c,d \ni a<x<b<c<y<d$

Put $U = (a,b)$ & $V = (c,d)$

But not regular:

Let $T = R - Q$ is closed in (R,τ), $1 \notin T$, suppose U,V are open $\ni 1 \in U$ & $T \subseteq V$

Case (1) $U = Q$ whenever $t \in T$ $\exists$ open interval $I \ni t \in I \subseteq V$ & $I \cap Q \neq \phi \Rightarrow U \cap V \neq \phi$

Case (2) $U = Q \cap (a,b)$ take $t \in (a,b)$ where $t \in T$ $\exists\, (c,d) \ni t \in (c,d) \subseteq V \Rightarrow t \in (a,b) \cap (c,d) = (e,f)$

so if $q \in (e,f) \cap Q \Rightarrow q \in U \cap V \neq \phi$

(4) Let X be a regular & T_0-space then X is T_3-space

Proof suppose X is a regular & T_0-space to show X is ...

Let $x,y \in X$ with $x \neq y$

since T_0-space $\Rightarrow \exists\, U$ (open) contains exactly one of x, ...

Say $\{x\}$ ie $x \in U$, $x \notin U^c$, $y \in U^c$ (closed)

$X-U$ is closed $x \notin X-U$ By regularity $\exists\, V, W \ni x \in V$, $X-U \subseteq W$

& $V \cap W = \phi$ $\therefore y \in X-U \subseteq W$ & $x \in V$ & $V \cap W = \phi$ ☒

$[0,\frac{1}{4})$ open because $= (-\frac{1}{4},\frac{1}{4}) \cap [0,1]$ is open in $[0,1]$

Every C.R space is R.

Let (X,T) be C.R space. Let $A \underset{\text{closed}}{\subseteq} X$ & $p \notin A$

to Try $\exists$ U_A, U_p two disjoint opensets $\ni A \subseteq U_A$, $p \in U_p$

since X is C.R so $\exists$ cts $f: X \longrightarrow [0,1] \ni f(A)=\{1\}$, $f(p)=0$

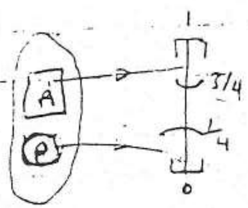
Take $U_p = f^{-1}(\underset{\text{open}}{[0,\frac{1}{4})})$

$U_A = f^{-1}(\underset{\text{open}}{(\frac{3}{4},1]})$

then U_A, U_p are opensets because f is cts

$[0,\frac{1}{4})$, $(\frac{3}{4},1]$ are open sets in $[0,1]$

$p \in U_p$, $A \subseteq U_A$, $U_p \cap U_A = \phi$



Let (X,T) be $\boxed{\text{Zero-dimintion}}$-space $\boxed{\text{show}}$ (X,T) is C.R

def: T has a base consisting of clopen set. Ex: (R, T_{dis}) is zero dim

$A \underset{\text{closed}}{\subseteq} X$ & $p \notin A$ $\Longrightarrow$ $p \in X - A$ (open) $\exists$ Base β consisting clopen sets $\exists$ V a basic open $p \in V \subseteq A^c$ by def of Base

Let $f: X \longrightarrow [0,1]$ as $f(p) = \begin{cases} 1 & \text{if } p \notin V \\ 0 & \text{if } p \in V \end{cases}$

so $f(p)=0$ & $f(A)=1$ $\Rightarrow$ C.R.

Q.E.D

Let (X,T) be a regular space & $x,y \in X$ with $x \neq y$

then either $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}} \cap \overline{\{y\}} = \phi$

suppose $\overline{\{x\}} \neq \overline{\{y\}}$ (say) $\exists t \in X \ni (t \in \overline{\{x\}}$, but $t \notin \overline{\{y\}}$

by regularity $\exists$ U, V (opensets) $\ni t \in U$ & $\overline{\{y\}} \subseteq V$ & $U \cap V = \phi$

so $U \cap \{x\} \neq \phi \Longrightarrow x \in U$

by regularity $\exists$ W (open) $\ni x \in W \subseteq \overline{W} \subseteq U$

$\overline{\{x\}} \subseteq \overline{W} \subseteq U$ while $\overline{\{y\}} \subseteq V$ & $U \cap V = \phi$ $\therefore \overline{\{x\}} \cap \overline{\{y\}} = \phi$

Q.E.D

---

(8) The following conduction are equivalent for a space X

1 – X is Regular

2 – $\forall x \in X$, $\forall$ open U with $x \in U$ $\exists$ V(open) $\ni x \in V \subseteq \overline{V} \subseteq U$

3 – $\forall x \in X$ has a nbd base consisting of closed subset

Proof

$(2 \to 3)$:- For $x \in X$ we take $B(x) = \{\overline{V} : x \in V \underset{\text{open}}{\subseteq} X\}$

Given a nbd N of $x \longrightarrow x \in N^{o}$

So by asumpption $\exists$ V(open) $\ni x \in V \subseteq \overline{V} \subseteq N^{o} \subseteq N$

to prove $(1 \longleftrightarrow 3)$

$\Longrightarrow$ Let X be regular so Let $x \in X$ & $x \in U$(open) then $\exists V \ni x \in V \subseteq \overline{V} \subseteq U$ [the family of all closed nbd of x is a nbd base at x]

$\Longleftarrow$ Let $C \underset{\text{closed}}{\subseteq} X$, $x \notin C$ then $X - C$ open & $x \in X - C$ $\exists$ a closed nbd G of x $\ni x \in G \subseteq X - C$. Put $U = G^c$ & $V = X - G$

then U, V are open, $x \in U$ & $C \subseteq V$ & $U \cap V = \phi$ so X is r.

[redacted]

(1) $(R, T_{L.r})$ not regular :- Let $C \underset{\text{closed}}{\subseteq} R$, $x \in R - C$ (say $C = [a,\infty)$)

L.r. U, V open $\ni x \in U$ & $C \subseteq V \Rightarrow V = R$

then $U \cap V \neq \phi$!! (open $\subseteq$ ...)

(2) (R, T_{cof}) not regular :- suppose is regular & $T_1 \longrightarrow T_3$-space & T_3 is T_2 #

(3) (R, T_{dis}) regular :- $x \in X$, $U \subseteq X$ with $x \in U \Rightarrow x \in \{x\} \subseteq \overline{\{x\}} \subseteq U$

(4) (R, T_S) is regular :- Let $x \in R$ & $U \underset{\text{open}}{\subseteq} R \ni x \in U$ $\exists$ (a,b) $\ni x \in (a,b) \subseteq U$

chose $c, d \in R \ni a < c < x < d < b$ & but $V = (c,d)$ then V is open $\ni x \in V \subseteq \overline{V} = [c,d] \subseteq (a,b) \subseteq U$

T_3-space is Topological property

Proof

Let $X \xrightarrow[\text{hom}]{f} Y$ (X is T_3), let $C \underset{\text{closed}}{\subseteq} Y$, $y \in Y - C$, then $f^{-1}(C) \underset{\text{closed}}{\subseteq} X$

$x \notin f^{-1}(C)$ where $f(x) = y$

By regularity of X $\exists$ U_1, V_1 opens $\ni x \in U_1$, $f^{-1}(C) \subseteq V_1$ & $U_1 \cap V_1 = \phi$

Put $U = f(U_1)$, $V = f(V_1)$ $\Rightarrow$ U, V opens in Y

$\ni y \in U$ & $C \subseteq V$ & $U \cap V = \phi$

Every subspace of T_3-space is T_3-space (hereditary)

Proof

Let (X, T) be T_3-space, (Y, T_Y) be subspace of X

then Y is T_1-space (since X is T_1-space & hereditary)

Now Let $C \underset{\text{closed}}{\subseteq} Y$ & $t \in (X) C$ $\Rightarrow$ $C = F \cap Y$ where $F \underset{\text{closed}}{\subseteq} X$

$t \in X - F$

by regularity of X $\exists$ U_1, V_1 opens in X $\ni t \in U_1$, $F \subseteq V_1$ & $U_1 \cap V_1 = \phi$

Put $U = U_1 \cap Y$ & $V = V_1 \cap Y$ $\Rightarrow$ U, V are open in Y

$t \in U$, $C \subseteq V$ & $U \cap V = \phi$ Q.E.D

The product space $\prod_{i=1}^{n} X_i$ is T_3-space $\Leftrightarrow$ X_i is T_3-space $\forall i$

Proof

Let $\prod X_i$ be T_3-space $X_i \cong Y_i \subseteq \prod X_i$ (Since hered + top-prop) $\Rightarrow X_i$ is T_3

Since X_i is T_1 space $\forall i = 1, \dots, n$ $\rightarrow \prod X_i$ is T_1-space

Let $x = (x_1, \dots, x_n) \in \prod_{i=1}^{n} X_i$ & $U \underset{\text{opn}}{\subseteq} \prod_{i=1}^{n} X_i$ with $x \in U$ $\exists$ a basic open set

$\ni x \in \prod_{i=1}^{n} U_i \subseteq U$ where $U_i \underset{\text{opn}}{\subseteq} X_i$ $\forall i = 1, \dots, n$ thus $x_i \in U_i \underset{\text{opn}}{\subseteq} X_i$ $\forall i = 1, \dots, n$

Since X_i is regular $\Rightarrow \exists$ V_i open in X_i $\ni x_i \in U_i \subseteq \overline{U_i} \subseteq U_i$

Since $V = \prod_{i=1}^{n} V_i$ then V is an open subset of $\prod_{i=1}^{n} X_i$ $\ni x \in V \subseteq \bar{V} \subseteq U$

(12) The product space $\prod_{\alpha \in \Delta} X_\alpha$ is T_3-space $\Leftrightarrow$ X_α is T_3-space

Proof $\Rightarrow$

Suppose $\prod X_\alpha$ is T_3-space then for $\beta \in \Delta$, $X_\beta \cong Y \subseteq \prod X_\alpha$

by hereditary & top property $\Rightarrow$ X_α is T_3-space

$\Leftarrow$

Since X_α is T_1-space $\Rightarrow$ $\prod X_\alpha$ is T_1-space

Now Let $x \in \prod_{\alpha \in \Delta} X_\alpha$ & $U = \prod_{\alpha \in \Delta} G_\alpha \subseteq \prod_{\alpha \in \Delta} X_\alpha$ basic open set where $G_\alpha = X_\alpha$ but for $\alpha = \alpha_1, \dots, \alpha_n$

So $x_\alpha \in V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$ (by X_α is regular)

take $V_{\alpha_1} \times V_{\alpha_2} \times \dots V_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \dots, \alpha_n} X_\alpha = V$ it will be open & $x \in V \subseteq \bar{V}$

(13) $\prod_{\alpha \in \Delta} X_\alpha$ is regular $\Leftrightarrow$ X_α is regular, $\forall \alpha \in \Delta$,

Proof $\Rightarrow$ Suppose $\prod X_\alpha$ is regular for $\beta \in \Delta$ $X_\beta \cong Y \subseteq \prod X_\lambda$ $\Rightarrow X_\alpha$ is regular

$\Leftarrow$ Suppose X_α is regular $\forall \alpha \in \Delta$

Let $x \in U \underset{\text{opn}}{\subseteq} \prod_{\alpha \in \Delta} X_\alpha$ $\exists$ a basic open set $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$

$\ni x \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$ $\forall i = 1, 2, \dots, n$

$\Rightarrow x_{\alpha_i} \in U_{\alpha_i} \underset{\text{opn}}{\subseteq} X_{\alpha_i}$ Since X_α is regular $\forall \alpha \in \Delta$

$\exists$ open set $V_{\alpha_i} \ni x_{\alpha_i} \in V_{\alpha_i} \subseteq \overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$ $\forall \alpha \in \Delta$

take $V = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i}) = \prod_{\alpha \in \Delta} V_\alpha$

then $x \in V \subseteq \bar{V} \subseteq \prod U_\alpha \subseteq U$ $\therefore \prod_{\alpha \in \Delta} X_\alpha$ is regular

(14) every subspace of CR is CR (hereditary)

Proof

Suppose X is CR & $Y \subseteq X$ take $A \underset{\text{closed}}{\subseteq} Y$ & $x \notin A$ then $A = B \cap Y$ where B closed

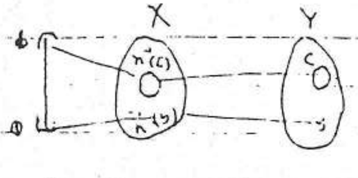
Let $x \in Y - A$, $x \notin B$ $\exists$ $f: X \rightarrow [0,1]$ $\ni f(x) = 0$ & $f(B) = 1$

then $f|_Y : X \rightarrow I$ (restriction of f in Y is still cts)

$x \in X - A$ & $A \subseteq Y$ $f(x) = 0$ & $f(A) = 1$

CR is Top-property

Proof: $X \overset{h}{\simeq} Y \Rightarrow Y$ is C.R



then $f(h^{-1}(c)) = 0$

$f(h^{-1}(y)) = 1$

so $f \circ h^{-1}$ is cts fn

The product space $\prod_{\alpha \in \Lambda} X_\alpha$ is CR $\Leftrightarrow$ X_α is CR, $\forall \alpha \in \Lambda$

Proof: $\Rightarrow$ $f_\beta \in \Lambda$ then $X_\beta \simeq Y \subseteq \prod_{\alpha \in \Lambda} X_\alpha$ since $\prod_{\alpha \in \Lambda} X_\alpha$ C.R

$\therefore Y$ is C.R $\Rightarrow X_\alpha$ is C.R $\forall \alpha \in A$

$\Leftarrow$ Suppose X_α is CR, $\forall \alpha \in A$, let $x \in U \underset{open}{\subseteq} \prod X_\alpha$ $\exists$ base open set: $\prod_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ $\ni x \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$ then $x_{\alpha_i} \in U_{\alpha_i} \subseteq X_{\alpha_i}$

$\exists$ cts fn $f_i : X_{\alpha_i} \to [0,1]$ $\ni f_i(x_{\alpha_i}) = 1$

$+ f_i(X_{\alpha_i} - U_{\alpha_i}) = 0$ $\forall \alpha \in \Lambda$

Define

$f : \prod_{\alpha \in \Lambda} X_\alpha \to R$ by $f(x) = g_{\alpha_1}(x) \cdots g_{\alpha_n}(x)$

defin $\boxed{g_i = f_i \circ \pi_{\alpha_i}\ \text{cts}}$

then $f(x) = g_1(x) \cdots g_n(x) = 1$

but $x \notin U$ + $x_{\alpha_i} \notin U_{\alpha_i}$, $1 < i < n$

$f_i(x_{\alpha_i}) = 0 \Rightarrow g_i(x) = 0 \Rightarrow f(x) = 0$

$\prod X_\alpha$ is Tychonoff $= T_{3\frac{1}{2}}$ $\Leftrightarrow$ X_α is Tychonoff space $\forall \alpha \in \Lambda$

Proof

$\Rightarrow$ let $\forall \alpha \in A$, $X_\alpha \simeq Y \subseteq \prod X_\alpha$ (Since hereditary + Top-prop)

Since $\prod X_\alpha$ (C.R) $\Rightarrow Y$ is C.R, X_α is CR $\forall \alpha \in A$

+ T_1 is Top-prop then X_α is Tychonoff space

$\Leftarrow$ let X_α is $T_{3\frac{1}{2}}$ $\forall \alpha \in A$ so $\prod X_\alpha$ is particular is T_1-space

Now

Let $p \in U \subseteq \prod X_\alpha$, $\exists$ basic open set $\prod_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$ $\ni p \in \prod_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \subseteq$

then $p_{\alpha_i} \in U_i \underset{open}{\subseteq} X_{\alpha_i}$

$\exists$ cts fns $f_i : X_{\alpha_i} \to R$ $\ni f_i(p_{\alpha_i}) = 1$ + $f_i(X_{\alpha_i} - U_i) = 0$

define $f : \prod_{\alpha \in \Lambda} X_\alpha \to R$ by $f(x) = g_1(x) \cdots g_n(x)$

define $g_i(x) = (f_i \circ \pi_{\alpha_i})(x)$

$\therefore f(p) = g_1(p) \cdots g_n(p) = 1$

but $x \notin U \Rightarrow x_{\alpha_i} \notin U_{\alpha_i}$, $1 < i < n$ $\Rightarrow f_i(x_{\alpha_i}) = 0 \Rightarrow g_i(x) = 0$

$\Rightarrow f(x) = 0$

(18) The following statement are equivalent

(1) X is CR

(2) whenever $x \in U \underset{open}{\subseteq} X$ $\exists$ cts $f : X \to I \ni f(x) = c + f(X - U) =$

(3) X is a sub base $S \ni$ whenever $x \in S \in S$ then $\exists$ cts fn $f : X \to I \ni f(x) = c + f(X - S) = 1$

Proof (1 $\Rightarrow$ 2) $\Rightarrow$ suppose X is CR, let $x \in U \underset{open}{\subseteq} X$ then $x \notin X - U$ (closed)

by assumption $\exists$ cts fn $f : X \to [0,1]$ $\ni f(x) = \ldots, f(X - U)$

$\Leftarrow$ let $C \underset{closed}{\subseteq} X$, $x \notin C$ then $x \in X - C$ (open) so $\exists$ cts fn

$f : X \to [0,1] \ni f(x) = 0$, $f(C) = 1$

(2-3) ie since every $S \in S$ is open it is clear take subbase $S = T$

(3 $\Rightarrow$ 1)

Let $x \notin C \underset{closed}{\subseteq} X$ Thus $x \in X - C$ (open) + hence $\exists$ finitely many $\ldots$

$\Rightarrow x \in \prod_{i=1}^{n} S_i \subseteq X - C$ by assumption $\exists$ cts fn $f_i : X \to I \ni$

$f_i(x) = 0$ + $f_i(X - S_i) = 1$

Let $f : X \to I$ defined by $f(t) = f_1(t) \vee \cdots \vee f_n(t)$ $\forall t \in X$

then f is cts + $f(x) = 0$

to show $f(C) = 1$ moreover if $z \in C \Rightarrow z \notin \prod_{i=1}^{n} S_i$ + $z \notin S_k$ for some $1 \le k \le$

this $f(z) = f_1(z) \vee \cdots \vee f_n(z) = 1$

Examples)

Sorgenfrey Line E is Tychonoff space.

E is T_2-space because $U=[x,a)$ & $V=[y,b)$ open in E, $U\cap V=\phi$ so E is T_2-space $\rightarrow$ T_1-space

$\boxed{C.R}$: Let $C \underset{cl}{\subseteq} E$ & $x \notin C$ & $(x \in E - C$ (open$))$ $\therefore x \in C^c$ open

$\exists$ basic open set $B(x) = [x,a)$: $\ni \boxed{x \in B(x) \subset E - C}$

Define $f : E \longrightarrow [0,1]$ by $f(B(x)) = 0$ & $f(E - B(x)) = 1$

The Γ – Moor plane is a Tychonoff space

T_2-space, case ① if $x,y \in 1^{st}$ kind take $U, V \ni U\cap V = \phi \ni x \in U, y \in V$

case ② if $x,y \in 2^{nd}$ kind take $x\in U$, $y \in V \ni U\cap V = \phi$

case ③ if $x \in 1^{st}$ & $y \in 2^{nd}$ also $x \in U$, $y \in V$ & $U\cap V = \phi$

$\boxed{C.R}$: Let $p \in U$ (open) then $\exists S \ni p \in S \subseteq U$

define $f : X \to R$ by $f(\Gamma - S) = 1$ & $f(p) = 0$ & $f(p) = \frac{\|p-q\|}{r}$

then $\Gamma - U \subseteq \Gamma - S$ & $f(\Gamma - U) = 1$ if $p \in 1^{st}$ kind

Define $f : X \to R$ by $f(\Gamma - S) = 1$ $\quad f(q) = \frac{\|p-q\|}{d}$ & $f(p) = 0$

we conclude that Γ is Tychonoff space.

Let X have weak Top induced by a family $\Phi = \{f_\alpha : \alpha \in \Lambda\}$ of functions $f_\alpha : X \longrightarrow X_\alpha$ (C.R) then X is C.R

Proof

Let S be a sub basic open set of X & Let $x \in S$ where $S = \{f^{-1}_{\alpha_0}(U_{\alpha_0}) ; U_{\alpha_0} \underset{open}{\subseteq} X_{\alpha_0}$ for some $\alpha_0 \in \Lambda\}$

$\Rightarrow f_{\alpha_0}(x) \in U_{\alpha_0} \subseteq X_{\alpha_0}$ (C.R)

$\exists$ cts $g : X_{\alpha_0} \longrightarrow I \ni g(f_{\alpha_0}(x)) = 0$ & $g(X_{\alpha_0} - U_{\alpha_0}) = 1$

Put $f = g \circ f_{\alpha_0} : X \longrightarrow I$ (clear the composition of cts)

$\ni f(x) = (g \circ f_{\alpha_0})(x) = g(f_{\alpha_0}(x)) = 0$

also if $y \in X - S = X - f^{-1}_{\alpha}(U_\alpha) \Rightarrow f(y) = g \circ f_\alpha(y) = g(f_\alpha(y))$

but $f_{\alpha_0}(y) \in X_{\alpha_0} - U_{\alpha_0}$ so $g(f_{\alpha_0}(y)) = 1$

Notes ① $C(X,Y)$ = the set of all cts fn from X into Y

② $C(X,R) = C(X)$ = The set of all cts fn $f : X \to R$

③ $C^*(X)$ = the set of all cts fn $f : X \longrightarrow [0,1]$

④ $C^*(X) \subseteq C(X)$ = the set of all bound cts real valued function from X into R

Define Cube: is the product space of Banach closed interval $\prod_{\alpha \in \Lambda} I_\alpha$

⑳ a space X is C.R $\Leftrightarrow$ X has weak Top induced by the family $C^*(X)$

Proof [X has sep. pts from closed & Bas. $\Rightarrow$ has weak Top ...]

$\Rightarrow$ Let X be C.R [It is enough to show $C^*(X)$ separated points from closed sets in X]

Let $C \underset{cl}{\subseteq} X$ & $x \notin C$ then $\exists$ (cts fn) $f : X \longrightarrow [0,1] \ni f(x) = 0$, $f(C) = 1$

i.e. $f(x) \notin \overline{f(C)} = \overline{\{1\}} = 1$

$\therefore (C^*(X))$ separated points from closed set in X

(therefor $\{f^{-1}(U) ; f \in C^*(X) ; U \subseteq [0,1]\}$ is base for some top on X which means that it is the weak Top induced by $C^*(X)$)

$\Leftarrow$ suppose weak Top induced by $C^*(X)$ where $\forall f \in C^*(X)$, $f : X \to I$ since I is C.R so X is also C.R. ☒

قبل صفحة (21) كتاب العمليات ...

$\rho : X \times X \to [0,\infty)$ (matrix is function) $\ni$
1. $\rho(x,y) = \rho(y,x)$
2. $\rho(x,y) = 0 \iff x = y$
3. $\rho(x,y) \le \rho(x,z) + \rho(z,y)$

defn (matric space)

$x \in X$, $\varepsilon > 0$, $B(x,\varepsilon) = \{ y \in X : \rho(x,y) < \varepsilon \}$ كرة

هذا الذي يتم بعده نفس النوع من التبولوجي يسمى Base ... metric Topology

(X,T) is metrizable if $\exists$ a metric ρ on $X \ni T = T_\rho$

T_ρ is Top induced by ρ metric

Every matrizable space is Tychonoff space.

Proof

1. To show every metrizable space is T_2-space

$x \neq y \in X$ then $\rho(x,y) > 0$ $\exists$ say $\varepsilon > 0$

take $U = B(x, \frac{\varepsilon}{2})$ + $V = B(y, \frac{\varepsilon}{2})$

$x \in U$ + $y \notin U$ or $y \in V$ + $y \notin V$ $\therefore$ T_2-space $\Rightarrow$ T_1-space

تحتاج لبرهان C.R التربليك

$\rho(x,A) = \inf\{ \rho(x,y) : y \in A \}$

2. To show C.R

Let $C \underset{\text{closed}}{\subseteq} X$, $x \notin C$ define $f: X \to \mathbb{R}$ by $f(x) = \rho(x,C) = \inf\{\rho(x,z) : z \in C\}$

1. f is cts
2. $f(x) = \rho(x,C) = a \neq 0$
3. $f(c) = \rho(c,C) = 0$

Q.E.D

(22) Let X is Tychonoff space $\iff$ X is homemorp to subspace of some cube

or (X can be embedded in a cube)

Proof

$\Rightarrow$ Suppose X be Tychonoff space $= T_{3\frac{1}{2}} = CR + T_1$

by CR 1. the X has weake Top induced by $C^*(X)$

2. $C^*(X)$ separates point from closed sets in X

by T_1

Let $x,y \in X$ + $x \neq y$ take $\{y\} \underset{\text{closed}}{\subseteq} X$ + $x \notin \{y\}$

$\exists f: X \to I \ni f(x) = 0$ while $f(y) = 1$

+ $f \in C^*(X)$ ... عبارة ...

we let $f: X \to I_f$ let $I_f = I$ $\quad x \xrightarrow{f} I_f$

$\Rightarrow e: X \to \prod_{f \in C(X)} I_f$ is an embedding $e^{-1}_{\pi I_f}$

Thus $X \cong$ to subspace of $\prod_{f \in C(X)} I_f$

$\Leftarrow$

assume $X \cong Y \subseteq \prod_{\alpha \in A} I_\alpha$, $(I_\alpha = I)$ $\forall \alpha \in I$

Where I_α is bounded interval $\forall \alpha \in A$

$\Rightarrow \prod_{\alpha \in A} I_\alpha$ is a Tychonoff

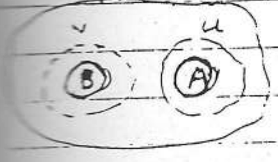
so Y is Tychonoff (by heredatiry)

$\therefore X$ is also a Tychonoff (by Top property)

(iii) Normal space & T_4 & Complet Normal & T_5

Dfns:

Normal space :- A Top-space (X,τ) is a normal space if Given any two disjont closed subsets A, B of X $\exists$ open sets U, V of X $\ni A \subseteq U$ & $B \subseteq V$ & $U \cap V = \phi$



T_4-space is ① X is normal ② X is T_1-spase

Examples —

ϕ closed & X closed
ϕ^c (open) & X^c X open

normal $\not\Rightarrow$ T_0 becase (R, T_{ind}) normal not T_0

Normal $\not\Rightarrow$ regular

Countar example: take $(R, T_{l.r})$

① is normal becase given any disjont closed subset can be find V & U open & $A \subseteq U$ & $B \subseteq V$ & $U \cap V = \phi$

but not regular: suppose C be closed $C = [a, \infty)$
$\forall x \notin C$ take $x = 0$ $C = [1, \infty)$ if $\exists$ open U
$\ni [1, \infty) \subseteq U \Rightarrow U = R$ but $x \in C$ # $U \cap V \neq \phi$

Theorms & Exs

① Every T_4-space is T_3-space

Proof

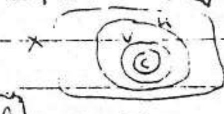
Let X be T_4-space & let $C \underset{\text{closed}}{\subseteq} X$, $x \notin C \Rightarrow \{x\}$ is closed (since X is T_1)
& $\{x\} \cap C = \phi$ then $\exists U, V$ opens $\ni \{x\} \subseteq U \Leftrightarrow x \in U$
& $C \subseteq V$ & $U \cap V = \phi$

② every T_4-space is Tychonoff space $T_{3\frac{1}{2}}$

Proof since X is T_4-space i.e. X is T_1 & Normal
Let $C \underset{\text{closed}}{\subseteq} X$ & $x \notin C \Rightarrow \{x\}$ is closed in X (beus T_1)
by normality (using Schr's Lemma) [illegible]
$\exists f : X \to [0,1] \ni f(\{x\}) = 0$ & $f(c) = 1$ is $C.R$
then X is Tychonoff

③ (Characterization of normalspace)

X is normal $\Leftrightarrow$ for any closed set C of X, for every open set U with $C \subseteq U$ $\exists$ open V
$\ni C \subseteq V \subseteq \bar{V} \subseteq U$



Proof $\Rightarrow$ X is normal, Let $C \underset{\text{closed}}{\subseteq} X$ & U open $\ni C \subseteq U$
$X - U$ is closed & $C \cap (X - U) = \phi$ since X is normal
$\exists U_1, U_2$ (opens) $\ni C \subseteq U_1$ & $X - U \subseteq U_2$ & $U_1 \cap U_2 = \phi$
Now $C \subseteq U_1 \subseteq \bar{U}_1 \subseteq U$. But $U_1 \subseteq U_2^c \Rightarrow \bar{U}_1 \subseteq \bar{U_2^c} = U_2^c \subseteq U$
$\therefore C \subseteq U_1 \subseteq \bar{U}_1 \subseteq U$

$\Leftarrow$ Let A, B any two disjont closed sets $A \cap B = \phi \Rightarrow A \underset{\text{closed}}{\subseteq} B^c$
using the assumption $\exists U$(open) $\ni A \subseteq U \subseteq \bar{U} \subseteq B^c$ & $A \subseteq U$ (open)
$B \subseteq (\bar{U})^c$ open $U \subseteq \bar{U} \Rightarrow U \cap (\bar{U})^c = \phi$ $\therefore X$ is normal

EE

67 67

لست hereditary ولكن

Every closed subspace of a Normal space is Normal

$Y \leq Y$

Let X be normal & Y subspace of X, let A, B be two disjoint closed subsets of Y

Y is closed $\Rightarrow$ A & B are disjoint subset of X by normality $\exists\, U_1, V_1$ opens in X $\ni A \subseteq U_1$ & $B \subseteq V_1$ & $U_1 \cap V_1 = \phi$

Put $U = U_1 \cap Y$, $V = V_1 \cap Y$ $\boxed{\text{then}}$ U & V are open in Y $\ni A \subseteq U$, $B \subseteq V$ & $U \cap V = \phi$ Q.E.D

Normality is a Top-property الثاني

Every closed cont image of normal space is normal

$X \xrightarrow{f} Y$ (X is normal), Let A, B be disjoint closed subsets of Y

$A_1 = f^{-1}(A)$ & $B_1 = f^{-1}(B)$ then A_1, B_1 are disjoint closed subsets in the normal space X. Thus

$\exists$ open subsets U_1, V_1 of X $\ni A_1 \subseteq U_1$, $B_1 \subseteq V_1$ & $U_1 \cap V_1 = \phi$

Take $U = Y - f(X - U_1)$, $V = Y - f(X - V_1)$ (U_1^c closed in X, V_1^c in X closed, $f(U_1^c)$ & $f(V_1^c)$ closed in Y)

then U & V are open in Y $\ni A \subseteq U$, $B \subseteq V$ & $U \cap V = \phi$ ⊠

Verification: $y \in A$ then $f^{-1}(y) \subseteq f^{-1}(A) = A_1 \subseteq U_1$ then $y \notin f(X - U_1)$

then $y \in U$, $U \cap V = [Y - f(X - U_1)] \cap [Y - f(X - V_1)]$

$= Y - [f(X - U_1) \cup f(X - V_1)]$

$= Y - f[(X - U_1) \cup (X - V_1)]$

$= Y - f[X - (U_1 \cap V_2)] = Y - Y = \phi$

The closed cont image of T_4-space is T_4-space

Proof ① to prove Y is T_1: Let $y \in Y$, $f(x) = \{y\}$ but $\{x\}$ is closed in X & X is T_1

& f is closed $\Rightarrow \{y\}$ is closed in Y then Y is T_1

② to prove Y is normal (G)

---

⑦ Product of two normal need not be normal

Solution

Counter example: Sorgenfrey Line E is normal

① R, $\beta = \{[a, b) : a, b \in R, a < b\}$, $(R, T(B))$ Sorgenfrey Line E is normal

Let A, B be disjoint closed subsets of E & $A \cap B = \phi$

For each $x \in A$ choose $[x, a_x) \ni x \in [x, a_x) \subseteq E - B$ (open)

Put $U = \bigsqcup_{x \in A} [x, a_x)$

Similarly $\forall y \in B$ choose $[y, b_y) \ni y \in [y, b_y) \subseteq E - A$ &

Put $V = \bigsqcup_{y \in B} [y, b_y)$

then U, V are open & $A \subseteq U$ & $B \subseteq V$ & $U \cap V = \phi$

Verification: $U \cap V \neq \phi$ then $\exists\, x \in A$ & $y \in B \ni [x, a_x) \cap [y, b_y) \neq \phi$ #

(*) $E \times E$ is not normal — $Y \underset{c}{\subseteq}$ ② Jones Lemma 2

suppose $E \times E$ is normal

take $Y = \{(x, -x) : x \in R\} \subseteq E \times E$ $\rightarrow$ Y is closed $E \times E$

Y is a discrete subspace, $\{p\} = U \cap Y$ open in Y, $\forall p \in Y$

(take) $D = \{(x, y) : x, y \in Q\}$ $\Rightarrow$ D is dense & countable

$|Y| = |R| = c > 2^{|D|}$ by Jones Lemma $E \times E$ is not normal

(*) Jones Lemma: Let X be space, Y be closed relative discrete subspace of X & D be dense subset of X $\ni |Y| \geq 2^{|D|}$ then X is not Normal

Proof

Given $D \underset{\text{dense}}{\subseteq} X$, $Y \underset{\text{closed}}{\subseteq} X$, $|Y| \geq 2^{|D|}$ بالتناقض

suppose X is normal, $\forall A \underset{\text{closed}}{\subseteq} Y$ $\boxed{\text{then}}$ both A & $Y - A$ are closed in X $\exists\, U(A)$ & $V(A)$ $\ni A \subseteq U(A)$, $Y - A \subseteq V(A)$ & $U(A) \cap V(A) = \phi$