Metric Space : ERWIN KREYSZIG 1978 Introductory functional analysis with 1

Def: let X be a non-empty set. A mapping application

$$d : X \times X \longrightarrow \mathbb{R} \quad \text{(the set of real numbers)}$$

is said to be a metric (or distance function) if d satisfies the following conditions:

M_1: $d(x,y) \geq 0 \quad \forall\, x,y \in X$.

M_2: $d(x,y) = 0 \iff x = y$.

M_3: $d(x,y) = d(y,x) \quad \forall\, x,y \in X$

M_4: $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$.

If d is a metric for X, then (X,d) is called metric space and $d(x,y)$ is called the distance between x and y.

Def: (Pseudo-metric)

A mapping $d: X \times X \longrightarrow \mathbb{R}$ is called a pseudo-metric (or semi-metric) for X if satisfies the above conditions M_1, M_3 and M_4 and the condition M_2' $d(x,x) = 0 \quad \forall\, x \in X$.

Thus for a pseudo metric $x = y \Rightarrow d(x,y) = 0$ but not conversely i.e $d(x,y) = 0$ even if $x \neq y$.

Remark: Every metric is a pseudo metric but converse is not necessarly true.

⊛ A subspace $(Y, \tilde{d})$ of (X, d) is obtained if we take subset $Y \subset X$ and restrict d to $Y \times Y$; thus the metric on Y is the restriction $\tilde{d} = d|_{Y \times Y}$. $\tilde{d}$ is called the metric induced on Y by d.

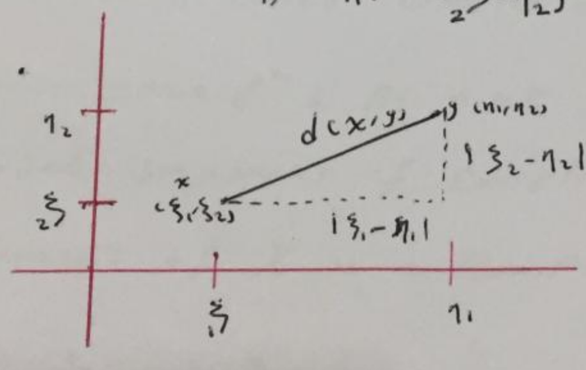
Examples:

1- Real line $\mathbb{R}$: This is the set of all real numbers, taken with the usual metric define by

$$d(x, y) = |x - y|.$$

2- Euclidean plane $\mathbb{R}^2$: The metric space $\mathbb{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers written $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, etc and Euclidean metric define by

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$$



3- Three-dimensional Euclidean space $\mathbb{R}^3$: This metric space consists of the set of ordered triples of real numbers $x = (\xi_1, \xi_2, \xi_3)$, $y = (\eta_1, \eta_2, \eta_3)$, etc and the Euclidean metric defined by

4. Euclidean space $\mathbb{R}^n$, unitary space $\mathbb{C}^n$, Complex plane $\mathbb{C}$

The previous examples are special cases of n-dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set of all ordered n-tuples of real numbers, written $x = (\xi_1, \xi_2, \dots, \xi_n)$, $y = (\eta_1, \eta_2, \dots, \eta_n)$ etc and the Euclidean metric defined by

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + \dots + (\xi_n - \eta_n)^2}$$

n-dimensional unitary space $\mathbb{C}^n$ is the space of all ordered n-tuples of complex numbers with metric defined by

$$d(x,y) = \sqrt{|\xi_1 - \eta_1|^2 + \dots + |\xi_n - \eta_n|^2}$$

When $n=1$ this is the complex plane $\mathbb{C}$ with the usual metric defined by $d(x,y) = |x-y|$.

$\mathbb{C}^n$ is sometimes called Complex Euclidean n-space.

5. sequence space ℓ^∞: As a set X we take the set of all bounded sequences of complex numbers; that is; every element of X is a complex sequence

$x = (\xi_1, \xi_2, \dots, \xi_i, \dots)$ briefly $x = (\xi_i)_{i=1,2,\dots}$

$|\xi_i| \le C_x$, where C_x is real number which may depend on x, but not depend on i, we choose the metric define by

$$d(x,y) = \sup_{i \in N} |\xi_i - \eta_i| , \quad N = \{1, 2, \dots\}$$

Note: Every metric is a pseudo-metric but a pseudo-metric is not necessarily a metric.

lemma: let (X, d) be a metric space, and $x, y, z \in X$. Then

$$d(x,y) \geq |d(x,z) - d(z,y)|$$

Proof: $d(x,z) \leq d(x,y) + d(y,z)$ by M_4

$\Rightarrow$ $d(x,z) - d(y,z) \leq d(x,y)$

$\Rightarrow$ $d(x,z) - d(z,y) \leq d(x,y)$ by M_3 --- ①

also, $d(z,y) \leq d(z,x) + d(x,y)$

$\Rightarrow$ $d(z,y) - d(z,x) \leq d(x,y)$

$\Rightarrow$ $d(z,y) - d(x,z) \leq d(x,y)$ --- (2)

From ① and ② we get

$$d(x,y) \geq |d(x,z) - d(z,y)|$$

□

Some Important inequalities:

[A]: $|z+w| \leq |z| + |w|$; $z, w \in \mathbb{C}$ The complex number

[B]: The generalization of inequality [A], we have

$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$.

[C] $$\frac{|z+w|}{1+|z+w|} \leq \frac{|z|}{1+|z|} + \frac{|w|}{1+|w|}, \quad z, w \in \mathbb{C}$$

[D] Holder's inequality

If a_i, b_i $i = 1, 2, \ldots, n$ are non-negative real numbers then $\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q \right)^{1/q}$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

[E] Cauchy - Schwarz inequality:

let $z = (z_1, z_2, ---, z_n)$ and $w = (w_1, w_2, ---, w_n)$ be two n-tuples of real or complex numbers, then

$$\sum_{i=1}^{n} |z_i w_i| \le \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2 \right)^{1/2}$$

[F] Minkowski's inequality:

If $p \geq 1$ and a_i, b_i $i = 1, 2, ---, n$ are non-negative number then

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}$$

[G] If $p \geq 1$ and a_i, b_i $i = 1, 2, ---, n$ are non-negative numbers then

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}.$$

[H] i. If a_i, b_i, $i = 1, 2, ---, n$ are non-negative real numbers and $0 < p \le 1$, then $\sum_{i=1}^{n} (a_i + b_i)^p \le \sum_{i=1}^{n} a_i^p + \sum_{i=1}^{n} b_i^p$.

ii. If z_i, w_i $i = 1, 2, ---, n$ are complex numbers, then

$$\left(\sum_{i=1}^{n} |z_i + w_i|^p \right)^{1/p} \le \left(\sum_{i=1}^{n} |z_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |w_i|^p \right)^{1/p}$$

This is Minkowski's inequality for complex numbers $p \geq 1$

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$$\sum_{i=1}^{n} |z_i + w_i|^p \le \sum_{i=1}^{n} |z_i|^p + \sum_{i=1}^{n} |w_i|^p \quad 0 < p \le 1$$

Problems

1- let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by setting

$d(x, y) = |x-y| \quad \forall\, x, y \in \mathbb{R}$. Then d is a metric for called the usuall metric.

The proof, from properties of absolute value, as follows,

M_1: Since $|x-y| \geq 0 \quad \forall\, x, y \in \mathbb{R}$, so that $d(x,y) \geq 0$.

M_2: ~~Since~~ if $|x-y| = 0 \Leftrightarrow x = y$, so that $d(x,y) = 0 \Leftrightarrow x = y$

M_3: $|x-y| = |(-1)(y-x)| = |-1|\,|y-x| = |y-x|$

$\forall\, x, y \in \mathbb{R}$, so that $d(x,y) = d(y,x)$.

M_4: $|x-y| = |x-z+z-y| \leq |x-z| + |z-y| \quad \forall\, x, y, z \in \mathbb{R}$

So that $d(x,y) \leq d(x,z) + d(z,y)$.

2- let $X \neq \varphi$ be any set. Then the mapping

$d: X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is metric for X called the discrete metric for X.

3- let $X = \mathbb{R}^n$ denote the set of all ordered n-tuple of real numbers for a fixed $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Define the mappings d_1, d_2, d_3 of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ by

i. $d_1(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{\frac{1}{2}}$

ii. $d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

iii. $d_3(x,y) = \max \{ |x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n| \}$

Show that d_1, d_2, d_3 are metrics on $\mathbb{R}^n$.

Solution: we need to prove that d_1 is metric on $\mathbb{R}^n$.

M_1: since $(x_i - y_i)^2 \geq 0 \Rightarrow \sum_{i=1}^{n} (x_i - y_i)^2 \geq 0$

$$\Rightarrow \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} \geq 0 \Rightarrow d(x,y) \geq 0$$

$\forall x, y \in \mathbb{R}^n$.

M_2 If $\left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} = 0 \Leftrightarrow (x_i - y_i)^2 = 0$

$\Leftrightarrow x_i - y_i = 0 \Leftrightarrow x_i = y_i \quad \forall i = 1, 2, \dots, n$

, so that $d(x,y) = 0 \Leftrightarrow x = y \quad \forall x, y \in \mathbb{R}^n$.

M_3: $\left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} ((-1)(y_i - x_i)^2 \right)^{\frac{1}{2}}$

$= \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{\frac{1}{2}}$, so that $d(x,y) = d(y,x)$

$\forall x, y \in \mathbb{R}^n$

M_4: let $z = (z_1, z_2, \dots, z_n)$ and to show that

$$d(x,y) \leq d(x,z) + d(z,y).$$

put $\alpha_i = x_i - z_i$, $\quad B_i = z_i - y_i$, $i = 1, 2, \dots, n$

$d(x,z) = \left(\sum_{i=1}^{n} (x_i - z_i)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{1}{2}}$

also $d(z,y) = \left(\sum_{i=1}^{n} (z_i - y_i)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} B_i^2 \right)^{\frac{1}{2}}$

$d(x,y) = \left(\sum_{i=1}^{n} (x_i - z_i + z_i - y_i)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} (\alpha_i + B_i)^2 \right)^{\frac{1}{2}}$

By using Minkowski's inequality, we have

$$\left(\sum_{i=1}^{n}(\alpha_i+\beta_i)^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n}\alpha_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}\beta_i^2\right)^{\frac{1}{2}}$$

Hence $d(x,y \le d(x,z) + d(z,y)$

$\Rightarrow$ d_1 on $\mathbb{R}^n$ is metric.

ii. we use the notation of (i). Thus

$$d_2(x,z) = \sum_{i=1}^{n}|\alpha_i| \quad \text{and} \quad d(z,y) = \sum_{i=1}^{n}|\beta_i|$$

and $d_1(x,y) = \sum_{i=1}^{n}|\alpha_i+\beta_i|$

Since $|\alpha_i+\beta_i| \le |\alpha_i| + |\beta_i|$, we have

$$d_2(x,y) = \sum_{i=1}^{n}|\alpha_i+\beta_i| \le \sum_{i=1}^{n}|\alpha_i| + \sum_{i=1}^{n}|\beta_i|$$

$$= d(x,z) + d(z,y).$$

iii. $d_3(x,z) = \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|\}$

$d_3(z,y) = \max\{|\beta_1|, |\beta_2|, \dots, |\beta_n|\}$

$d_3(x,y) = \max\{|\alpha_1+\beta_1|, |\alpha_2+\beta_2|, \dots, |\alpha_n+\beta_n|\}$

Since $|\alpha_i+\beta_i| \le |\alpha_i| + |\beta_i|$, it easy to see that

$$\max\{|\alpha_1+\beta_1|, \dots, |\alpha_n+\beta_n|\} \le \max\{|\alpha_1|, \dots, |\alpha_n|\} + \max\{|\beta_1|, \dots, |\beta_n|\}$$

Hence $d_3(x,y) \le d_3(x,z) + d_3(z,y)$

4. let X be the set of all real valued bounded + cont. functions defined on $[0,1]$. we defined the mapping $d: X \times X \to \mathbb{R}$ by $d(f,g) = \int_0^1 |f(x)-g(x)|\,dx \quad \forall f,g \in X$

and $x \in [0,1]$, show that d is metric for X.

5- Give an example of a pseudo-metric which is not a metric.

Sol. let $R[0,1]$ denote the class of all Rieman integrable functions f for $[0,1]$ into $\mathbb{R}$

Consider the mapping $d : R[0,1] \times R[0,1] \longrightarrow \mathbb{R}$ defined by $d(f,g) = \int_0^1 |f-g|(x)\, dx = \int_0^1 |f(x) - g(x)| dx$

Then d is pseudo-metric for $R[0,1]$ which not a metric. Hence M_1, M_2 and M_3 are abrious, to Prove M_4 i.e $d(f,g) \leq d(f,h) + d(h,g)$ $\forall f,g,h \in X$

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx = \int_0^1 | f(x) - h(x) + h(x) - g(x)| dx$$

$$\leq \int_0^1 | f(x) - h(x) | dx + \int_0^1 | h(x) - g(x) | dx$$

$$= d(f,h) + d(h,g).$$

Thus if $f = g \Rightarrow d(f,g) = 0$, but $d(f,g) = 0$ it is not necessary that $f = g$. For example

$$f(x) = \begin{cases} 2 & \text{for } x = 0 \\ 0 & \text{for } 0 < x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } 0 < x \leq 1 \end{cases}$$

Then $f(x) - g(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } 0 < x \leq 1 \end{cases}$

Since $f-g$ has only one point of discont. at $x=0$

it follows that $f-g$ is Riemann integrable and

$$d(f,g) = \int_0^1 |f(x) - g(x)|\,dx = \int_0^1 (f(x) - g(x))\,dx$$

$$= \int_0^1 (f-g)(x)\,dx = 0 = \int_0^1 0\,dx$$

But $f \neq g$ Hence d is pseudo-metric for $R[0,1]$.

open set and closed set:

Def: Given a point $x_0 \in X$ and real number $r > 0$, we define three types of sets:

a. $B(x_0; r) = \{x \in X \,/\, d(x, x_0) < r\}$ open ball.

b. $\overline{B}(x_0, r) = \{x \in X \,/\, d(x, x_0) \leq r\}$ closed ball.

c. $S(x_0, r) = \{x \in X \,/\, d(x, x_0) = r\}$ sphere.

We see that $S(x_0; r) = \overline{B}(x_0; r) - B(x_0; r)$.

In all three cases, x_0 is called center and r the radius.

Def: A subset M of metric space X is said to be open if it contains a ball about each of its points. A sub-set K of X is said to be closed if its complement in X is open, that is $K^c = X - K$ is open.

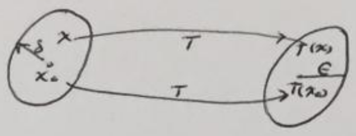
open ball $\equiv$ open set & closed ball $\equiv$ closed set.

Def: Continuous mapping

let $X = (X, d)$ and $Y = (Y, \bar{d})$ be a metric spaces. A mapp-ing $T: X \longrightarrow Y$ is said to be continuous at a point $x_0 \in X$ if $\forall \epsilon > 0$, $\exists\, \delta > 0 \ni \bar{d}(T(x), T(x_0)) < \epsilon$ for all x

satisfying $d(x, x_0) < \delta$.

T is said to be continuous if it is continuous at every point of X.



Theorem: A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

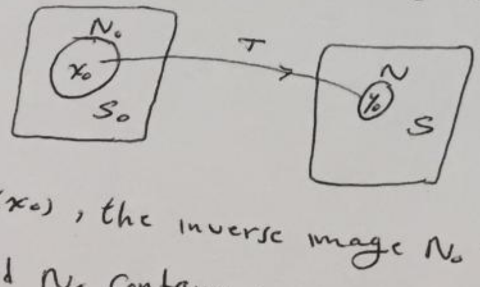
Proof: suppose that T is cont^s. and let $S \subset Y$ be open and S_0 the inverse image of S.

If $S_0 = \phi$ it is open. let $S_0 \neq \phi$, for any $x_0 \in S_0$ and let $y_0 \in Y$ and $y_0 = T(x_0)$.

since S is open it contains an ϵ-neighborhood N of y_0

since T is conts. x_0 has δ-neighborhood N_0 of which is mapped into N. since $N \subset S$, we have $N_0 \subset S_0$, so that S_0 is open because $x_0 \in S_0$ was arbitrary.

Conversely: $\Leftarrow$ Assume that the inverse image of every set in Y is an open set in X: Then for every $x_0 \in X$ and any ϵ-neighborhood



N of $T(x_0)$, the inverse image N_0 of N is open, since N is open and N_0 contains x_0. Hence N_0 contains δ-neighborhood of x_0 which mapped into N because N_0 is mapped into N.
$\Rightarrow T$ is conts. at x_0, $x_0 \in X$ was arbitrary, T is conts. $\square$

Convergence :

Definition : A sequence (x_n) in metric space $X = (X, d)$ is said to converge or to be convergent to the limit $x \in X$ if, for $\epsilon > 0$, there exist a positive integer N such that $d(x_n, x) < \epsilon \quad \forall n > N$.

In other words $\{x_n\}$ is convergent to the limit $x \in X$ iff $d(x_n, x) \to 0$ as $n \to \infty$ or simply $x_n \to x$.

If (x_n) is not convergent, it is said to be divergent.

let us first show that two familiar properties of a convergent sequence (uniqueness of limit and boundedness).

Def.: Let (X, d) be a metric space and let $\phi \neq A \subset X$ the diameter of A, denoted by $\delta(A)$ is defined by

$$\delta(A) = \sup \{ d(x, y) \,/\, x, y \in A \}$$

Def.: Let (X, d) be a metric space, we say that X is bounded if $\exists\ M > 0$ (real) such that $d(x, y) \leq M$ for every pair of points x and y of X. A metric space which is not bounded is said to be unbounded.

Thus a metric space X is bounded if its diameter is finite.

Example: let $X = \mathbb{R}$ and $d(x,y) = |x-y|$.

This metric space is unbounded, since the diameter of $\mathbb{R}$ is infinite.

A discrete metric space (X, d) where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is bounded, since $\delta(X) = 1$.

Lemma (Boundedness, limit):

let $X = (X, d)$ be a metric space. Then:

i. A convergent sequence in X is bounded and its limit is unique.

ii. If $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x,y)$.

Proof: suppose that (x_n) is Convergent seq. $\forall \epsilon > 0$
$\Rightarrow \exists N \ni d(x_n, x) < \epsilon \quad \forall n > N$

taking $\epsilon = 1 \Rightarrow d(x_n, x) < 1 \quad \forall n > N$

Hence by triangle inequality, for all n, we have

$d(x_n, x) < 1 + a$, where $a = \max\{d(x_1, x), \dots, d(x_N, x)\}$.

Thus by definition of boundedness, we obtain (x_n) is bounded.

Now, Assume that $x_n \to x$ & $x_n \to z$, we have
$0 \le d(x,z) \le d(x, x_n) + d(x_n, z) \to 0 + 0$
and the uniqueness $x = z$ of the limit follows.

$\square$

ii- we have $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$

$$\Rightarrow \quad d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$$

since $d(x_n, x) \longrightarrow 0$ and $d(y_n, y) \longrightarrow 0$

$$\Rightarrow \quad |d(x_n, y_n) - d(x, y)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$\Rightarrow \quad d(x_n, y_n) \longrightarrow d(x, y). \quad \text{as } n \longrightarrow \infty.$$

$\square$

Definition: A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_m, x_n) < \epsilon \quad \text{for every } m, n > N.$$

Definition: The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Theorem: Every convergent sequence in metric space is Cauchy sequence.

Proof: let (x_n) be a convergent seq. in metric space X i.e $x_n \longrightarrow x \Rightarrow \forall \epsilon > 0 \; \exists N = N(\epsilon)$ such that $d(x_n, x) < \epsilon/2 \quad \forall n > N$

we obtain, for $m, n > N$

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence (x_n) is Cauchy sequence.

$\square$

Theorem: let M be a non-empty subset of a metric space (X, d) and $\overline{M}$ its closure. Then

i. $x \in \overline{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

ii. M is closed if and only if the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.

Proof: (i) suppose that $x \in \overline{M}$ and to prove $\exists\ x_n \in M$ $\ni$ $x_n \to x$.

we have $\overline{M} = M \cup M^c$ (the set of all accumulation points of M)

if $x \in M$, then sequence of that type is. $(x, x, \ldots)$
$\Rightarrow x_n \to x$

if $x \notin M \Rightarrow x \in M^c \Rightarrow$. x is accumulation of M. Hence for $n = 1, 2, \ldots$, the ball $B(x; 1/n)$ contains an $x_n \in M$ and $x_n \to x$ because $1/n \to 0$ as $n \to \infty$.

Conversely $\Leftarrow$ Assume that $x_n \in M$ and $x_n \to x$
then $x \in M$ or every neighborhood of x contains points $x_n \ni x_n \neq x$, so that x is accumulation point of M
$\Rightarrow x \in \overline{M}$. □

ii. suppose that M is closed $\Rightarrow M = \overline{M}$, from (i)
$x \in \overline{M} \Rightarrow \exists\ x_n \in M \ni x_n \to x \Rightarrow x \in M$
because $M = \overline{M}$

Conversely : immediately from (i). □

Theorem: A subspace M of complete metric space X is itself complete if and only if the set M is closed in X.

Proof: let M be complete subspace of Complete space X and to prove M is closed i.e $M=\overline{M}$.

From above theorem, let $x \in \overline{M}$ $\exists$ $x_n \in M \ni x_n \to x$

$\Rightarrow$ (x_n) is Cauchy sequence in M.

M is Complete $\Rightarrow$ $x_n \to x \in M$ {by def. of Complete}

$\Rightarrow$ $\overline{M} \subset M$ --(1) and we have always

$M \subset \overline{M}$ --(2), From (1) and (2), we get $M=\overline{M}$

$\Rightarrow$ M is closed.

Conversely $\Leftarrow$: let M is closed sub set of X and (x_n) be Cauchy in M. Then $x_n \to x \in X$ which implies $x \in \overline{M}$ $\Rightarrow$ $x \in M$.

Hence (x_n) is Cauchy seq. and Converge in M i.e. $x_n \to x \in M$ $\Rightarrow$ M is Complete.

$\square$

Theorem: (Continuous mapping). A mapping $T: X \to Y$ of metric space (X, d) into metric space $(Y, \bar{d})$ is conts. at a point x_0 if and only if $x_n \to x_0 \Rightarrow T(x_n) \to T(x_0)$.

Proof: Suppose that T is conts. at x_0. Then for a given $\epsilon > 0$ there is a $\delta > 0$ such that

$d(x, x_0) < \delta \Rightarrow \bar{d}(T(x), T(x_0)) < \epsilon$

let $x_n \to x_0$. Then there is an N such that for all $n > N$

we have $d(x_n, x_0) < \delta$

Hence for all $n > N$, $\bar{d}(T(x_n), T(x_0)) < \epsilon$

This mean $T(x_n) \longrightarrow T(x)$.

Conversely $\Leftarrow$: we assume that $x_n \to x_0 \Rightarrow T(x_n) \to T(x_0)$ and to prove that T is cont^s. at x_0.

suppose this is false. Then there is $\epsilon > 0$ such that for every $\delta > 0$, there is an $x_n \neq x_0$ satisfying

$d(x_n, x_0) < \delta$ but $\bar{d}(T(x_n), T(x_0)) \geq \epsilon$

In particular, for $\delta = 1/n$ there is an x_n satisfying $d(x_n, x_0) < 1/n$ but $\bar{d}(T(x_n), T(x_0)) \geq \epsilon$

Clearly $x_n \to x_0$ but $T(x_n) \not\to T(x_0)$. This C! Contradicts $T(x_n) \longrightarrow T(x_0)$ and prove the theorem.

$\Box$

Examples:

1. Euclidean space $\mathbb{R}^n$ and Unitary space $\mathbb{C}^n$ are Complete.

Proof: we first consider $\mathbb{R}^n$. The metric on $\mathbb{R}^n$ (the Euclidean metric) is define by
$$d(x, y) = \left(\sum_{j=1}^{n} (\xi_j - \eta_j)^2 \right)^{1/2}$$
where $x = (\xi_j)$ and $y = (\eta_j)$. We consider any Cauchy sequence (x_m) in $\mathbb{R}^n$, writting $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_n^{(m)})$

Since (x_m) is Cauchy, for every $\epsilon > 0$ there is an N such that
$$d(x_m, x_r) = \left(\sum_{j=1}^{n} (\xi_j^{(m)} - \xi_j^{(r)})^2 \right)^{1/2} < \epsilon \quad \forall m, r > N$$

Squaring of two sided for $m, r > N$ and $j = 1, 2, \ldots, n$ $\Rightarrow$

$$(\xi_j^{(m)} - \xi_j^{(r)})^2 < \epsilon^2 \Rightarrow |\xi_j^{(m)} - \xi_j^{(r)}| < \epsilon$$

This shows that for each fixed j, $(1 \le j \le n)$, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is cauchy sequence of real number. It converges by theorem [The real number and Complex plane are Complete metric spaces].

say, $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$. Using these n limits, we define $x = (\xi_1, \ldots, \xi_n)$ clearly $x \in \mathbb{R}^n$

with $r \to \infty$, $d(x_m, x) < \epsilon$ $m > N$.

This shows that x is limit of (x_m) and proves Completeness of $\mathbb{R}^n$ because (x_m) was an arbitrary Cauchy sequence.

Completeness of $\mathbb{C}^n$ follows by the same of proof. □

2. The space ℓ^∞ is complete.

Proof: let (x_m) be cauchy sequence in the space ℓ^∞, where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \ldots)$. since the metric on ℓ^∞ is given by $d(x, y) = \sup_j |\xi_j - \eta_j|$,

where $x = (\xi_j)$ and $y = (\eta_j)$ and (x_m) is Cauchy, for any $\epsilon > 0$ there is an N such that for that all $m, n > N$

$$d(x_m, x_n) = \sup_j |\xi_j^{(m)} - \xi_j^{(n)}| < \epsilon$$

for every fixed j, $|\xi_j^{(m)} - \xi_j^{(n)}| < \epsilon$ ~ ⊛ $m, n > N$

Hence for every fixed j, the seq. $(\xi_j^{(1)}, \xi_j^{(2)}, \dots)$ is a cauchy sequence of real numbers. It converges by $(\mathbb{R})$ say, $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$. Using these infinitely many limits $\xi_1, \xi_2, \dots$, we define $x = (\xi_1, \xi_2, \dots)$ and show that $x \in \ell^\infty$ and $x_m \longrightarrow x$

From ⊛ with $|\xi_j^{(m)} - \xi_j| < \epsilon \quad m > N$

Since $x_m = (\xi_j^{(m)}) \in \ell^\infty$, there is real number K_m such that $|\xi_j^{(m)}| \leq K_m$ for all j.

We have

$$|\xi_j| \leq |\xi_j - \xi_j^{(m)}| + |\xi_j^{(m)}| < \epsilon + K_m \qquad m > N$$

This inequality holds for every j,

$\Rightarrow |\xi_j|$ is bounded seq $\Rightarrow x = (\xi_j) \in \ell^\infty$

Hence $d(x_m, x) = \sup\limits_j |\xi_j^{(m)} - \xi_j| < \epsilon \qquad m > N$

This show that $x_m \longrightarrow x$. Since (x_m) was an arbitrary cauchy sequence, ℓ^∞ is Complete.

□

3. let X be the set of all continuous real-valued functions on $J = [0,1]$ and let $d(x,y) = \int_0^1 |x(t) - y(t)|\, dt$.

This metric space (X, d) is not Complete.

Proof: The functions x_m in Fig-(1) form a cauchy sequence because $d(x_m, x_n)$ is the area of the triagnal in Fig. (2) and for every given $\epsilon > 0$,

$d(x_m, x_n) < \epsilon$ when $m, n > 1/\epsilon$.

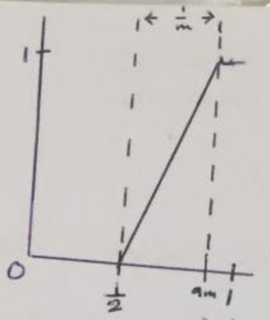


Fig. (1)

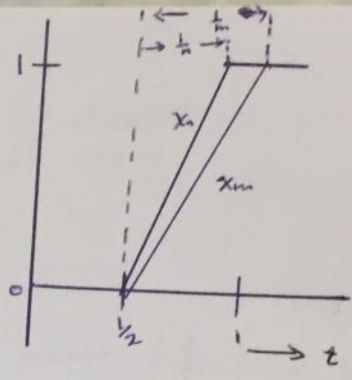


Fig. (2)

let us show that this cauchy sequence dose not Converge. We have $x_m(t) = 0$ if $t \in [0, \frac{1}{2}]$, $x_m(t) = 1$ if $t \in [a_m, 1]$ where $a_m = \frac{1}{2} + \frac{1}{m}$. Hence for every $x \in X$

$$d(x_m, x) = \int_0^1 | x_m(t) - x(t) | dt$$

$$= \int_0^{1/2} |x(t)| dt + \int_{1/2}^{a_m} | x_m(t) - x(t) | dt + \int_{a_m}^{1} | 1 - x(t) | dt.$$

Since the integrands (دالة التكامل) are non-negative, so is each integral on the right. Hence $d(x_m, x) \to 0$ would imply that each integral approaches zero and, since x is cont^s, we should have

$$x(t) = 0 \text{ if } t \in [0, \tfrac{1}{2}] \qquad x(t) = 1 \text{ if } t \in (\tfrac{1}{2}, 1]$$

But this is impossible for a cont^s. function.

Hence (x_m) dose not Converge, that is, dose not have a limit in X. This proves that X is not Complet.

$\square$