الكورس الثاني

الوحدات	الموضوع	المفردات	عدد الساعات
الوحدة الخامسة	بعض التوزيعات الاحتمالية (التوزيعات المتقطعة)	مفهوم التوزيع المنتظم المتقطع، الدالة التوزيعية، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل. توزيع ثنائي الحدين السالب، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	٧ أسابيع
		توزيع برنولي، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		توزيع ثنائي الحدين، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		التوزيع الهندسي الزائد، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		توزيع بواسون، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		التوزيع متعدد الحدود، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
الوحدة السادسة	بعض التوزيعات الاحتمالية (التوزيعات المستمرة)	مفهوم التوزيع المنتظم المستمر، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	٧ أسابيع
		التوزيع الطبيعي، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		التوزيع المعياري، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		نظرية الغاية المركزية.	
		التوزيع الأسي، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		توزيع كاما، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		توزيع بيتا، الوسط الحسابي والتباين، الدالة المولدة للعزوم حول نقطة الاصل.	
		توزيع مربع كاي + توزيع t + توزيع F.	

المراجع والمصادر:

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Chapter Four

Axiomatic Approach of Probability Theory

4.1 Probability Defined on Events

In this Chapter we develop a mathematical model of an experiment, which is defined mathematically. by three things.

1. Specifying the sample space S on which probability statements are made;
2. defining the events of interest; and
3. defining a numerical measure for a probability statement, which is assigned to the events.

The sample space S may be a finite or countably infinite set, then S is discrete. Some experiments have an uncountably infinite sample space which is called *continuous*.

The events may be either discrete or continuous, as with the sample space.

To each event E defined on a sample space S We shall assign a non negative real number which is called *probability* of E., denoted by P (E)

4.2. Axioms of Probability

In 1933, Kolmogorov gave the following definition of probability :

Let S be a sample space associated with a statistical experiment. Let F be a σ – field, which is a collection of subsets of S. A set function P defined on F is Called a *probability measure* (or simply probability) if it satisfies the following axioms:

Axiom 1.

$o \leq P(A) \leq 1$, for all $A \in F$,

Axiom 2.

$P(S) = 1$

Axiom 3.

For any sequence of mutually exclusive events $A_1, A_2,$ of F , that is, events for which $A_i A_j = \phi$ for $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Axiom 3. states that for any sequence of mutuall exclusive events, the probability of at least one of these even occurring is just the sum of their respective probabilitie This property is called *countable additivity*

Definition 4.2.1.

The triple (S, F, P) is called a *probability space,* where S is the sample space, F is a σ - field of subsets of S and P probability measure on F.

The probability assigned to S is 1, by Axiom 2 .

Sometimes other events $A \in F$ with $P(A) = 1$ will happen. If a statement holds for all points in A with $P(A) = 1$, then it is customary to say that the statement is true almost surely.

Proposition 4.2.1.

$P(\phi) = 0$

Proof.

Let $A_n = \phi$ for n = 1.2, ... Then $\bigcup_{n=1}^{\infty} A_n = \phi$. Then by Axiom 3, we have:

$$P(\phi) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(\phi)$$

This is true only when $P(\phi) = 0$.

Proposition 4.2.2.

If $A_1, A_2, \dots, A_n$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Proof :

Let $A_i = \phi$ for $i = n+1, n+2, \dots$. By Axiom (3) we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i)$$

$$+ P(\phi) + P(\phi) + \cdots = \sum_{i=1}^{n} P(A_i).$$

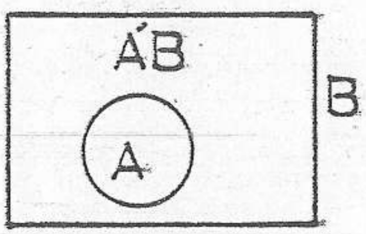
Theorem 4.2.1.

1. For any event $A \in F$, $P(\text{not } A) = P(A') = 1 - P(A)$.
2. For any two events $A \in F$ and $B \in F$, If $A \subset B$, then $P(A) \le P(B)$.

Proof:

(1) We have $S = A \cup A'$, but A and A' are disjoint, then $P(S) = P(A) + P(A')$ by Proposition 4.2.2. By Axiom (2), we have $P(S) = 1$. So
$P(A) + P(A') = 1$
or $P(A') = 1 - P(A)$.

(2) we have $B = A \cup (A'B)$ (see Fig 4.1) union of mutually exclusive events.
Then by Proposition 4.2.2, we have



A⊂B
Fig. 4.1

$P(B) = P(A) + P(A'B)$ ..... (4.2.1)
The result follows because $P(A'B) \geq 0$.

From Equation (4.2 1) we have
$P(A'B) = P(B/A) = P(B) - P(A)$, if $A \subset B$
In general, For any two events A and B, We have
$P(B / A) = P(B) - P(AB)$ .... (4.2.2)

Theorem 4.2.2.

For any two events $A \in F$ and $B \in F$ we have
$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B) - P(AB)$.

Proof:

From Fig 4.2, we have
$A \cup B = (A/B) \cap (A\ B) \cup (B/A)$ which are unions of mutually exclusive events. Then by Proposition 4.2.2, we have

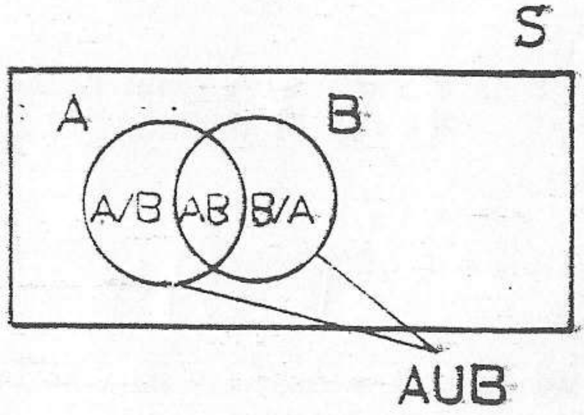


Fig 2.4

Some Special Distribution :-

① Uniform Distribution (Discrete) $X \sim U(K)$

It is said that arandom variable X has a Uniform distribution with parameter K if X has a discrete distribution for which the p.m.f. is as follows:

$$P[X=x] = f(x) = \frac{1}{K} I_{[1,2,\dots,K]}(x) = \begin{cases} \frac{1}{K} & \text{for } x=1,2,\dots,K \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \mu_x = E(x) = \frac{K+1}{2}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{K^2-1}{12} = \frac{K-1}{6} \cdot \mu_x \quad \text{where } \mu_x = \frac{1}{K}\sum_{1}^{K} x = \frac{1}{K}\frac{K(K+1)}{2}$$

$$\text{M.g.f} = M_X(t) = E(e^{tX}) = \frac{1}{K}\sum_{X=1}^{K} e^{tX} = \frac{1}{K}\sum_{X=1}^{K} \cancel{Z}^x \quad \text{where } Z = e^t$$

$$= \frac{\cancel{Z}}{K}\frac{1-Z^K}{1-Z} = \begin{cases} \dfrac{e^t(1-e^{tK})}{K(1-e^t)} & t>0 \\ 0 & \text{o.w.} \end{cases}$$

② Bernoulli Distribution: $X \sim Ber(P)$

It is said that a random variable X has a Bernoulli dist. with parameter P if X has adiscrete distribution for which the p.m.f is as follows:

$$P[X=x] = f(x) = P^x q^{1-x} I_{[0,1]}(x) = \begin{cases} P^x q^{1-x} & \text{for } x=0,1 \\ & q=1-P \\ & 0 \le P \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = M_x = E(x) = P$$

$$var(x) = Pq$$

$$\text{M.g.f.} = M_X(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} P^x q^{1-x} = \begin{cases} Pe^t + q & t>0 \\ 0 & \text{o.w.} \end{cases}$$

Comment :

1. $F_X(x)$ is disContinuous at $x = 1, 2, \dots, k$.

2. The size of the step or jump $1, 2, \dots, k$ equals $\frac{1}{k}$.

3. Raw moments

$$\mu_1' = E(x) = \sum_{x=1}^{k} x \, P[X=x] = \sum_{x=1}^{k} x \cdot \frac{1}{k} = \frac{1}{k} \frac{k(k+1)}{2} = \frac{k+1}{2}$$

$$\mu_2' = E(x^2) = \frac{(k+1)(2k+1)}{6}$$

$$\mu_3' = E(x^3) = \left(\frac{k(k+1)}{2}\right)^2 \cdot \frac{1}{k}$$

$$\mu_4' = E(x^4) = \frac{k+1}{30}\left(6k^3 + 9k^2 + k - 1\right)$$

4. Central Moments

$$M_1 = 0$$

$$M_2 = Var(X) = E(x^2) - (E(X))^2 = \frac{k^2-1}{12}$$

$$M_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = 0$$

$$M_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{(k^2-1)(3k^2-7)}{240}$$

5. Skewness Coefficient

$$B_1 = \frac{M_3^2}{M_2^3} = 0 \Rightarrow \gamma_1 = \sqrt{B_1} = 0$$ unskewed. symmetric.

6. Kurtios Coefficient

$$B_2 = \frac{M_4}{M_2^2} = \frac{3}{5}\left(1 - \frac{4}{k^2-1}\right)$$

$\therefore \gamma_2 = B_2 - 3 \Rightarrow \gamma_2 < 0$ Platykurtios for all k.

③ Binomail Distribution

It is said that a random variable X has a Binomail dist. with parameters n and p if X has a discrete dist. for which the p.m.f is as follows:

$$P[X=x] = f(x) = f(x|n,p) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x=0,1,\dots,n \\ & q=1-p \\ & 0 \le p \le 1 \\ 0 & \text{o.w.} \end{cases}$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$ and $X \sim B(n,p)$.

$\text{Mean} = \mu_x = np$

$\text{var}(x) = npq$

$M_x(t) = (pe^t + q)^n$.

Example: Suppose that a r.v. X has a uniform dist. on the six integers 2, 3, 4, 5, 6, 7. Find the m.g.f, mean, variable. of this distribution.

Solution

$X \sim U(6)$

$$f(x) = \begin{cases} \frac{1}{6} & \text{for } x = 2,3,4,5,6,7 \\ 0 & \text{o.w.} \end{cases}$$

$\text{Mean} = \dfrac{k+1}{2} = \dfrac{7}{2}$

$\text{Var} = \dfrac{k^2-1}{12} = \dfrac{35}{12}$

$M_x(t) = \dfrac{e^t(1-e^{6t})}{6(1-e^t)}$

Comment :

1. $E\left(\frac{x}{n}\right) = p$, and $var\left(\frac{x}{n}\right) = \frac{pq}{n}$

2. $Mean = E(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np \sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$

$$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y} = np \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y} =$$

(let $y = x-1 = 0$, let $m = n-1$)

$$= np\,(p+q)^m = np.$$

by using Binomail Theorem$\left(\sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = (a+b)^n \right)$

3. $Variance = E(x^2) - (E(x))^2 = E(x(x-1)+x) - (np)^2$

$$= np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq.$$

4. Moment Generating Function $(m.g.f) = M_x(t) = E(e^{tx})$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

5. Raw Moment

$\mu'_1 = E(x) = np$

$\mu'_2 = E(x^2) = np + n^2p^2 - np^2$

$\mu'_3 = E(x^3) = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$\mu'_4 = E(x^4) = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

6. Central Moment

$M_1 = 0$, $M_2 = npq$, $M_3 = npq(q-p)$, $M_4 = 3(npq)^3 + npq(1-6pq)$

7. Skewed and Kurtios Coefficients.

$B_1 = \frac{(q-p)^2}{npq} \Rightarrow \gamma_1 = \sqrt{B_1} = \frac{q-p}{\sqrt{npq}}$, $B_2 = 3 + \frac{1-6pq}{npq} \Rightarrow \gamma_2 = \frac{1-6pq}{npq}$

Example: if $X \sim B(7, \frac{1}{2})$ Find

ⓐ M.g.f, Mean and variance of the r.v. X.

ⓑ $P[0 \leq x \leq 1]$, $P[X=5]$, Skewedness and Kurtosis Coefficient

Solution: $n=7$, $P=\frac{1}{2}$, $q=\frac{1}{2}$

ⓐ $$M_x(t) = (Pe^t + q)^n = \begin{cases} \left(\frac{1}{2}e^t + \frac{1}{2}\right)^7 & t>0 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = E(x) = 7 \cdot \frac{1}{2} = \frac{7}{2} = 3.5$$

$$\text{Variance} = npq = 7 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{4}$$

ⓑ $$P[0 \leq x \leq 1] = \binom{7}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{7-0} + \binom{7}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{7-1} = \frac{8}{2^7}$$

$$P[X=5] = \binom{7}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{7-5} = \frac{21}{2^7}$$

$$\gamma_1 = 0$$

$$\gamma_2 = \frac{1 - 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{7\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = -0.28 < 0$$ platykurtosis for all K.

Example if x be a r.v. has Binomail distributed as $X \sim B(9, \frac{1}{3})$

Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$.

Solution: $$f(x) = \begin{cases} \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x} & x = 0,1,2,\dots,9. \\ 0 & \text{o.w.} \end{cases}$$

$n=9$, $P=\frac{1}{3}$, $q=\frac{2}{3}$

$\text{Mean} = nP = 3$

$\text{Variance} = npq = 2 \Rightarrow \sigma = \sqrt{2}$

$$P(3-2\sqrt{2} < X < 3+2\sqrt{2}) = P(0.17 < x < 5.83) = \sum_{x=1}^{5} f(x)$$

$$= \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 + \dots + \binom{9}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^4 =$$

$$=$$

(4) Hyper Geometric Distribution : $X \sim HG(A, B, n)$

It is said that X has a hypergeometric distribution with parameters A, B and n if a random variable X has a discrete distribution which the p.m.f is as follows: $X \sim HG(r, r_1, n)$

$$P[X=x] = f(X|A,B,n) = \begin{cases} \dfrac{\binom{B}{x}\binom{A-B}{n-x}}{\binom{A}{n}} = \dfrac{\binom{r_1}{x}\binom{r-r_1}{n-x}}{\binom{r}{n}} & \text{for } x = 0,1,2,\dots,n \quad x \geq \max\{0, n-B\} \quad x \leq \min\{n, A\} \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \mu_x = \frac{nB}{A} = \frac{nr_1}{r}$$

with replacement

$$\text{Variance} = \sigma_x^2 = npq\frac{A-n}{A-1} \quad \text{where } p = B/A.$$

$$= n\left(\frac{r_1}{r}\right)\left(1-\frac{r_1}{r}\right) = n\left(\frac{r_1}{r}\right)\left(1-\frac{r_1}{r}\right)\left(1-\frac{n-1}{r-1}\right)$$ without replacement $p = \frac{r_1}{r}$.

(Pascal Distribution)

(5) Geometric Distribution : $X \sim G(p) \sim NB(1, p)$

$$P[X=x] = f(x) = \begin{cases} p(1-p)^x = pq^x & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \mu_x = q/p$$

$$\text{var}(x) = q/p^2$$

$$\text{M.g.f} = M_x(t) = E(e^{tx}) = p\sum_{x=0}^{\infty}(qe^t)^x = p\sum_{x=0}^{\infty} Z^x = p\left(\frac{1}{1-Z}\right)$$

$$= \begin{cases} \dfrac{p}{1-qe^t} & t < \log\left(\frac{1}{q}\right) \\ 0 & \text{o.w.} \end{cases}$$

Comment :- Bernoulli

1. Raw moment

$\mu_1' = P$, $\mu_2' = P$, $\mu_3' = P$, $\mu_4' = P$

2. Central Moment

$M_1 = 0$, $M_2 = pq$, $M_3 = pq(q-p)$, $M_4 = pq(1-3pq)$

3. Skewdness Coefficient

$$\gamma_1 = \sqrt{B_1} = \sqrt{\frac{(q-p)^2}{pq}} = \frac{q-p}{\sqrt{pq}}$$

4. Kurtibs Coefficient

$$\gamma_2 = B_2 - 3 = \frac{1-3pq}{pq} - 3 = \frac{1}{pq} - 6$$

Example :- Assume $X \sim Ber(P)$, obtain Coefficient of skewness and Kurtosis of X for $P = 0.35$, $P = 0.85$.

Solution $P = 0.35$, $q = 0.65$

$$\gamma_1 = \sqrt{B_1} = \frac{q-p}{\sqrt{pq}} = 0.629$$ positively skewness

$$\gamma_2 = B_2 - 3 = \frac{1}{pq} - 6 = -1.6$$ playtkurtibs .

$P = 0.85$, $q = 0.15$

$$\gamma_1 = \frac{-0.7}{0.357} = -2$$ Negatively skewness

$$\gamma_2 = B_2 - 3 = 4.84$$ loptkurtibs .

6) Negative Binomail distribution: $X \sim NB(r, P)$

It is said that a r.v. X has a negative Binomail distribution with parameters r and p if X has a discrete distribution for which the p.m.f is as follows:

$$f(x|r,p) = P[X=x] = \begin{cases} \binom{-r}{x} p^r (-q)^x & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \binom{r+x-1}{x} p^r q^x & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \mu_x = \frac{rq}{p}$$

$$Var(x) = \sigma^2 = \frac{rq}{p^2}$$

$$m.g.f = M_x(t) = \begin{cases} \left(\frac{p}{1-qe^t}\right)^r & \text{for } t < \log\left(\frac{1}{q}\right) \\ 0 & \text{o.w.} \end{cases}$$

Comments:

① Raw Moment

$$E(x) = M_x'(t=0) = \frac{rq}{p}$$

$$E(x^2) = M_x''(t=0) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p}$$

$$E(x^3) = M_x'''(t=0) =$$

$$E(x^4) = M_x''''(t=0) =$$

② Central moment

$$M_1 = 0, \quad M_2 = Var(x) = \frac{rq}{p^2}, \quad M_3 = \frac{q(1+q)}{p^3}, \quad M_4 = \frac{q}{p^4}(p^2+9q)$$

③ skewedness and Kurtios Coefficient

$$B_1 = \frac{(1+q)^2}{q} \Rightarrow \gamma_1 = \frac{1+q}{rq} > 0, \quad B_2 = \frac{p^2}{q} + 9 \Rightarrow \gamma_2 = \frac{p^2}{q} + 6 > 0$$

lept kurtios

Positively skewedness

⑦ Poisson Distribution $X \sim Poi(\lambda)$

It is said that a r.v. X has a poisson Distribution with parameter λ if X has a discrete distribution for which the p.m.f ~~is~~ as Follows

$$f(x|\lambda) = P[X=x] = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & \text{for } x=0,1,\dots, \text{ and } \lambda>0 \\ 0 & o.w. \end{cases}$$

$$Mean = \mu_1' = E(x) = \sum_{x=0}^{\infty} x\,\frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}\cdot e^{\lambda} = \lambda$$

$$variance = \mu_2' - (\mu_1')^2 = \lambda^2+\lambda-\lambda^2 = \lambda$$

$$M.g.f = \begin{cases} e^{\lambda(e^{t}-1)} & t>0, \lambda>0 \\ 0 & o.w. \end{cases}$$

Comment:

① Raw moment

$\mu_1' = \lambda$, $\mu_2' = \lambda^2+\lambda$, $\mu_3' = \lambda^3+3\lambda^2+\lambda$, $\mu_4' = \lambda^4+6\lambda^3+7\lambda^2+\lambda$

② Central Moment

$M_1 = 0$, $M_2 = \lambda$, $M_3 = \lambda$, $M_4 = 3\lambda^2+\lambda$

③ skewedness Coefficient

$B_1 = \dfrac{1}{\lambda} \Rightarrow \gamma_1 = \dfrac{1}{\sqrt{\lambda}} > 0$ positively skewedness

④ Kurtios Coefficient

$B_2 = 3+\dfrac{1}{\lambda} \Rightarrow \gamma_2 = \dfrac{1}{\lambda} > 0$ leptokurtios

⑤ $\displaystyle\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!} = e^{\lambda}$, $\lambda>0$.

7 The Uniform Distribution (Continuous) $X \sim U(a,b)$

let X be r.v. from Interval $[a,b]$ $-\infty < a < x < b < \infty$
and has the p.m.f of x as follow

$$f(x) = \frac{1}{b-a} I_{x \in [a,b]}(x) = \begin{cases} \frac{1}{b-a} & \text{for } a<x<b \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \mu_x = \frac{a+b}{2}$$

$$var(x) = \frac{(b-a)^2}{12}$$

$$m.g.f = M_x(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

if $a=0 \wedge b=1$ Then $X \sim U(0,1) \Rightarrow \text{Mean} = \frac{1}{2}, \sigma^2 = \frac{1}{12}$

$$M_x(t) = \begin{cases} \frac{e^t}{t} & t>0 \\ 0 & \text{o.w.} \end{cases}$$

Comments:

① Raw moment:

$\mu_1' = E(x) = \frac{a+b}{2}$, $\mu_2' = \frac{b^2+ab+a^2}{3}$, $\mu_3' = \frac{b^3+ab^2+a^2b+a^3}{4}$

$\mu_4' = \frac{b^5-a^5}{5(b-a)}$, $E(x^r) = \frac{1}{b-a}\int_a^b x^r dx = \frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}$, $r=1,2,$ —

② Central moment $M_r = E(x-\mu)^r = \frac{(b-a)^r}{2^r(r+1)}$ if r is even, $=0$ if r is odd

$M_1 = 0$, $M_2 = \mu_2' - (\mu_1')^2 = \frac{(b-a)^2}{12}$, $M_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$
$= 0$

$$M_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4}{80}$$

③ skewedness Coefficient $\gamma_1 = \sqrt{B_1} = \sqrt{\frac{M_3^2}{M_2^3}} = 0$ unskewedness symmetric

④ Kurtosis Coefficient $\gamma_2 = B_2 - 3 = \frac{M_4}{M_2^2} - 3 = 1.8 - 3 = -1.2 < 0$ Platykurtic

Example if $X \sim Poi(\lambda)$ and $P[X=1] = P[X=2]$, find $P[X=4]$?

Solution

$$P[X=1] = P[X=2]$$

$$\Rightarrow \quad \lambda e^{-\lambda} = \frac{\lambda^2}{2} e^{-\lambda}$$

$$\Rightarrow \quad \lambda = 2$$

$$\therefore P[X=4] = \frac{2^4 e^{-2}}{4!} = \begin{cases} \frac{e^{-2}}{3} & x=4 \\ 0 & o.w. \end{cases}$$

Example if M.g.f of X is $e^{4(e^t-1)}$. find the Mean and the variance of X and $P[X=3]$.

Solution: $\lambda = 4$ because this m.g.f of poisson Distribution

$$f(x) = \begin{cases} \frac{4^x e^{-4}}{x!} & x=0,1,2,\dots \\ 0 & o.w. \end{cases}$$

The mean = The variance = $\lambda = 4$.

$$P[X=3] = \begin{cases} \frac{4^3 e^{-4}}{3!} & x=3 \\ 0 & o.w. \end{cases}$$

② Normal Distribution (Laplace - Gausse) $X \sim N(\mu, \sigma^2)$

The random variable X has a normal dist. if its p.d.f is defined by

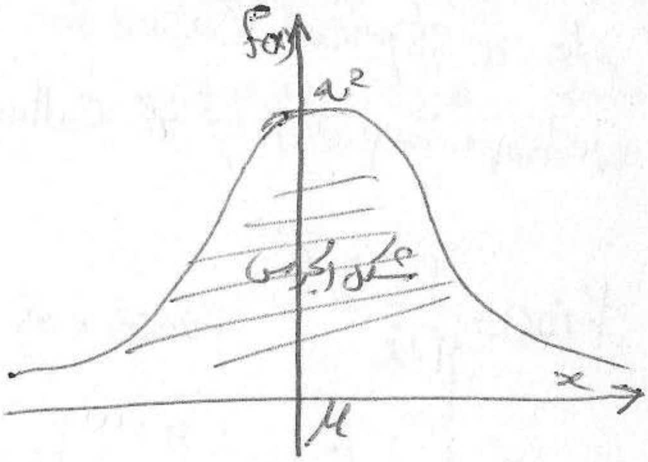
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \underset{x \in (-\infty, \infty)}{I(x)}$$

$$= \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & \text{for } -\infty < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

μ is mean
σ is standard divation

we Prove that $F(x)$ is p.d.f.

Now we evalute the integral



$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$

$$\therefore I = \int_{-\infty}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma} \cdot dz \Rightarrow I > 0 \text{ if } I^2 = 1 \text{ then } I = 1.$$

$$I^2 = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dy\, dx$$

let $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$.

$$\therefore I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi} \cdot 2\pi = 1.$$

Then $I = o$ $\therefore$ $F(x)$ is p.d.f of X.

Mean = μ

$Var(x) = \sigma^2$

m.g.f of X = $M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

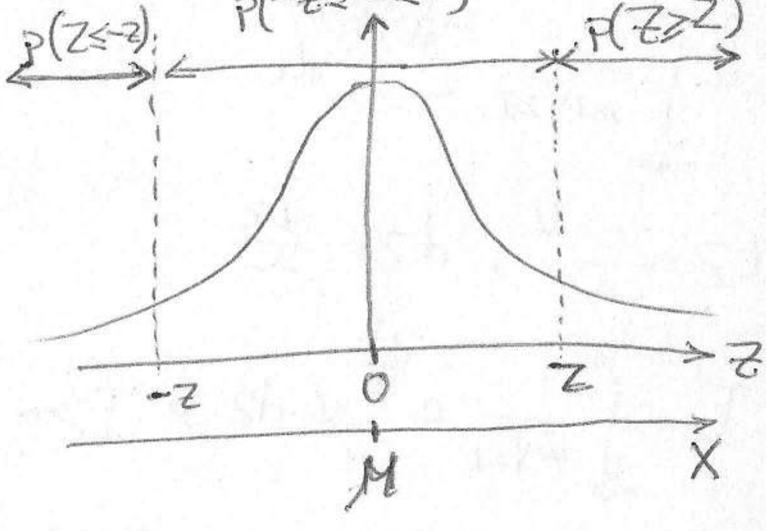
$\gamma_1 = 0$ symmetric (no skewedness)

$\gamma_2 = 3$ Normal. (Kurtosis)

③ Standard Normal Distribution $Z \sim N(0,1)$

As a espical Case for normal dist. when $\mu = 0$, & $\sigma^2 = 1$.
we have a p.d.f of Z Called standard dist. of Z

$$f(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} & -\infty < z < \infty \\ 0 & o.w. \end{cases}$$



Mean = 0 , Var(Z) = 1.

m.g.f of Z = $M_Z(t) = e^{\frac{t^2}{2}}$

when $Z = \frac{x - np}{\sqrt{npq}} \Rightarrow N(0,1) \sim B(n,p)$.

① if p & $q \geq 0.5$ or $npq \geq 9$

② if $n \geq 20$, $np \geq 10$, $nq \geq 10$

$$\lim_{n \to \infty} P(a \le Z \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}} dz.$$

Also, we have a relation between Normal of Poisson distributions ① $\lim_{\lambda \to \infty} P\left(a \leq \frac{x-\lambda}{\sqrt{\lambda}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}} dz$

② if $\lambda \geq 10$ or $\lambda \geq 15$.

④ The exponential Distribution $X \sim \exp\left(\frac{1}{\theta}\right) \sim \exp(\lambda)$

let X be a r.v. is defined by:

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} I_{x \in [0,\infty)}(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \text{For } 0 \leq x < \infty \\ 0 & \text{o.w} \end{cases} = \begin{cases} \lambda e^{-\lambda x} & \lambda > 0 \\ 0 & \text{o.w.} \end{cases}$$

Mean if $\boxed{\frac{1}{\theta} = \lambda} = \theta = \frac{1}{\lambda}$

variance $= \theta^2 = \frac{1}{\lambda^2}$

m.g.f of X $= \begin{cases} \frac{1}{1-\theta t} & t < \frac{1}{\theta} \\ 0 & \text{o.w.} \end{cases} = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ 0 & \text{o.w.} \end{cases}$

Median $= \mu \ln(2) < \mu$

$= \frac{\ln(2)}{\theta}$

Raw moment $= \mu'_r = E(X^r) = \frac{r!}{\lambda^r} = r!\,\theta^r$ where $r = 1, 2, \ldots$

The integral of this p.d.f is 1, from the definition of gamma function

$$\int_0^\infty x^{\alpha-1}e^{-\beta x}\,dx = \frac{\Gamma\alpha}{\beta^\alpha}$$

<u>Raw moment</u>

$$\mu_r' = E(x^r) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^r} = \frac{\Gamma(\alpha+r)}{\beta^r\,\Gamma\alpha}$$

$$\text{Mean} = E(x) = \frac{\alpha}{\beta}$$

$$\text{Variance} = \frac{\alpha}{\beta^2}$$

$$\text{m.g.f of } X = M_X(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$$

it can be seen that an exponential distribution with parameter β is the same as gamma distribution with parameter $\alpha=1$ and β.

$$\text{Mean} = \frac{1}{\beta}$$

$$\text{Variance} = \frac{1}{\beta^2}$$

$$\text{m.g.f of } X = \frac{\beta}{\beta-t} \quad \text{for } t<\beta.$$

⑤ Gamma Distribution $X \sim \text{Gamma}(\alpha, \beta)$

For any positive number t, the value Γt be defined

$$\Gamma t = \int_0^\infty x^{t-1} e^{-x}\, dx . \quad t > 0$$ this function Γ is

called the gamma function and this integration is finite.

Properties of gamma function

① $\Gamma t = (t-1)\Gamma t-1$

④ Mode of $X = \dfrac{\beta(\alpha-1)}{\beta}$

② $\Gamma n = (n-1)!\,\Gamma(1) = (n-1)!$

③ $\Gamma \frac{1}{2} = \sqrt{\pi}$

Proof that when $x = \frac{1}{2}y^2$ and $dx = y\,dy$.

$$\Gamma \tfrac{1}{2} = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx = 2^{\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}y^2}\, dy = 2^{1/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} = \sqrt{\pi}.$$

it is said that a r.v. X has a Gamma distribution with parameters α and β ($\alpha > 0$ and $\beta > 0$) if X has a p.d.f $f(x|\alpha,\beta)$ as follows:

$$f(X|\alpha,\beta) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x} & \text{For } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$= \{$$

Example Calculate the ~~following~~ following:

$$\int_0^\infty t^4 e^{-t}\, dt = \Gamma(5) = 24$$

$$\int_0^\infty x^5 e^{-x}\, dx = \Gamma(7) = 720$$

$$\int_0^\infty \frac{e^{-x}}{x^{1/2}}\, dx = \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Gamma(7) = 720$$

$$\Gamma(4.5) = \Gamma(3.5+1) = 3.5\,\Gamma 3.5 = 3.5(2.5)\Gamma 1.5 = (3.5)(2.5)(1.5)\Gamma\tfrac{1}{2}$$
$$= (3.5)(2.5)(1.5)\sqrt{\pi}$$

$$\Gamma(2.5) = 1.5\,\Gamma 0.5 = 1.5\sqrt{\pi}\,.$$

Remarks ① if ~~X~~ X has a Pareto distribution with parameter x_0 and α ($x_0 > 0$ and $\alpha > 0$) if has a Continuous distribution for which the p.d.f is

$$f(x/x_0,\alpha) = \begin{cases} \dfrac{\alpha x_0^{\alpha}}{X^{\alpha+1}} & \text{for } x > x_0 \\ 0 & x \leq x_0 \end{cases}$$

② if X has a pareto dist. then $\log(X/x_0) \sim Exp(\alpha)$

③ if X has a weibull distribution with parameters $a > 0$ and $b > 0$ if X has a p.d.f as

$$f(x|a,b) = \begin{cases} \dfrac{b}{a^b} x^{b-1} e^{-(x/a)^b} & \text{for } x > 0 \\ 0 & X \leq 0 \end{cases}$$

④ if X has a weibull dist. then $X^b \sim Exp(\beta = a^{-b})$.

⑥ Beta Distribution :- $X \sim Beta(\alpha, B)$

It is said that a r.v. X has a beta distribution with parameters ($\alpha > 0$ and $B > 0$) if X has a p.d.f as

$$f(x/\alpha, B) = \begin{cases} \frac{\Gamma(\alpha+B)}{\Gamma(\alpha)\Gamma(B)} x^{\alpha-1}(1-x)^{B-1} & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

where $$\int_0^1 x^{\alpha-1}(1-x)^{B-1}dx = \frac{\Gamma(\alpha)\Gamma(B)}{\Gamma(\alpha+B)} = B(\alpha, B) \text{ beta function}$$

From the definition of the gamma function, it follows that

$$\Gamma(\alpha)\Gamma(B) = \int_0^\infty u^{\alpha-1}e^{-u}du \int_0^\infty v^{B-1}e^{-v}dv = \int_0^\infty\int_0^\infty u^{\alpha-1}v^{B-1}e^{-(u+v)}du\,dv.$$

let $x = \frac{u}{u+v}$ and $y = u+v$ Then $u = xy$ and $v = (1-x)y$

if $\alpha = 1$ and $B = 1$ ~~it can seen the~~ the Beta distribution $\Longrightarrow$ Uniform (0,1)

(i.e) $Beta(1,1) \sim U(0,1)$

① Mean $= \frac{\alpha}{\alpha+B}$

② Variance $= \frac{\alpha B}{(\alpha+B)^2(\alpha+B+1)}$

③ Raw moment $E(X^r) = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{(\alpha+B)(\alpha+B+1)\cdots(\alpha+B+r-1)}$

Example Calculate that following $B(\alpha,\beta)=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$B(3,4)=\frac{(3-1)!\,(4-1)!}{(7-1)!}=\frac{1}{60}$$

$$B(\tfrac{1}{2},\tfrac{1}{2})=\pi$$

$$B(n,2)=\frac{(n-1)!\,1!}{(n+1)!}=\frac{1}{n(n+1)}$$

$$B(n,1)=\frac{1}{n}$$

Example Find the Expected Ratio of defective Production and the variance if this Ratio following this distribution function as
$$f(x)=\begin{cases}6(1-x)^5 & 0<x<1\\ 0 & \text{o.w.}\end{cases}$$

Answer
$\because B(n,1)=\frac{1}{n} \Rightarrow 6=1/B(1,6) \Rightarrow X\sim Beta(1,6)$

$\alpha=1,\ \beta=6 \Rightarrow Mean=\mu=\dfrac{\alpha}{\alpha+\beta}=\dfrac{1}{2}$.

$$Variance=\sigma^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{1}{20}$$

Example Suppose that X has a beta distribution with parameters α and β. Show that $(1-X)$ has a beta distribution with parameters β and α.

Answer By using the definition of Beta distribut.

⑦ Chi-Square Distribution : $X \sim Chi(r) \sim \chi^2_{(r)}$

where r degree of freedom.

let $X \sim Gamma\left(\alpha = \frac{r}{2}, \beta = \frac{1}{2}\right) \Rightarrow X \sim \chi^2_{(r)}$

let $X_1, X_2, \dots, X_r$ are random variables, each one has standard normal distribution $X_i \sim N(0,1)$, then $X = X_1^2 + X_2^2 + \dots + X_r^2$ has chi-square distributed which has a p.d.f as follows:

$$f(x) = \begin{cases} \dfrac{x^{\frac{r}{2}-1} e^{-x/2}}{\Gamma\left(\frac{r}{2}\right)(2)^{\frac{r}{2}}} & \text{for } x>0 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Mean} = \frac{\alpha}{\beta} = r$$

$$\text{Variance} = \frac{\alpha}{\beta^2} = 2r$$

$$\text{M.g.f of } X = M_X(t) = \begin{cases} (1-2t)^{-\frac{r}{2}} & \text{for } t > \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

Remark

① $\left(N(0,1)\right)^2 \sim \chi^2_{(1)}$

② Mode of $X = r-2$

③ $\gamma_1 = \sqrt{\frac{2}{r}} > 0$ Positive skewness

④ Raw moments $E(X^n) = 2^n \dfrac{\Gamma\left(\frac{r}{2}+n\right)}{\Gamma\left(\frac{r}{2}\right)}$

Example: if $X \sim \chi^2_{(n)}$ and $Y = \ln x$, find the m.g.f of Y around the origin point.

Solution

m.g.f of Y = $M_Y(t) = E(e^{tY}) = E(e^{t\ln x}) = E(e^{\ln x^t}) = E(X^t)$

$$= \frac{1}{\Gamma(\frac{n}{2}) 2^{n/2}} \int_0^\infty x^t \cdot x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}}} \int_0^\infty x^{(\frac{n}{2}+t)-1} e^{-\frac{x}{2}} dx$$

The integration is represent Gamma distribution with parameters Gamma$(\alpha = \frac{n}{2}+t, \beta = \frac{1}{2})$ and the value of this integration is $\Gamma(\frac{n}{2}+t)(\frac{1}{2})^{\frac{n}{2}+t}$

finally, $M_Y(t) = \frac{1}{\Gamma(\frac{n}{2}) \cdot 2^{n/2}} \cdot \Gamma(\frac{n}{2}+t) \cdot (\frac{1}{2})^{\frac{n}{2}+t} = \frac{\Gamma(\frac{n}{2}+1)}{2^t \Gamma(\frac{n}{2})}$

Example let $X_1 \sim \chi^2_{(3)}$ & $X_2 \sim \chi^2_{(5)}$ find the distribution of $Y = X_1 + X_2$ and calculate Mean, variance and skewed of Y. also $P(2<Y<6)$. (i.e) X_1 & X_2 are indep.).

Solution

$M_Y(t) = E(e^{t(X_1+X_2)}) = E(e^{tX_1}) \cdot E(e^{tX_2}) = (1-2t)^{-r_1} \cdot (1-2t)^{-r_2} = (1-2t)^{-(r_1+r_2)} = (1-2t)^{-8}$

Mean of $Y = r = 8$, $\sigma^2_y = 2r = 16$, $\gamma_1 = \sqrt{\frac{2}{8}} = \sqrt{\frac{1}{4}} > 0$ positive skewedness

8) t- Distribution (student) $X \sim t(r)$

if $n < 30$ and Z is an r.v. that is $N(0,1)$, V is an random variable that $\chi^2_{(r)}$ and Z, V are independent

then
$$T = \frac{Z}{\sqrt{\frac{V}{r}}} = \begin{cases} \dfrac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} & \text{for } -\infty < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Remarks

① Mean $= 0$

② Variance $= \dfrac{r}{r-2}$, $r > 2$

③ if $r = 1 \Rightarrow t \sim$ Chaushy distribution

④ M.g.f is doesn't exist.

⑤ $X_1 \sim N(0,1)$, $X_2 \sim \chi^2_{(r)} \Rightarrow t = \dfrac{X_1}{\sqrt{\frac{X_2}{r}}}$; where X_1, & X_2 are indep.

⑥ if $n \to \infty \Rightarrow t_{(r)} \sim N(0,1)$

⑦ $\gamma_2 = \dfrac{3(r-2)}{r-4} - 3 = \beta_2 - 3$ where $\dfrac{n-2}{n-4} > 1$.

The F-fisher Distribution

let $X_1 \sim \chi^2_{(n_1)}$ independent to $X_2 \sim \chi^2_{(n_2)}$ such that

$F = \dfrac{X_1/n_1}{X_2/n_2}$ is distributed and is called F-fisher dis. with degree of freedom n_1 & n_2 as follows.

$$f(F) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \cdot \frac{F^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}.$$

Rawmoment

$$E(F^r) = \left(\frac{n_2}{n_1}\right)^r E(x_1^r)\,E(X_2^{-r})$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\Gamma\left(\frac{n_1}{2}+r\right)\cdot\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\frac{n_1}{2}\cdot\Gamma\frac{n_2}{2}} \qquad \text{where } \frac{n_2}{2} > r,\ r=1,2,\dots$$

$$\text{Mean} = E(F) = \left(\frac{n_2}{n_1}\right)\frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\frac{n_1}{2}\cdot\Gamma\frac{n_2}{2}} = \frac{n_2}{n_1}\cdot\frac{\left(\frac{n_1}{2}\right)\Gamma\frac{n_1}{2}\,\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\frac{n_1}{2}\left(\frac{n_2}{2}-1\right)\Gamma\left(\frac{n_2}{2}-1\right)}$$

$$= \frac{n_2}{n_2-2}, \quad n_2 > 2.$$

$$\text{Var}(F) = 2\left(\frac{n_2}{n_2-2}\right)^2 \cdot \frac{n_1+n_2-2}{n_1(n_2-4)}.$$

$$\text{Mode of } F \text{ is} = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$