Calculus And Analytic Geometry

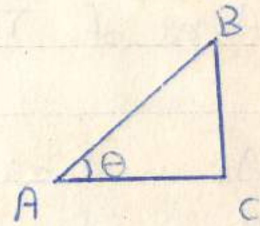
Notations and Abbrevations

$\alpha \equiv$ Alpha , $\beta \equiv$ Beta , γ or $\Gamma \equiv$ Gamma , δ or $\Delta \equiv$ Delta
$\theta \equiv$ Theta , $\lambda \equiv$ Lembda, $\eta \equiv$ Eata , $\xi \equiv$ Zeeta , $\mu \equiv$ Mu.
σ or $\Sigma \equiv$ Segma , π or $\Pi \equiv$ Pi , ϕ or $\Phi \equiv$ Fi , ψ or $\Psi \equiv$ Epsi

$=$ Equal , $\equiv$ Idintical , $\geq$ Greater than or equal ,
$\leq$ less than or equal , $\Rightarrow$ or $\ni$ implies , $\longrightarrow$ appraoch

Some Trigonometric Identities

$\sin\theta = \frac{BC}{AB}$, $\cos\theta = \frac{AC}{AB}$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{BC}{AC}$



$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{AC}{BC}$, $\sec\theta = \frac{AB}{AC} = \frac{1}{\cos\theta}$

$\csc\theta = \frac{AB}{BC} = \frac{1}{\sin\theta}$

$-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$
$-\infty \leq \tan\theta \leq \infty$ and $-1 \leq \cot\theta \leq 1$

$\{\sec\theta \leq -1$ or $\sec\theta \geq 1\}$ and $\{\csc\theta \leq -1$ or $\csc\theta \geq 1\}$
$\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = \tan^2\theta + 1$, $\csc^2\theta = \cot^2\theta + 1$
$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \sin\theta_2\cos\theta_1$
$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$
$\sin 2\theta = 2\sin\theta\cos\theta$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

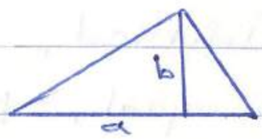
Spicial Angles :

$\sin 30 = \sin\frac{\pi}{6} = \frac{1}{2}$, $\tan 45 = \tan\frac{\pi}{4} = 1$, $\sin 0 = 0$, $\sin 90 = \sin\frac{\pi}{2} = 1$

$\cos 0 = 1$, $\cos 90 = \cos\frac{\pi}{2} = 0$.

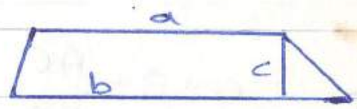
Areas and Volumes

Area of tringle $= \frac{a \times b}{2}$



Area of parallelgram $= a \times b$

Area of Trapezoid $= \frac{(a+b) \times c}{2}$



Area of circle $= \pi r^2$

Area of sector $= \frac{1}{2} r^2 \theta$ (θ = angle of sector)

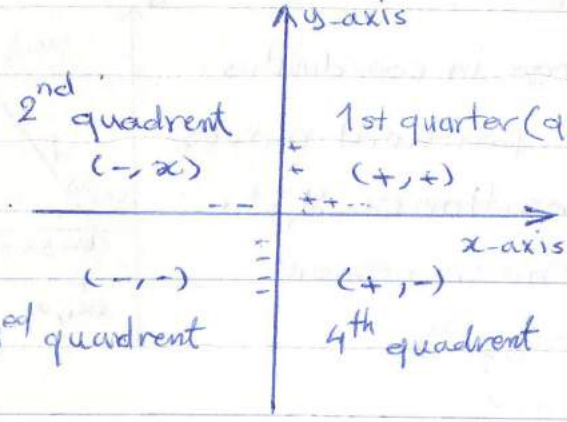
circumference $= 2\pi r$

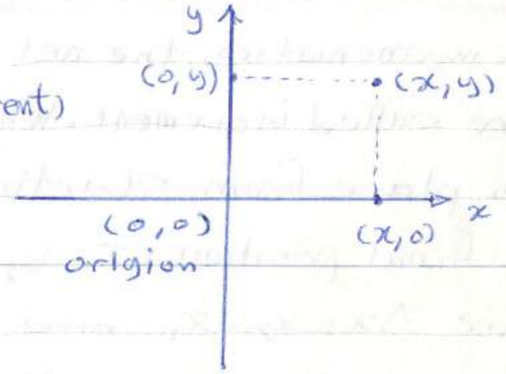
Volume of right circular Cylinder $= \pi r^2 h$

Volume of sphere $= \frac{4}{3}\pi r^3$

Surface Area $= 4\pi r^2$

The Cartesian Coordenates



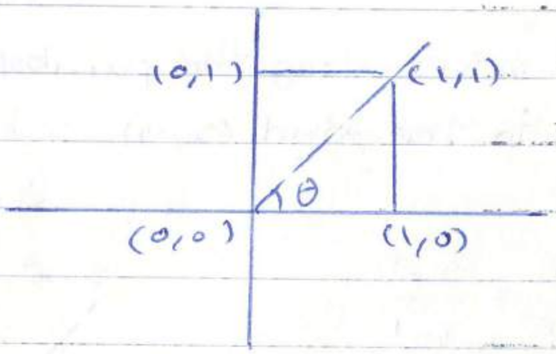


Ex(1) A line is drawn from point (0, 0) to point (1, 1). What acute angle does it make with +ve x-axis?

Solution:

$$\tan\theta = \frac{1}{1} = 1$$

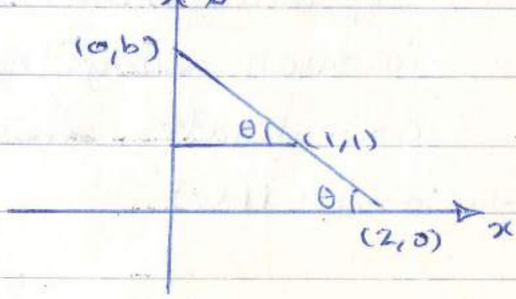
$$\theta = \frac{\pi}{4} = 45^\circ$$



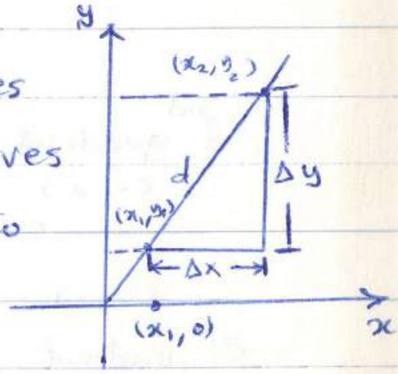
Ex(2) The line passing through the points (1, 1) and (2, 0) cuts the y-axis at (0, b). Find b.

$$\tan\theta = \frac{b}{2} = \frac{b-1}{1}$$

$$\Rightarrow b = 2$$



Increments and Distance:



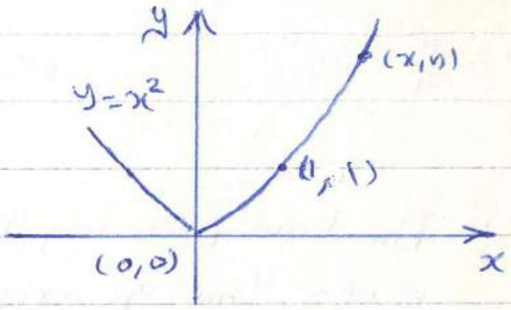
In mathematics, the net change in coordinates are called increment. When a partical moves in plane from starting position (x_1, y_1) to a final position (x_2, y_2), the increment are $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$

From pythagorean theorem:

$$\text{distance} \equiv d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Ex(3) A partical moves along the parabola $y = x^2$ from point $(1,1)$ to the point (x, y), $x \neq 1$. Sketch and show that

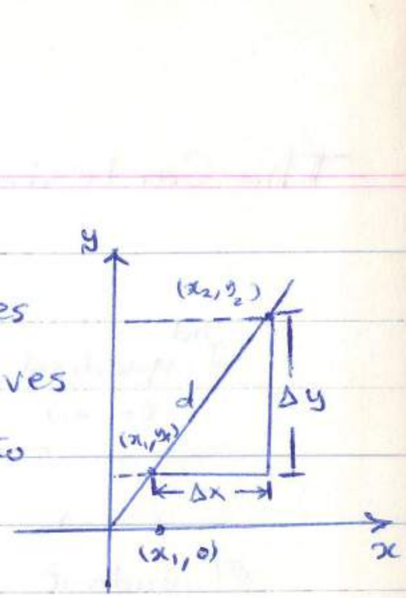
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - 1}{x - 1}$$
$$= \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = x + 1$$



Ex(4) A partical moves from point $(-2, 5)$ to the y-axis in such away that $\Delta y = 3\Delta x$. Find its new coordinates and the moving distence.

Solution. (H.W)

Increments and Distance:



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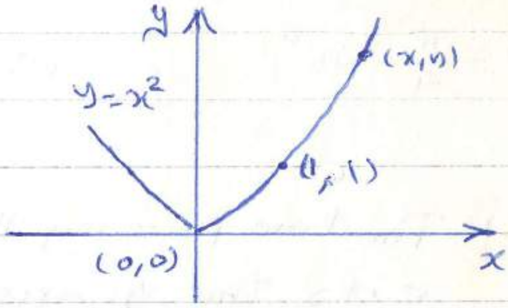

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y-1}{x-1}$$
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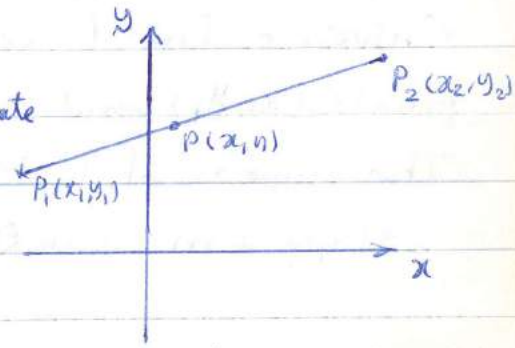
Solution. (H.W)

Divide a line segment by a ratio

If the point $p(x,y)$ is h away from $P_1(x_1,y_1)$ to $P_2(x_2,y_2)$, then the coordinate of $p(x,y)$ are:



$$x = x_1 + h(x_2 - x_1)$$
$$y = y_1 + h(y_2 - y_1)$$

Note If $p(x,y)$ is the midpoint of the segment P_1P_2 then

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

Ex(5). Find the coordinates of the point which divide the line segment from $(2,-3)$ to $(-1,4)$ in the ratio $\frac{3}{5}$.

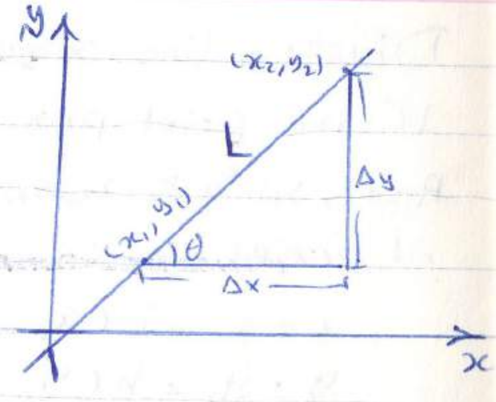
Solution: H.W

Slope of straight line

Given a line L passing through the points (x_1, y_1) and (x_2, y_2).

The slope m of L is

$$\text{slope} \equiv m = \tan\theta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



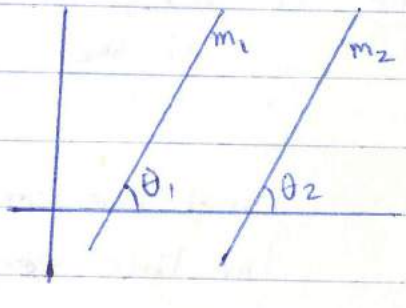
Notes:

(i) If two lines L_1 and L_2 are parallel with slopes m_1 and m_2 then $m_1 = m_2$

Proof: since $\theta_1 = \theta_2$

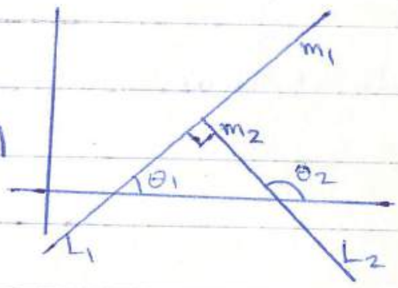
$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$



ii) If two lines L_1 and L_2 are orthogonal with slope m_1 and m_2, then $m_1 \cdot m_2 = -1$

Proof:



$$\theta_1 + \frac{\pi}{2} + \pi - \theta_2 = \pi$$

$$\theta_1 + \frac{\pi}{2} = \theta_2$$

$$\tan\left(\theta_1 + \frac{\pi}{2}\right) = \tan\theta_2$$

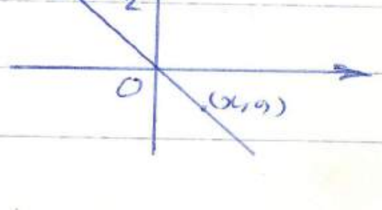
$$\Rightarrow -\cot\theta_1 = \tan\theta_2$$

$$-\frac{1}{\tan\theta_1} = \tan\theta_2 \Rightarrow \tan\theta_1 \tan\theta_2 = -1 \Rightarrow m_1 \cdot m_2 = -1$$

Ex(6) Sketch the line L that passing through the origion with slope $m = -\frac{1}{2}$. If the point (x, y) lies on L, show that $y = -\frac{x}{2}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow -\frac{1}{2} = \frac{y - 0}{x - 0} = \frac{y}{x}$$

$$\Rightarrow y = -\frac{x}{2}.$$



Equation of a Straight line.

The eq. of st. line is $ax+by+c=0$

where a, b, c are constants.

Ways of finding the equation of st. line:

(1) From a given slope m and a given point (x_1, y_1)

$$y - y_1 = m(x - x_1).$$

(2) From two given points (x_1, y_1) and (x_2, y_2)

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

(3) From a given slope m and y-intercept at $(0, b)$

$$y = mx + b$$

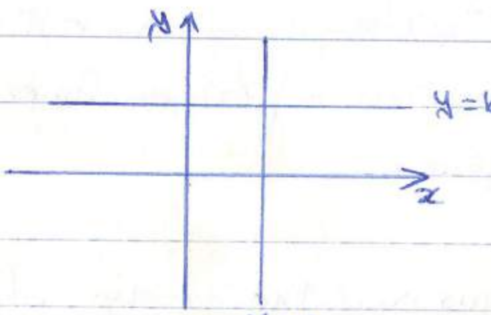
(4) From given x-Int at $(a, 0)$ and y-Int at $(0, b)$

$$\frac{x}{a} + \frac{y}{b} = 1$$

Note:

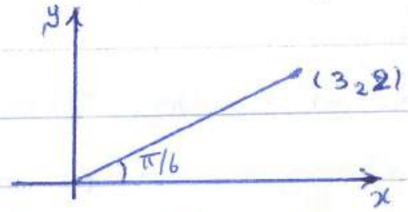
$x = a$, $a \neq 0$ is a st. line eq. parallel to y-axis

$y = b$, $b \neq 0$ " " " " , x-axis



Ex(7) Find the eq. of the st. line which make an angle $\frac{\pi}{6}$ with the x-axis and passing through the pt. $(3,2)$

Solution:



$$m = \tan\theta = \tan\frac{\pi}{6} = \tan 30 = \frac{1}{\sqrt{3}}$$

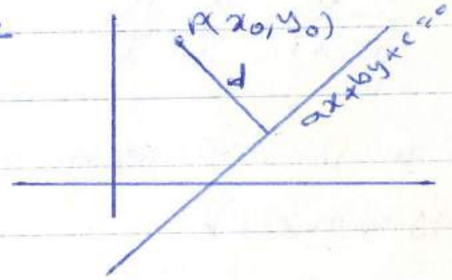
$$m = \frac{y-y_1}{x-x_1} \quad \text{or} \quad y-y_1 = m(x-x_1)$$

$$y-2 = \frac{1}{\sqrt{3}}(x-3) \Rightarrow \sqrt{3}\,y - 2\sqrt{3} = x-3$$

$$\Rightarrow \sqrt{3}\,y - x + 3 - 2\sqrt{3} = 0$$

Perpendiculer distence:

From a given point to a given line



$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

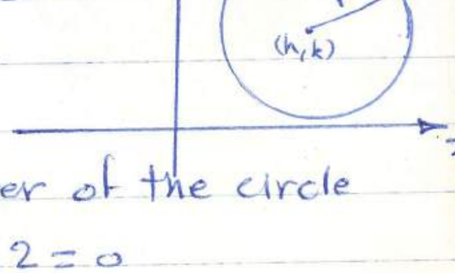
Circle: Is the locus of all points in plane whose distance from a fixed point is constant. The fixed point is called the center of the circle and denoted by $c(h,k)$ and the constant distance is called the radius of the circle and denoted by r.

The eq. of the circle with center at $c(h,k)$ and radius r is

$$r^2 = (x-h)^2 + (y-k)^2 \quad \text{---}(1)$$

if $h=k=0$, then eq (1) becomes

$$x^2+y^2=r^2$$



Ex(8): Find the radius and the center of the circle whose eq. $x^2+y^2-2x+6y-2=0$

- Inequalities : (المتباينات)

If a and b are real no.^{es}, then one of the following is true

$a > b$ or $a = b$ or $a < b$

Notes: If $a > b$ then $-a < -b$

If $a > b$ then $\frac{1}{a} < \frac{1}{b}$, $a \neq 0$ & $b \neq 0$

- Intervals :

Defn: An interval is a set of real no^{es} x having one of the following forms:

i) Open interval : $a < x < b \equiv (a,b)$

ii) Close interval : $a \leq x \leq b \equiv [a,b]$

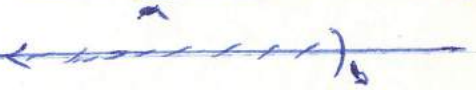
(iii) Half open from the left or half closed from right : $a < x \leq b \equiv (a,b]$

iv) Half close from the left or half open from right : $a \leq x < b \equiv [a,b)$

Note :

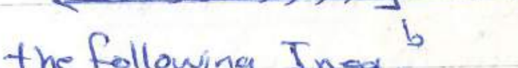
$a < x < \infty \equiv a < x \equiv (a, \infty)$

$a \leq x < \infty \equiv a \leq x \equiv [a, \infty)$



$-\infty < x < b \equiv x < b \equiv (-\infty, b)$

$-\infty < x \leq b \equiv x \leq b \equiv (-\infty, b]$



Ex(10) Find the solution set of the following Ineq.

(1) $12x - 21 < 27 + 4x$, (4) $x^3 - x > 0$

(2) $x^2 - x - 12 < 0$ (5) $x^2 + 2x + 2 > 0$

(3) $2x^2 + 5x + 2 > 0$ (6) $\frac{x-1}{x^2+x-6} < 0$

Absolute Value

Defn The absolute value of a real no. x is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties of absolute values:

1. $|x \cdot y| = |x| \cdot |y|$ and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
2. $|x| = |-x|$
3. $|x+y| \leq |x| + |y|$
4. $|x| < a$ mean $-a < x < a$
5. $|x| \leq a$ mean $-a \leq x \leq a$
6. $|x| > a$ mean $x < -a$ or $x > a$
7. $|x| \geq a$, mean $x \leq -a$ or $x \geq a$

Ex(11) Find the solution set of the following ineq.

(1) $\left|\frac{3x+1}{2}\right| < 1 \Rightarrow -1 < \frac{3x+1}{2} < 1 \Rightarrow -2 < 3x+1 < 2$

$\Rightarrow -3 < 3x < 1 \Rightarrow -1 < x < 1/3$

(2) $|x-1| \geq 5 \Rightarrow x-1 \leq -5$ or $x-1 \geq 5 \Rightarrow x \leq -4$ or $x \geq 6$

Graphs and Functions:

Defn The solution set of an eq. in two unknown consists of all points in the plane whose coordinates satisfy the eq. A gemetrical representation of the locus is called the graph of the equation.

Ex(12)

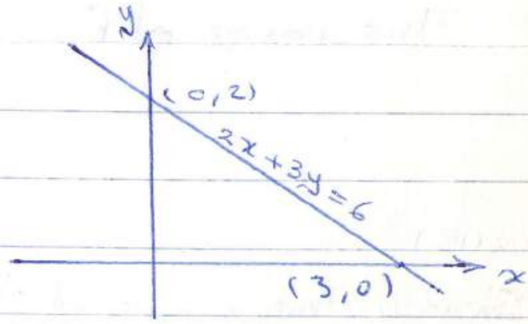
Sketch the graph of the following eq^{ns}.

(1) $2x+3y=6$ (2) $y=\begin{cases} x & 0\le x\le 1 \\ 2-x & 1<x\le 2 \end{cases}$ (3) $y=\begin{cases} -x & x<0 \\ x^2 & 0\le x\le 1 \\ 1 & 1<x \end{cases}$

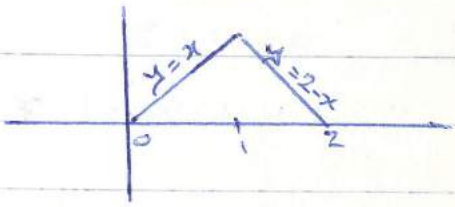
(4) $y=|x^2-1|$ (5) $16x^2+25y^2=400$

Solution

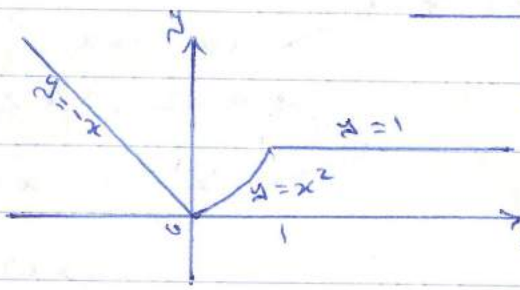
(1) $2x+3y=6$



(2) $y=\begin{cases} x & 0\le x\le 1 \\ 2-x & 1<x\le 2 \end{cases}$



(3) $y=\begin{cases} -x & x<0 \\ x^2 & 0\le x\le 1 \\ 1 & 1<x \end{cases}$

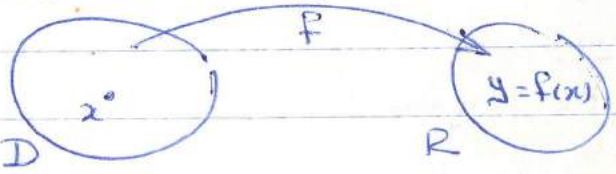


(4) $y=|x^2-1|=$

(5) $16x^2+25y^2=400$

Defn (Function): A function f from a set D to a set R is a rule that assigns a single element $y \in R$ to each element $x \in D$.

Note: The element $y \in R$ denoted by $f(x)$, the set D is called the domain of f, and the set R is called the range of f.

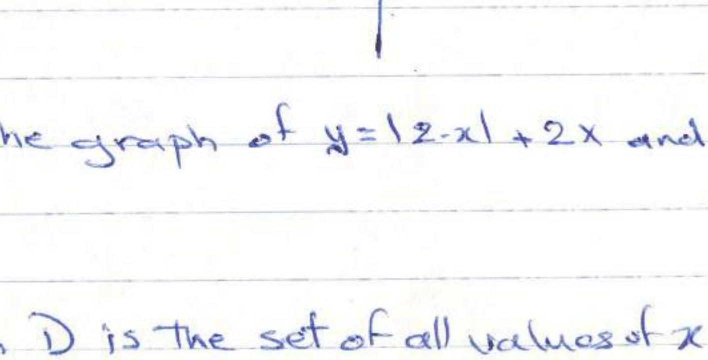


Ex(13)

Sketch the graph of the function $y = f(x) = |x+2| + x$ for $-5 \le x \le 2$

$$y = f(x) = |x+2| + x = \begin{cases} x+2+x & x+2 \ge 0 \\ -(x+2)+x & x+2 < 0 \end{cases}$$

$$= \begin{cases} 2x+2, & x \ge -2 \\ -2, & x < -2 \end{cases}$$



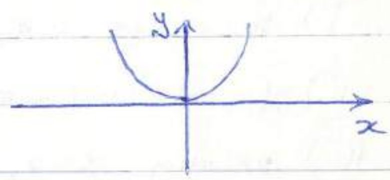
Ex(14) Sketch the graph of $y = |2-x| + 2x$ and express x in terms of y.

Note

The domain D is the set of all values of x for which y is defi
The range R is the set of all values of y for which x define.

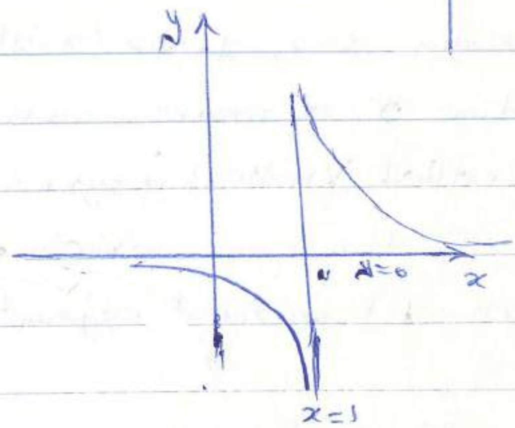
Ex. 15): Find the domain and the range of the following:

1) $y = f(x) = x^2$, D: all x , R: $y \geq 0$



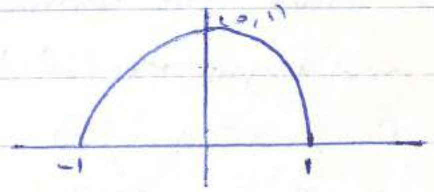
2) $y = \dfrac{1}{x-1}$, D: $x \neq 1$

$x = \dfrac{y+1}{y}$ R: $y \neq 0$



3) $y = \sqrt{1-x^2}$; D: $-1 \leq x \leq 1$

R: $0 \leq y \leq 1$



4) $y = f(x) = \sqrt{x^2 - 4x + 3}$

$x^2 - 4x + 3 \geq 0 \Rightarrow$ D: $x \leq 1$ or $x \geq 3$

$y^2 = x^2 - 4x + 3 \Rightarrow x^2 - 4x \cancel{+ 3 - y^2 = 0}$

$x = \dfrac{4 \mp \sqrt{16 - 4(3 - y^2)}}{2} = \dfrac{4 \mp \sqrt{4 + 4y^2}}{2} = 2 \mp \sqrt{1 + y^2}$, R: all y

(5) $y = \sqrt{2 - \sqrt{x}}$

For $\sqrt{x}$ it must be $x \geq 0$

$2 - \sqrt{x} \geq 0 \Rightarrow 2 \geq \sqrt{x} \Rightarrow 4 \geq 4$

$\therefore$ D: $0 \leq x \leq 4$

$x = (2 - y^2)^2$, R: all y

Intercepts, Symmetry and Asymptotes: التقاطع، التناظر، المحاذات:

1. To fin. x-intercepts, set $y=0$ and solve for y

To find y-intercepts, set $x=0$ and solve for x.

2) The locus is symmetric w.r.t. the

i) x - axis $(x,-y) \Leftrightarrow (x,y)$

ii) y - axis $(-x,y) \Leftrightarrow (x,y)$

iii) origin $(-x,-y) \Leftrightarrow (x,y)$

3) i) Aline $x=a$ near which a locus goes off to infinity is called Virtical asymmtotic

ii) A line $y=b$ near which a locus goes off to infinity is called Horizintal asymmtotic.

Ex(16): Find the domain, range, intercepts, symmetry and asymptoties if they exist for the followings.

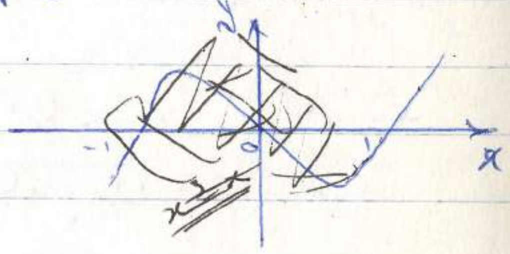
(1) $y=f(x)=x^2-x$, D: all x , R: all y

$(0,0)$, $(1,0)$ ~~[illegible]~~ are x-intercepts

$(0,0)$ is the y-intercept

Symmetry ~~w.r.t origin only~~

No asymtotes



(2) $y=f(x)=\dfrac{1}{x^2-1}$, D: $x \neq \mp 1$

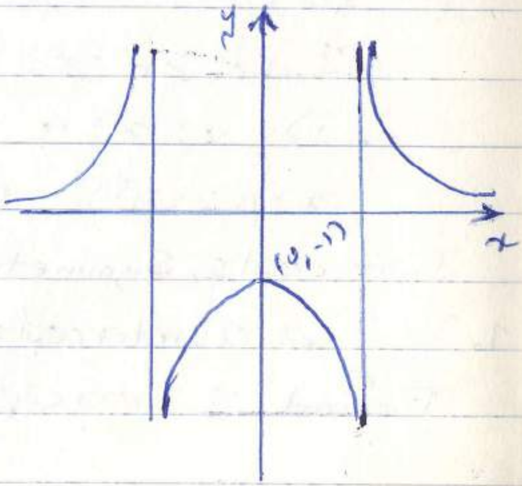
$x=\mp\sqrt{\dfrac{y+1}{y}}$, R: $y>0$ or $y\leq -1$

$(0,-1)$ is y-intercept

symm. w.r.t y-axis only

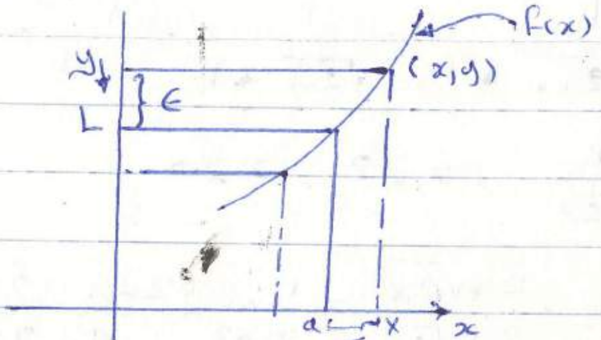
$x=\mp 1$ V. Asymptotes

$y=0$ H. Asymptote



Limit and Continuouty

<u>Defn</u>: (Limit), We say that $f(x)$ tends to the no. L as x tends to the no. a if and only if, $\forall \epsilon > 0$, $\exists \delta > 0$, such that $|f(x) - L| < \epsilon$ for all x for which $0 < |x - a| < \delta$
we write $f(x) \rightarrow L$ as $x \rightarrow a$



Ex(1)

Let $f(x) = 2x + 5$. Evalute $f(x)$ at $x = 1.1, 1.01, 1.001, \ldots$

$f(1.1) = 2(1.1) + 5 = 7.2$

$f(1.01) = 7.02$

$f(1.001) = 7.002$

$\vdots$

we see that $f(x)$ tends to 7 as x tends 1

and we say: $f(x) \xrightarrow{\text{appraoch}} 7$ as $x \xrightarrow{\text{appraoch}} 1$

<u>Note</u> Limit of $f(x)$ is L as x approaches a $\equiv \lim\limits_{x \to a} f(x) = L$

Ex(2): If $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$, $x \neq 2$. Find $\lim\limits_{x \to 2} f(x)$

غير معرفة

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim = \frac{(4) - 6 + 2}{2 - 2} = \frac{0}{0} \text{ (undefined)}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}} = \lim_{x \to 2}(x - 1) = 2 - 1 = 1$$

Ex(3) Evaluate the following limits, if they exist:

(a) $\lim_{x \to -1} \frac{\sqrt{2+x} - 1}{x+1}$, $x \neq -1$, $x \geq -2$

$$\lim_{x \to -1} \frac{\sqrt{2+x} - 1}{x+1} \cdot \frac{\sqrt{2+x}+1}{\sqrt{2+x}+1} = \lim_{x \to -1} \frac{(2+x-1)}{(x+1)(\sqrt{2+x}+1)}$$

$$= \lim_{x \to -1} \frac{1}{\sqrt{2+x}+1} = \frac{1}{\sqrt{2-1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

(b) $\lim_{x \to 2} \frac{2-x}{2-\sqrt{2x}}$, $x \neq 2$, $x \geq 0$

$$\lim_{x \to 2} \frac{2-x}{2-\sqrt{2x}} \cdot \frac{2+\sqrt{2x}}{2+\sqrt{2x}} = \lim_{x \to 2} \frac{(2-x)(2+\sqrt{2x})}{4-2x} = \lim_{x \to 2} \frac{2+\sqrt{2x}}{2}$$

$$= \frac{2+\sqrt{4}}{2} = 2$$

(c) $\lim_{x \to 2} \frac{x^4 - 2x^2 - 8}{x^2 - 4}$ $x \neq 2$

(d) $\lim_{x \to a} \frac{\sqrt{x^2+1} - \sqrt{a^2+1}}{x-a}$, $x \neq a$

(e) $\lim_{x \to 3} \frac{\sqrt{3x} - 3}{x^2 - 9}$, $x \neq 3$

(f) $\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{x+2} - \frac{1}{2}\right)$

(g) $\lim_{x \to 0} \frac{(1+x)^{3/2} - 1}{x}$

Theorems on limits

1) Uniquness on limit

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$

2) If c is constant, then if $f(x) = c$, then $\lim_{x\to a} f(x) = c$

3) If $f(x) = x$ then $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$.

4) If $f(x) = f_1(x) + f_2(x) + \dots + f_n(x)$ and $\lim_{x\to a} f_i(x) = L_i$, $i = 1, 2, \dots, n$

then $\lim_{x\to a} f(x) = \lim_{x\to a} [f_1(x) + f_2(x) + \dots f_n(x)]$

$$= \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \dots + \lim_{x\to a} f_n(x)$$

$$= L_1 + L_2 + \dots + L_n = \sum_{i=1}^{n} L_i$$

(5) Limit of product

If $f(x) = f_1(x) . f_2(x) \dots f_n(x)$, $\lim_{x\to a} f_i(x) = L_i$, $i = 1, 2, \dots, n$

$\lim_{x\to a} f(x) = \lim_{x\to a} [f_1(x)] . \lim_{x\to a} f_2(x) \dots \lim_{x\to a} f_n(x)$

$$= L_1 . L_2 \dots L_n = \prod_{i=1}^{n} L_i$$

(6) If $f(x) = \frac{h(x)}{g(x)}$, and $\lim_{x\to a} h(x) = L_1$, $\lim_{x\to a} g(x) = L_2$

$$\lim_{x\to a} f(x) = \lim \frac{h(x)}{g(x)} = \frac{\lim_{x\to a} h(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$$

Ex(4) Evalute the following limits

i) $\lim\limits_{x \to 1} \frac{x^3 - 1}{x - 1}$, $x \neq 1$

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x^2 + x + 1) = 3$$

ii) $\lim\limits_{h \to 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right)$, $h \neq 0$

$$\lim_{h \to 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) = \lim_{h \to 0} \frac{1}{h} \left(\frac{x - x - h}{(x+h)x} \right) = \lim_{h \to 0} \frac{-h}{h(x+h)x}$$

$$= -\lim_{h \to 0} \frac{1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}$$

iii) $\lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$, $h \neq 0$

One Sided and Two sided limits (Right & Left limits)

Some times, the values of a function $f(x)$ tend to different limits as x tends a from different sides. When this happens, we call the limit of $f(x)$ as x approaches a from the right by the Right hand limit and denoted by $\lim_{x \to a^+} f(x) = L$.

and the limit of $f(x)$ as x approaches a from the left by the Lift hand limit and denoted by

$$\lim_{x \to a^-} f(x) = L$$

Note: From uniquness theorem of the limit, we know that if the limit exist then it is unique, so that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ & $\lim_{x \to a^-} f(x) = L$

Ex(5): $f(x) = \sqrt{x}$, $D: x \geq 0$. Find $\lim_{x \to 0} f(x)$

Ex(6): $f(x) = \sqrt{1-x}$, Find $\lim_{x \to 1} f(x)$

Ex(7) : $f(x) = \frac{x}{|x|}$. Find $\lim_{x \to 0} f(x)$.

Ex(8) : $f(x) = \frac{x\sqrt{x^2+1}}{|x|}$, $x \neq 0$ Find $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$.

Ex(9) : $f(x) = |x-1|$, Find $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1} f(x)$

Ex(10) $f(x) = \dfrac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$. What is the domain.

Find $\lim\limits_{x \to 2^-} f(x)$, $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2} f(x)$

Limits at Infinity

We note that when the limit of a function $f(x)$ exist as x approaths infinity, we write $\lim\limits_{x \to \infty} f(x) = L$

Also we write

$\lim\limits_{x \to +\infty} f(x) = L$ for +ve values of x and $\lim\limits_{x \to -\infty} f(x) = L$ for -ve values of x. For one-sided and two-sided limits, we have $\lim\limits_{x \to \infty} f(x) = L$ iff $\lim\limits_{x \to +\infty} f(x) = L$ and $\lim\limits_{x \to -\infty} f(x) = L$

Some useful limits:

(1) if k is constant, then $\lim\limits_{x \to +\infty} k = k$ and $\lim\limits_{x \to -\infty} k = k$.

(2) $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$; $\lim\limits_{x \to +\infty} \dfrac{1}{x} = 0$ $\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

(3) $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$, $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

Ex(11) Find the following limits:

(1) $\lim\limits_{x \to \infty} \frac{x}{2x+3} =$

(2) $\lim\limits_{x \to \infty} \frac{2x^2+3x+5}{5x^2-4x+1} =$

(3) $\lim\limits_{x \to \infty} \frac{2x^2+1}{3x^3-2x^2+5x} =$

(4) $\lim\limits_{x \to \infty} \frac{2x^3+2x-1}{x^2-5x+2} =$

(5) $\lim\limits_{x \to \infty} \sqrt{x} =$

(6) $\lim\limits_{x \to -\infty} (2x + \frac{3}{x})$

(7) $\lim\limits_{x \to 2^-} \frac{1}{x^2-4}$, $\lim\limits_{x \to 2^+} \frac{1}{x^2-4}$

(8) $\lim\limits_{x \to \infty} (\sqrt{x^2+1} - x)$

(9) $\lim\limits_{x \to \infty} (\sqrt{x^2+2x} - x)$

(10) $\lim\limits_{x \to 0} (2 + \frac{\sin x}{x})$

More about An Asymptotes

Given $y=f(x)$. A line $y=mx+b$ is an asymptote for $f(x)$ if (1) $m=\lim\limits_{x\to\infty}\frac{f(x)}{x}$ (2) $b=\lim\limits_{x\to\infty}(f(x)-mx)$

Ex(12)

Find the asymptotes of the following functions

(1) $y=f(x)=x+\frac{1}{x}=\frac{x^2+1}{x}$

$x=0$, is V-Asy.

$x^2+1=yx \Rightarrow x^2-yx+1=0 \Rightarrow x=\frac{y\mp\sqrt{y^2-4}}{2}$

No H-Asy

Let $y=mx+b$ be an asy

$m=\lim\limits_{x\to\infty}\frac{f(x)}{x}$

$b=\lim\limits_{x\to\infty}(f(x)-mx)$

(2) $y=f(x)=\frac{x^2-3}{2x-4}$, $x=2$ is V. Asy.

$y=f(x)$, find x in terms of y to conclude H-asy.

let $y=mx+b$ be an asymtote.

$m=\lim\frac{f(x)}{x}$

$b=\lim(f(x)-mx)$

Sandwich Theorem:

If $g(x) \leq f(x) \leq h(x)$ and if:

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \text{ then } \lim_{x \to a} f(x) = L$$

Ex(13)

find $\lim_{x \to \infty} f(x)$. if $\frac{2x+3}{x} \leq f(x) \leq \frac{2x^2+5x}{x^2}$

$$\lim_{x \to \infty} \frac{2x+3}{x} = \lim_{x \to \infty} (2 + \frac{3}{x}) = 2+0 = 2$$

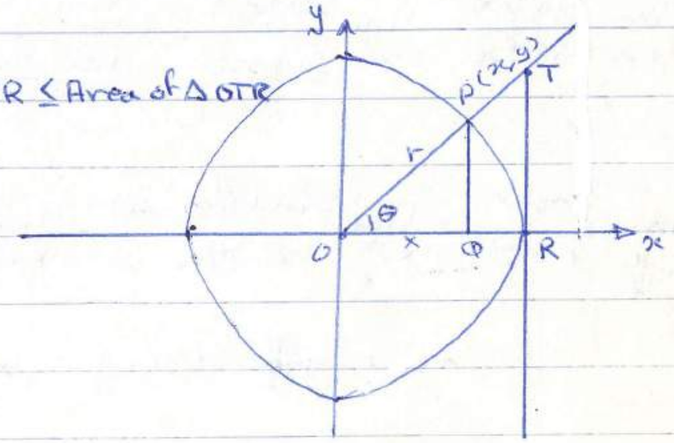
$$\lim_{x \to \infty} \frac{2x^2+5x}{x^2} = \lim (2 + \frac{5}{x}) = 2+0 = 2$$

Then by sandwich theorem $\lim_{x \to 2} f(x) = 2$

Theorem (1): If θ is measured in radian then

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Proof:



Area of $\Delta OPQ \leq$ Area of sector $OPR \leq$ Area of ΔOTR

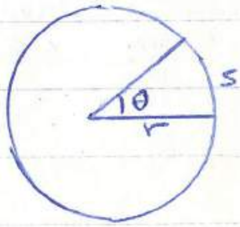
$\frac{1}{2}(OQ)(PQ) \leq \frac{1}{2}r^2\theta \leq \frac{1}{2}(OR)(TR)$

$(r\cos\theta)(r\sin\theta) \leq r^2\theta \leq r(r\tan\theta)$

$$\cos\theta \sin\theta \leq \theta \leq \frac{\sin\theta}{\cos\theta}$$

$$\frac{\cos\theta}{\sin\theta} \leq \frac{1}{\theta} \leq \frac{1}{\cos\theta\sin\theta}$$

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq \frac{1}{\cos\theta}$$



note: $\theta = \frac{s}{r}$

$$\lim_{\theta \to 0} \cos\theta \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq \lim_{\theta \to 0} \frac{1}{\cos\theta}$$

Since $\lim_{\theta \to 0} \cos\theta = \cos 0 = 1$ and $\lim_{\theta \to 0} \frac{1}{\cos\theta} = \frac{1}{\cos 0} = \frac{1}{1} = 1$

$\therefore \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (by Sandwich theorem).

Theorem (2)

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$$

Proof: $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)}$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \lim_{\theta \to 0} \frac{\sin\theta}{1+\cos\theta} = 1 \cdot \frac{\sin 0}{1+\cos 0} = \frac{0}{1+1} = 0$$

Ex(14): Find the following limits:

1. $\lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim \frac{\sin 3x}{3x} = 3(1) = 3$ (as $x \to 0 \Rightarrow 3x \to 0$)

2. $\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \frac{\lim_{x\to 0} \frac{\sin 5x}{x}}{\lim_{x\to 0} \frac{\sin 3x}{x}} = \frac{5 \lim_{x\to 0} \frac{\sin 5x}{5x}}{3 \lim_{x\to 0} \frac{\sin 3x}{3x}} = \frac{5(1)}{3(1)} = \frac{5}{3}$

3. $\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim_{x\to \frac{\pi}{2}} \frac{\sin(\pi/2 - x)}{x - \pi/2} = \lim_{x\to \frac{\pi}{2}} \frac{-(\sin(x - \pi/2))}{x - \pi/2}$

as $x \to \frac{\pi}{2} \Rightarrow x - \frac{\pi}{2} \to 0$

then $\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{x - \pi/2} = -\lim_{x - \frac{\pi}{2} \to 0} \frac{\sin(x - \pi/2)}{x - \pi/2} = -1$

4) $\lim_{x\to 0} \frac{\tan x}{x} =$

5) $\lim_{x\to 0} \frac{\sin 2x}{2x^2 + x}$

6) $\lim x \sin \frac{1}{x}$

7) $\lim \frac{\sin x}{|x|}$

Defn (Continuous function)

A function $f(x)$ is said to be cont. at $x = a$ if :

1. $f(a)$ is defined
2. $\lim\limits_{x \to a} f(x) = f(a)$

Ex (15)

(a) Every polynomial (متعددة حدود) of the form:

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ is cont. for all x.

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = \dfrac{x+3}{(x-3)(x-2)}$

(d) $f(x) = \dfrac{\sin x}{x}$

(e) $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

(f) $f(x) = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4} & , x \neq 2 \\ \dfrac{5}{4} & \end{cases}$

Exercises:

I) Find the following limits:

1. $\lim_{x \to 0} (x^2 - 2x + 1)$

2. $\lim_{\Delta x \to 0} 2x + \Delta x$

3. $\lim_{x \to 1} |x - 1|$

4. $\lim_{x \to 0} \frac{3x^3 + 8x^2}{3x^4 - 16x^2}$

5. $\lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4}$

6. $\lim_{x \to 1} \frac{x^n - 1}{x - 1}$, n is +ve integer

7. $\lim_{x \to \infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5}$

8. $\lim_{r \to \infty} \frac{8r^2 + 7r}{4r^2}$

9. $\lim_{x \to \infty} \frac{x + \sin x}{x + \cos x}$

10. $\lim_{x \to \infty} (1 + \cos\frac{1}{x})$

11. $\lim_{x \to 0^+} \frac{5}{2x}$

12. $\lim_{x \to 3} \frac{x - 3}{x^2}$

13. At what points the function:

$$y = f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 1 & 0 < x \end{cases}$$

is continuous?

In 14 - 16 find discont. pts.

14. $y = \frac{1}{(x+2)^2}$

15. $y = \frac{x+1}{x^3 - 4x + 3}$

16. $y = \frac{\cos x}{x}$

Find limits if exist for followi

17. $\lim_{x \to \infty} \frac{x + \sin x}{2x + 5}$

18. $\lim_{x \to \infty} \frac{x \sin x}{x + \sin x}$

19. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

20. $\lim_{x \to 0^+} \frac{|x|}{x}$

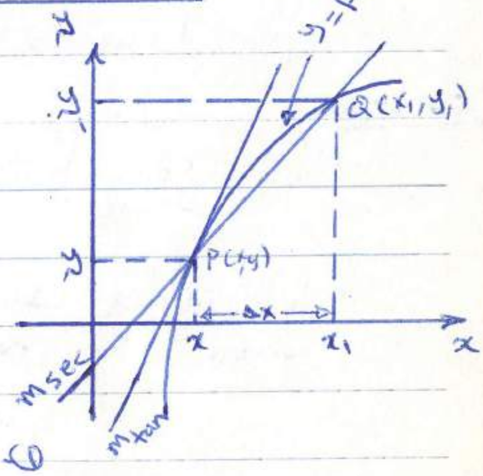
21. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2}$

The slope of the curve $y=f(x)$ and derivatives

Let $p(x,y)$ be a fixed point on the curve $y=f(x)$. Let $Q(x_1,y_1)$ be a moving point on the curve $y=f(x)$.

Let m_{sec} be the slope of the secant line PQ

let m_{tan} be the slope of the tangent at p.



$$m_{sec}=\frac{y_1-y}{x_1-x}=\frac{f(x_1)-f(x)}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \cdots (1)$$

As the point Q approaches P, we see Δx approaches zero

that is, as $Q \rightarrow P$, we have $\Delta x \rightarrow 0$

this mean:

$$m_{tan}=\lim_{\Delta x\to 0} m_{sec}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad \Delta x\neq 0 \quad \cdots (2)$$

The function m_{tan} is a function of x defined at every pt. x at which the limit of eq(2) exist

Usually we denote the slope tangent by $f'(x)$.

Defn (Derivative)

* The derivative of a function $y=f(x)$ is the function $f'(x)$ whose value at each x is defined by the rule

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad \Delta x\neq 0, \quad \text{provided this limit exist.}$$

Ex. Find the eq. of tangent and normal lines to the curve $f(x) = x^2$ at $x = 3$

<u>solution</u>

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x$$

which is the slope of the tangent at any point (x, y).

$m_1 = f'(3) = (2)(3) = 6$ = slope of the tangent at $(3, 9)$

$m_1 = \frac{y - y_1}{x - x_1} \Rightarrow 6 = \frac{y - 9}{x - 3} \Rightarrow y = 6x - 4$ eq. of tangent line.

$m_2 = -\frac{1}{m_1} = -\frac{1}{3}$, slope of normal at $(3, 9)$

$-\frac{1}{3} = \frac{y - 9}{x - 3} \Rightarrow 3y = -x + 30$ eq. of normal line.

Ex. If $f(x) = \sqrt{x}$. Find $f'(x)$ using the definition of derit.

Ex Given a function f whose domain is the set of real numbers and has the property $f(x+y) = f(x) \cdot f(y), \forall x$ ... & $f(0) \neq 0$. (a) Show that $f(0) = 1$.

(b) If $f'(x)$ exist. show that $f'(x) = f(x) \cdot f'(0)$

Ex Given a function f satisfies the following conditions:

(1) $f(x+y) = f(x) \cdot f(y)$. (2) $f(x) = 1 + x g(x)$

(3) $\lim_{x \to 0} g(x) = 1$. Show that $f'(x) = f(x)$.

Derivatives المشتقات

Definition: Let $y = f(x)$ be a function of x.

If $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ defined and exists,

then we call the derivative of f at x, (or f is differentiable at x). it denoted by:

$$f'(x),\ y',\ \frac{dy}{dx} \text{ or } \frac{d f(x)}{dx}.$$

Rules of Derivatives

- Rule (1)

If $y = f(x) = c$, where c is constant ثابت

then $\frac{dy}{dx} = 0$

Ex $y = 5$, $y' = 0$

- Rule (2) If n is integer and $y = x^n$, then

$$f'(x) = \frac{dy}{dx} = n x^{n-1}$$

Ex (a) $f(x) = x^4$, $f'(x) = 4x^3$

(b) $f(x) = x^{-2}$, $f'(x) = -2x^{-3}$

- Rule (3) If $f(x) = c\,u(x)$, where c is constant and $u(x)$ is a function of x. Then

$$f'(x) = c.\,u'(x) = c.\frac{d f(x)}{dx}.$$

Ex (a) $f(x) = 2x^3$, $f'(x) = 2(3)x^2 = 6x^2$

(b) $f(x) = -2x^4$ $f'(x) = -8x^3$

Rule (4) If $u_i(x)$, $i=1,2,\dots,n$ are differentiable functions of x and $f(x)=u_1(x)+u_2(x)+\dots+u_n(x)$ then $f'(x)=u_1'(x)+u_2'(x)+\dots+u_n'(x)$

Ex $f(x)=2x^3-3x^5+1$

$$f'(x)=6x^2-15x^4$$

Rule (5) If $f(x)=u(x)\cdot v(x)$, where $u(x), v(x)$ are functions of x, then

$$f'(x)=u(x).v'(x)+v(x).u'(x)$$

Ex (a) $y=(x^2+1)(2x^3+x)$

$$\frac{dy}{dx}=(x^2+1)(6x^2+1)+(2x)(2x^3+x)$$

(b) $y=x^2(2x-3)$

$$\frac{dy}{dx}=x^2(2)+(2x)(2x-3)$$

Rule (6) If $f(x)=\dfrac{u(x)}{v(x)}$, $v(x)\neq 0$

$$\frac{df(x)}{dx}=\frac{v(x).u'(x)-u(x).v'(x)}{(v(x))^2}$$

Ex

$$y=\frac{2x^3+3x+1}{x^2+1}$$

$$\frac{dy}{dx}=\frac{(x^2+1)(6x^2+3)-(2x^3+3x+1)(2x)}{(x^2+1)^2}$$

Rule(7) If $f(x) = [u(x)]^n$, where n is real number and $u(x)$ is differentiable on x. Then

$$\frac{df}{dx} = n\,[u(x)]^{n-1} \cdot u'(x).$$

Ex (a) $f(x) = (2x^2 + 3x - 1)^3$,

$$f'(x) = 3(2x^2 + 3x - 1)^2 (4x + 3)$$

- Implicit Differentiation الاشتقاق الضمني

في بعض الحالات يكون من غير الممكن كتابة الدالة $f(x,y) = 0$ بالصيغة $y = f(x)$ فلإيجاد المشتقة بالنسبة dy/dx نجد المشتقة ضمنياً ونجد منها قيمة dy/dx من المعادلة كما يلي:

Ex Find dy/dx, if $y^3 - 3x^2y + x^3 = 5$

$$3y^2y' - 3(x^2y' + 2xy) + 3x^2 = 0$$

$$y'(3y^2 - 3x^2) - 6xy + 3x^2 = 0$$

$$y' = \frac{6xy - 3x^2}{3y^2 - 3x^2}$$

Ex Find dy/dx if $xy + 2x^3y^{-2} + x = 1$

$$xy' + y + 2(3x^2y^{-2} - 2y^{-3}y'x^3) + 1 = 0$$

$$y' = \frac{-y - 6x^2y^{-2} - 1}{x - 4x^3y^{-3}}$$

The second and Higher Derivatives المشتقة الثانية والعليا

لتكن $y = f(x)$ دالة للمتغير x .

المشتقة $y' = \frac{df}{dx}$ هي المشتقة الاولى لـ y بالنسبة لـ x.

فيكون المشتقة الثانية معرفة بالشكل التالي :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

بشكل عام للدالة القابلة للاشتقاق $y = f(x)$ تكون المشتقة النونية (nth-derivative) لـ y بالنسبة لـ x :

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Ex If $y = 2x^4 + 3x^2 - x$

$$\frac{dy}{dx} = 8x^3 + 6x - 1$$

$$\frac{d^2y}{dx^2} = 24x^2 + 6$$

$$\frac{d^3y}{dx^3} = 48x$$

Ex Find $\frac{d^2y}{dx^2}$ ~~at~~ at $x=0$ if

$$y = (3x^2+5)^2 + (x^3-1)^{-4}$$

$$y' = (3x^2+5)^2\left(-4(x^3-1)^{-5}(3x^2)\right) + 2(3x^2+5)(6x)(x^3-1)^{-4}$$

$$y'' = (3x^2+5)^2\left[-12\left\{x^2\left[-5(x^3-1)^{-6}(3x^2)\right] + 2x(x^3-1)^{-5}\right\}\right] + $$
$$+2(3x^2+5)(6x)\left(-12x^2(x^3-1)^{-5}\right)$$
$$+12\left[(3x^2+5x)\left(-4(x^3-1)^{-5}(3x^2)\right) + (6x+5)(x^3-1)^{-4}\right]$$

$$y''(x)\Big|_{x=0} = 0+0+0+12\left[0+(5)(1)\right] = 60$$

(b) If $y = (2x+1)^2 + x^3(2x+7)^{-2}$

find $y''(1)$

Chain Rule قاعدة السلسلة

اذا كانت $y = f(x)$ هي دالة للمتغير x وكانت
$x = g(t)$ هو دالة للمتغير t فان y هي
دالة للمتغير t ايضاً ، $y = f(g(t))$ وتكون
مشتقة y بالنسبة لـ t هي :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

EX If $y = x^3 - 2x^2 + 3$ and $x = t^2 + 2$
find $\frac{dy}{dt}$ at $t = 2$

Solution:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= (3x^2 - 4x)(2t)$$

when $t = 2 \rightarrow x = (2)^2 + 2 = 6$

$$\left.\frac{dy}{dt}\right|_{t=2} = (3(6)^2 - 4(6))(2(2)) = 336$$

OR $$\frac{dy}{dt} = (3(t^2+2)^2 - 4(t^2+2))(2t)$$

$$\left.\frac{dy}{dt}\right|_{t=2} = (3(2^2+2)^2 - 4(2^2+2))(2(2)) = 336$$

L'Hopital Rule   قاعدة هوبيتال

اذا كانت ($f(x) = g(x) = 0$ أو ∞) وكانت f, g دالتين لـ x مستمرة في فترة I ،

فلايجاد $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ يمكننا اشتقاق البسط والمقام

مرة او عدة مرات ما دامت الصيغة $\frac{0}{0}$ او $\frac{\infty}{\infty}$

تظهر عند التعويض . وكما يلي :

Ex

(a) $$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{2x}{1} = 4$$

ظهرت $\frac{0}{0}$ عند التعويض فنشتق البسط والمقام

(b) $$\lim_{x \to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x \to 1} \frac{3x^2-3}{3x^2-2x-1}$$

$$= \lim_{x \to 1} \frac{6x}{6x-2} = \frac{6}{4} = \frac{3}{2}$$

(c) $$\lim_{x \to 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = \frac{-1}{8}$$

(d) $$\lim_{x \to \infty} \frac{2x^2+4x+1}{x^2-7} = \lim_{x \to \infty} \frac{4x+4}{2x} = \lim_{x \to \infty} \frac{4}{2} = 2$$

Transental Functions

The Trigonometric functions الدوال المثلثية

Theorem

1. If $y = f(x) = \sin x$ then $\dfrac{dy}{dx} = \cos x$

~~2. If y = [illegible]~~

2. If $y = f(x) = \cos x$ then $\dfrac{dy}{dx} = -\sin x$

3. If $y = f(x) = \tan x$ then $\dfrac{dy}{dx} = \sec^2 x$

4. If $y = f(x) = \cot x$ then $\dfrac{dy}{dx} = -\csc^2 x$

5. If $y = f(x) = \sec x$ then $\dfrac{dy}{dx} = \sec x . \tan x$

6. If $y = f(x) = \csc x$ then $\dfrac{dy}{dx} = -\csc x . \cot x$

Now, if $u(x)$ is differentiable on x and

1. $y = \sin u$ then $\dfrac{dy}{dx} = \cos u . \dfrac{du}{dx}$

2. $y = \cos u$ then $\dfrac{dy}{dx} = -\sin u . \dfrac{du}{dx}$

3. $y = \tan u$, then $\dfrac{dy}{dx} = \sec^2 u . \dfrac{du}{dx}$

4. $y = \cot u$, then $\dfrac{dy}{dx} = -\csc^2 u . \dfrac{du}{dx}$

5. $y = \sec u$, then $\dfrac{dy}{dx} = \sec u . \tan u . \dfrac{du}{dx}$

6. $y = \csc u$, then $\dfrac{dy}{dx} = -\csc u . \cot u . \dfrac{du}{dx}$

Examples:
find $\frac{dy}{dx}$ for the following functions:

1) $y = \sin(x^2+2x-5)$

$$dy/dx = \cos(x^2+2x-5)\cdot(2x+2)$$
$$= 2(x+1)\cos(x^2+2x-5)$$

2) $y = \sin^2(x^2+\frac{1}{x^2})$

$$\frac{dy}{dx} = 2\sin(x^2+\frac{1}{x^2})\cdot\cos(x^2+\frac{1}{x^2})\cdot(2x-\frac{2}{x^3})$$

3 $y = \tan^{-3}(2x)\cdot\cos(x^2+1)$

$$y' = -\tan^{-3}(2x)\cdot\sin(x^2+1)(2x) + \cos(x^2+1)\cdot$$
$$(-3\tan^{-4}(2x)\cdot\sec^2(2x)\cdot 2)$$

Ex
find

1) $\lim_{x\to0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$

2) $\lim_{x\to0} \frac{\sin 2x}{2x^2+x} = \lim_{x\to0} \frac{2\cos 2x}{4x+1} = 2$

3) $\lim_{x\to0} \frac{\tan 3x}{\sin x} = \lim_{x\to0} \frac{3\sec^2 3x}{\cos x} = \frac{3}{1} = 3$

4) $\lim_{x\to0} \frac{x-\sin x}{x^3} = \lim_{x\to0} \frac{1-\cos x}{3x^2}$

$$= \lim_{x\to0} \frac{\sin x}{6x} = \lim_{x\to0} \frac{\cos x}{6} = \frac{1}{6}$$

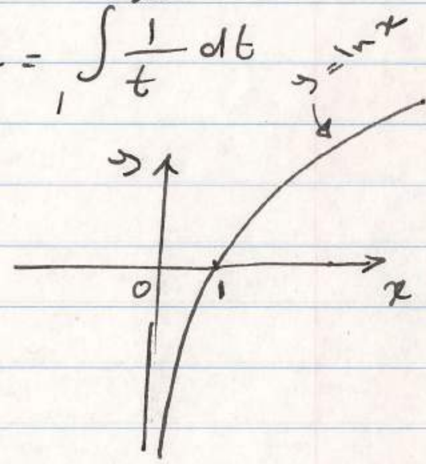
The Logarithmic functions

Consider $x = b^y \Leftrightarrow y = \log_b x$

If $b=10$, $y = \log_{10} x = \log x$

If $b = e = 2.7183\ldots$, we write $\log_e x = \ln x$ which is called natural logarithm.

Definition
For $x > 0$, define $\ln x = \int_1^x \frac{1}{t}\,dt$



Properties of N. logarithm

1. $\ln(a.b) = \ln a + \ln b$
2. $\ln(a/b) = \ln a - \ln b$
3. $\ln(1) = 0$
4. $\ln(a^r) = r \ln a$

Derivative of Natural Logarthm

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

Also, if $y = \ln u(x)$ then $\frac{dy}{dx} = \frac{1}{u(x)} . u'(x)$

Ex find dy/dx for following functions,

1. $y = \ln(x^3 - 3x + 1) \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x + 1}$

2. $y = x^{\sin x} \Rightarrow \ln y = \sin x . \ln x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x . \frac{1}{x} + \ln x . \cos x$

$$\therefore \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$$

$$= x^{\sin x} \cdot \left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$$

3. $$y = \frac{(\sin^{-3} x + x^{3/4})^{5/2}}{x^{2/3} + 2x} \xRightarrow[\ln]{\text{take}}$$

$$\ln y = \ln(\sin^{-3} x + x^{3/4})^{5/2} - \ln(x^{3/2} + 2x)$$

$$= \frac{5}{2}\ln(\sin^{-3} x + x^{3/4}) - \ln(x^{3/2} + 2x)$$

$$\rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{5}{2} \cdot \frac{-3\cos x \cdot \sin^{-4} x + \frac{3}{4} x^{-1/4}}{\sin^{-3} x + x^{3/4}} - \frac{\frac{3}{2} x^{1/2} + 2}{x^{3/2} + 2x}$$

4. $$y = (\ln x)^x \xrightarrow{\ln} \ln y = x \ln(\ln x)$$

$$\xrightarrow{d/dx} \frac{1}{y}\frac{dy}{dx} = \cancel{x} \cdot \frac{1}{\ln x} \cdot \frac{1}{\cancel{x}} + \ln(\ln x)$$

$$= y\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

<u>Ex</u> Evaluate:

1 $$\lim_{x \to \infty} \frac{\ln x}{x} \xRightarrow[\text{Hopitals}]{\text{by}} \lim_{x \to \infty} \frac{1/x}{1} = 0$$

2 $$\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} =$$ by L'Hopital's

$$= \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \cancel{\frac{1}{x}}}{\cancel{\frac{1}{x}}} = \lim_{x \to \infty} \frac{1}{\ln x} = 0$$

Ex. Solve for x if $3^x = 2^{x+1}$

<u>Sol:</u> take logarithm for both sides

$$x \ln 3 = (x+1)\ln 2 \Longleftrightarrow x(\ln 3 - \ln 2) = \ln 2$$

$$\rightarrow x = \frac{\ln 2}{\ln 3 - \ln 2}.$$

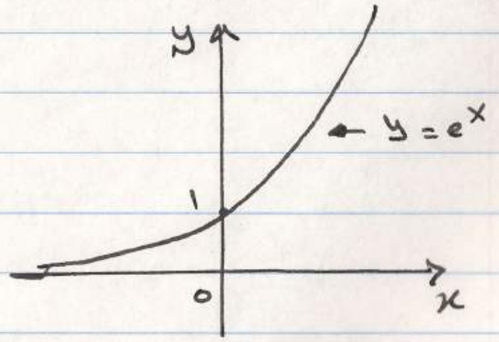
The exponential function

The exponential function is defined as inverse of the Natural logarithmic function. denoted by e

That is

for $-\infty < x < \infty$, we define $y = e^x \longleftrightarrow x = \ln y$

$y > 0$



Properties of e^x.

1. $e = 2.7183\ldots$
2. $e^{x+y} = e^x \cdot e^y$
3. $e^{x-y} = e^x / e^y$
4. $e^{\ln x} = x$
5. $\ln e^x = x$

بسّط

Ex Simplify the following exprissions:

1. $e^{\ln 2} = 2$

2. $e^{\ln \sin x} = \sin x$

3. $\ln\left(\frac{e^{3x}}{5}\right) = \ln(e^{3x}) - \ln 5 = 3x - \ln 5$

4. $e^{\ln 2 + 3\ln x} = e^{\ln 2} \cdot e^{\ln x^3} = 2x^3$.

Ex solve for y if:

$$\ln(y-1) - \ln y = 2x$$

$$\ln\left(\frac{y-1}{y}\right) = 2x \xrightarrow[e]{\text{take}}$$

$$\frac{y-1}{y} = e^{2x} \longrightarrow y-1 = ye^{2x} \longrightarrow y(1-e^{2x}) = 1$$

$$\therefore y = \frac{1}{1-e^{2x}}$$

Derivative of the exponential functions

If $y = e^x$ then $\frac{dy}{dx} = e^x$

Proof

$$y = e^x \leftrightarrow x = \ln y \xrightarrow{d/dx} 1 = \frac{1}{y}\frac{dy}{dx}$$

$$\longrightarrow \frac{dy}{dx} = y = e^x.$$

Now; if $u = u(x)$ is diff. on x then if

$$y = e^{u(x)} \longrightarrow \frac{dy}{dx} = e^{u(x)} \cdot u'(x)$$

Ex Find $\frac{dy}{dx}$ of the following:

1. $y = e^{x^2 + \sin 2x}$ $\longrightarrow \frac{dy}{dx} = e^{x^2 + \sin 2x} \cdot (2x + 2\cos 2x)$

2. $y = e^{\sec x} \cdot \sec e^x$

$$\frac{dy}{dx} = e^{\sec x} \cdot \sec e^x \cdot \tan e^x \cdot e^x + \sec e^x \cdot e^{\sec x} \cdot \sec x \cdot \tan x,$$

Ex If $y = \sin x$ find $\frac{d\, e^{2y}}{d \ln x^2}$

Let $u = e^{2y}$, $v = \ln(x^2)$

$$\therefore \frac{d e^{2y}}{d \ln x^2} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$$

$$= 2e^{2y} \cdot \cos x \cdot \frac{x^2}{2x} = x e^{2y} \cdot \cos x.$$

Applications on Derivatives — تطبيقات على المشتقة .

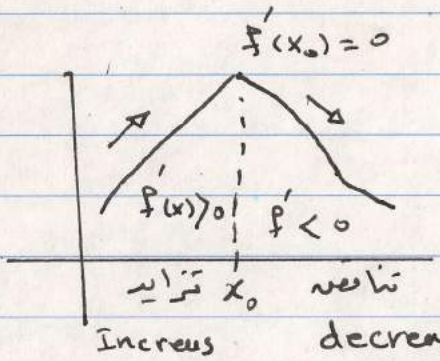
. Curve Skeching — . رسم الدوال

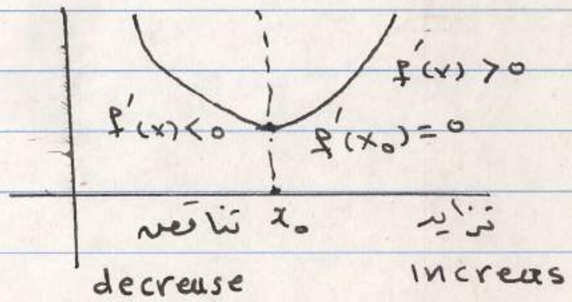
لتكن $y=f(x)$ دالة مستمرة في الفترة I وقابلة للاشتقاق
في كل نقاط الفترة I فان:

1. $f(x)$ متزايدة في I اذا كانت $\forall x\in I$, $f'(x)>0$
2. $f(x)$ متناقصة في I اذا كانت $\forall x\in I$, $f'(x)<0$

. Defn Critical point x_0 of a function $f(x)$ is the value of x where $f'(x_0)=0$

لتكن $f(x)$ دالة مستمرة عند النقطة x_0
تكون النقطة x_0 نقطة نهاية عظمى محلية اذا كانت الفترة المجاورة لها من جهة اليمين متناقصة ومن جهة اليسار متزايدة
والعكس اذا كانت الفترة المجاورة للنقطة x_0 متناقصة من اليسار ومتزايدة من جهة اليمين تسمى x_0 نقطة نهاية صغرى محلية
كما موضح بالشكل:

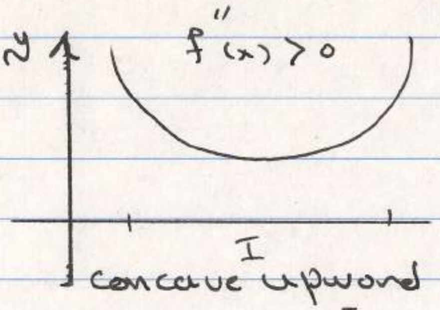




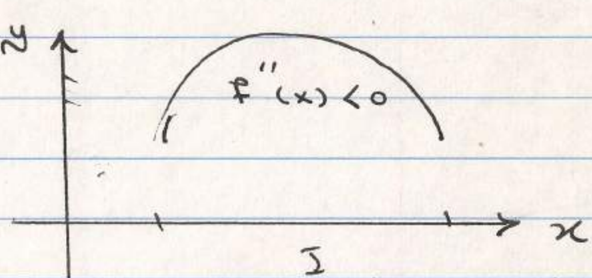
x_0 هي نقطة نهاية عظمى محلية — x_0 هي نقطة نهاية صغرى محلية

. اختبار المشتقة الثانية:
لتكن $f(x)$, $f'(x)$ و $f''(x)$ دوال مستمرة في I فان:

1. $f''(x)>0$, $\forall x\in I$, Then the graph of $f(x)$ is concave upword on I تقعر
2. $f''(x)<0$, $\forall x\in I$, Then the graph of $f(x)$ is concave down word on I تحدب

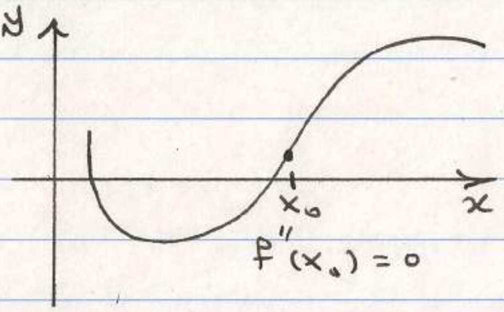


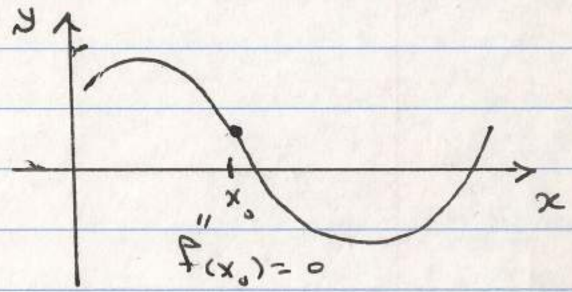
concave upword
تقعر



concave downword
تحدب

Defn: A point on the curve $y = f(x)$ where the concavity changes from up to downword (or vise versa) is called a point of inflection. نقطة انقلاب





Ex. 1 Discuss and Sketch the graph of function

$y = x^2 + 4$

- Domain and range: $y = x^2 + 4$ D : all $\mathbb{R}$

R : ~~all $\mathbb{R}$~~

$y \geq 4$

نقاط التقاطع - Intercepts

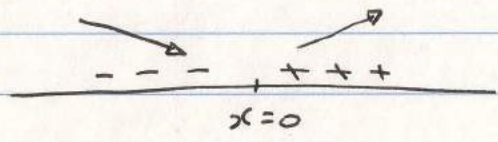
x - intercepts set $y = 0 \rightarrow x^2 + 4 = 0 \rightarrow x^2 = -4$

$\therefore$ no x - intercepts

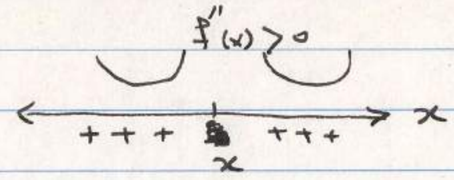
y - intercepts set $x = 0 \rightarrow y = 0 + 4 = 4$

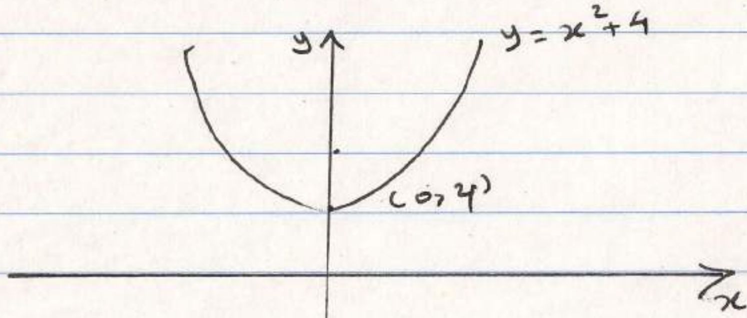
$(0, 4)$ is y - intercept.

1st der. test - $\frac{dy}{dx} = 2x \longrightarrow \frac{dy}{dx} = 0 \longleftrightarrow 2x = 0 \longleftrightarrow x = 0$



$f''(x) = 2$, $f''(x) \neq 0 \to$ no inflection points
and $f''(x) > 0$ concave upword

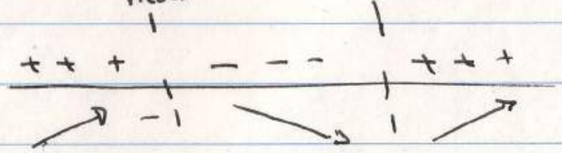
$f''(x)$ 



<u>Ex. 2</u> Discuss and sketch the function
$f(x) = x^3 - 3x + 2$

Solution:

- $D : \mathbb{R}$, $R : \mathbb{R}$
- Intercepts : x_int. at $y = 0 \to x^3 - 3x + 2 = 0$
 $\to (x-1)^2(x+2) = 0 \to x = 1, -2$
 $\therefore (1,0), (-2,0)$ are x-int.
- y-int., at $x = 0 \to y = 0 - 3(0) + 2 = 2$
 $\therefore (0, 2)$ is the y-int.
- 1st derivative test
 $\frac{dy}{dx} = 3x^2 - 3$, $\frac{dy}{dx} = 0 \longleftrightarrow 3x^2 - 3 = 0 \to$
 $x^2 = 1 \to$
 $x = \mp 1$

sigh of $f'(x)$ 

- 2d. derivative test
 $\frac{d^2y}{dx^2} = 6x \longrightarrow \frac{d^2y}{dx^2} = 0 \longleftrightarrow 6x = 0 \longrightarrow x = 0$

sigh of $f''(x)$