LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 1#

➢ Outline :-

✓ Discrete distributions

1- Bernoulli distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Relation to the binomial distribution

Solved exercises

Bernoulli distribution

Suppose you perform an experiment with two possible outcomes: either success or failure. Success happens with probability p, while failure happens with probability 1-p. A random variable that takes value 1 in case of success and 0 in case of failure is called a Bernoulli random variable (alternatively, it is said to have a Bernoulli distribution).

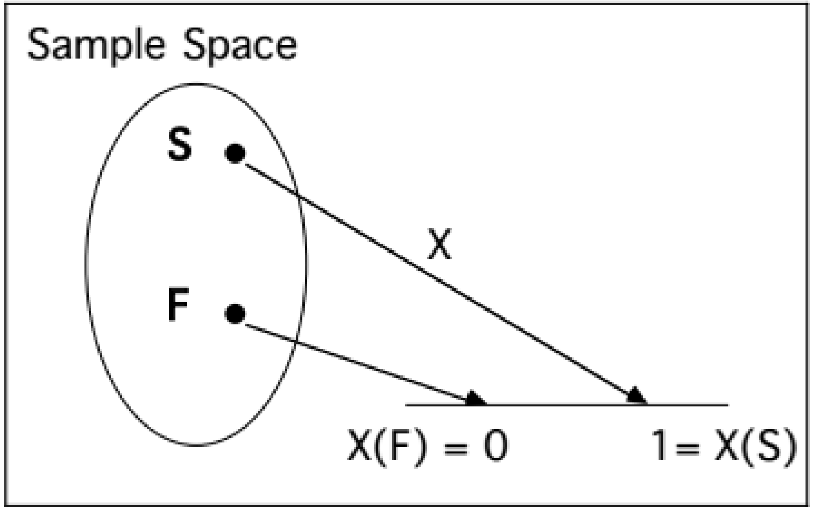
Bernoulli distribution

Definition:

The random variable *X* is called the Bernoulli random variable if its probability mass function is of the form

$$f(x) = p^x (1-p)^{1-x}, \qquad x = 0, 1$$

where p is the probability of success.



We denote the Bernoulli random variable by writing $X \sim BER(p)$.

Bernoulli distribution

Proof :

Non-negativity is obvious. We need to prove that the sum of f(x) over its support equals 1. This is proved as follows:

$$\sum_{x=0}^{1} f(x) = f(0) + f(1)$$

$$= 1 - p + p = 1$$

Bernoulli distribution

Example :

What is the probability of getting a score of not less than 5 in a throw of a six-sided die?

Answer: Although there are six possible scores $\{1, 2, 3, 4, 5, 6\}$, we are grouping them into two sets, namely $\{1, 2, 3, 4\}$ and $\{5, 6\}$. Any score in $\{1, 2, 3, 4\}$ is a failure and any score in $\{5, 6\}$ is a success. Thus, this is a Bernoulli trial with

$$P(X = 0) = P(\text{failure}) = \frac{4}{6} \qquad \text{and} \qquad P(X = 1) = P(\text{success}) = \frac{2}{6}.$$

Hence, the probability of getting a score of not less than 5 in a throw of a six-sided die is $\frac{2}{6}$.

Bernoulli distribution

Theorem :

If X is a Bernoulli random variable with parameter p, then the mean, variance and moment generating functions are respectively given by:

$$\mu_X = p$$

$$\sigma_X^2 = p(1-p)$$

$$M_X(t) = (1-p) + pe^t.$$

Bernoulli distribution

Proof:

The mean of the Bernoulli random variable is

$$\mu_X = \sum_{x=0}^{1} x\, f(x)$$
$$= \sum_{x=0}^{1} x\, p^x\,(1-p)^{1-x}$$
$$= p.$$

Similarly, the variance of X is given by

$$\sigma_X^2 = \sum_{x=0}^{1} (x-\mu_X)^2\, f(x)$$
$$= \sum_{x=0}^{1} (x-p)^2\, p^x\,(1-p)^{1-x}$$
$$= p^2\,(1-p) + p\,(1-p)^2$$
$$= p\,(1-p)\,[p + (1-p)]$$
$$= p\,(1-p).$$

Next, we find the moment generating function of the Bernoulli random variable

$$M(t) = E\left(e^{tX}\right)$$
$$= \sum_{x=0}^{1} e^{tx}\, p^x\,(1-p)^{1-x}$$
$$= (1-p) + e^t\, p.$$

Bernoulli distribution

Characteristic function

Definition Let X be a random variable. The characteristic function $\phi(t)$ of X is defined as

$$\begin{aligned}\phi(t) &= E\left(e^{i\,t\,X}\right)\\ &= E\left(cos(tX) + i\,sin(tX)\right)\\ &= E\left(cos(tX)\right) + i\,E\left(sin(tX)\right).\end{aligned}$$

The probability density function can be recovered from the characteristic function by using the following formula

$$f(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{-i\,t\,x}\,\phi(t)\,dt.$$

Bernoulli distribution

Characteristic function

The characteristic function of a Bernoulli random variable X is

$$\varphi_X(t) = 1 - p + p\exp(it)$$

Proof. Using the definition of characteristic function:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \sum_{x \in R_X} \exp(itx)\, p_X(x) \\
&= \exp(it \cdot 1) \cdot p_X(1) + \exp(it \cdot 0) \cdot p_X(0) \\
&= \exp(it) \cdot p + 1 \cdot (1-p) \\
&= 1 - p + p\exp(it)
\end{aligned}$$

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Bernoulli distribution

Distribution function

The distribution function of a Bernoulli random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Proof. Remember the definition of distribution function:

$$F_X(x) = \mathrm{P}(X \leq x)$$

and the fact that X can take either value 0 or value 1. If $x < 0$, then $\mathrm{P}(X \leq x) = 0$, because X can not take values strictly smaller than 0. If $0 \leq x < 1$, then $\mathrm{P}(X \leq x) = 1 - p$, because 0 is the only value strictly smaller than 1 that X can take. Finally, if $x \geq 1$, then $\mathrm{P}(X \leq x) = 1$, because all values X can take are smaller than or equal to 1. ∎

1

Solved exercises

Let X be a Bernoulli random variable with parameter $p = 1/2$. Find its tenth moment.

Solution

The moment generating function of X is

$$M_X(t) = \frac{1}{2} + \frac{1}{2}\exp(t)$$

The tenth moment of X is equal to the tenth derivative of its moment generating function, evaluated at $t = 0$:

$$\mu_X(10) = \mathrm{E}\left[X^{10}\right] = \left.\frac{d^{10}M_X(t)}{dt^{10}}\right|_{t=0}$$

But

$$\begin{aligned}
\frac{dM_X(t)}{dt} &= \frac{1}{2}\exp(t) \\
\frac{d^2M_X(t)}{dt^2} &= \frac{1}{2}\exp(t) \\
&\vdots \\
\frac{d^{10}M_X(t)}{dt^{10}} &= \frac{1}{2}\exp(t)
\end{aligned}$$

so that:

$$\begin{aligned}
\mu_X(10) &= \left.\frac{d^{10}M_X(t)}{dt^{10}}\right|_{t=0} \\
&= \frac{1}{2}\exp(0) = \frac{1}{2}
\end{aligned}$$

2

Solved exercises

Let X and Y be two *independent* Bernoulli random variables with parameter p. Derive the probability mass function of their sum: Z = X + Y?

Solution

The probability mass function of X is

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability mass function of Y is

$$p_Y(y) = \begin{cases} p & \text{if } y = 1 \\ 1 - p & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

The support of Z (the set of values Z can take) is

$$R_Y = \{0, 1, 2\}$$

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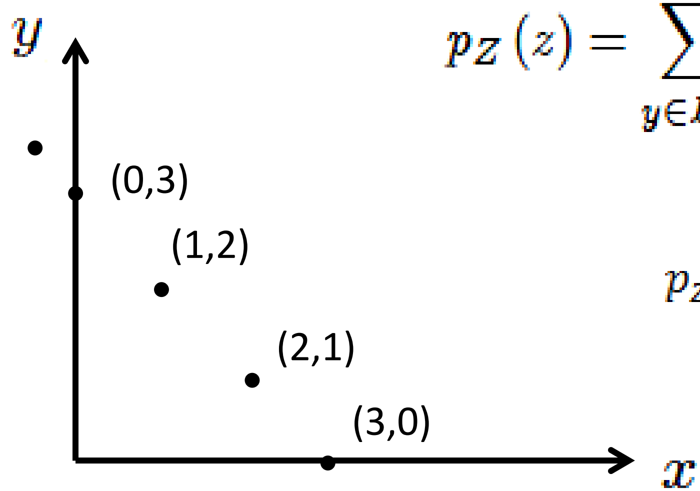
The formula for the probability mass function of a sum of two independent variables

$$p_Z(z) = \sum_{y \in R_Y} p_X(z-y)\, p_Y(y)$$

What is that?



$$p_Z(3) = \cdots + P(X=0, Y=3) + P(X=1, Y=2) + \cdots$$

$$= \cdots + P_X(0)P_Y(3) + P_X(1)P_Y(2) + \cdots$$

In general

$$p_Z(z) = \sum_{Y} P(X = z-y, Y = y)$$

sum

Independent

$$p_Z(z) = \sum_{Y} P_X(z-y)P_Y(y)$$

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When $z = 0$, the formula gives:

$$\begin{aligned} p_Z(0) &= \sum_{y \in R_Y} p_X(-y)\, p_Y(y) \\ &= p_X(-0)\, p_Y(0) + p_X(-1)\, p_Y(1) \\ &= (1-p)(1-p) + 0 \cdot p = (1-p)^2 \end{aligned}$$

When $z = 1$, the formula gives:

$$\begin{aligned} p_Z(1) &= \sum_{y \in R_Y} p_X(1-y)\, p_Y(y) \\ &= p_X(1-0)\, p_Y(0) + p_X(1-1)\, p_Y(1) \\ &= p \cdot (1-p) + (1-p) \cdot p = 2p(1-p) \end{aligned}$$

When $z = 2$, the formula gives:

$$\begin{aligned} p_Z(2) &= \sum_{y \in R_Y} p_X(2-y)\, p_Y(y) \\ &= p_X(2-0)\, p_Y(0) + p_X(2-1)\, p_Y(1) \\ &= 0 \cdot (1-p) + p \cdot p = p^2 \end{aligned}$$

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Therefore, the probability mass function of Z is

$$p_Z(z) = \begin{cases} (1-p)^2 & \text{if } z = 0 \\ 2p(1-p) & \text{if } z = 1 \\ p^2 & \text{if } z = 2 \\ 0 & \text{otherwise} \end{cases}$$

See you in the next lecture

LECTURE 2#

➢ Outline :-

✓ Discrete distributions

1- Binomial distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

Relation to the binomial distribution

Solved exercises

Exercises

Binomial distribution

Consider an experiment having two possible outcomes: either success or failure. Suppose the experiment is repeated several times and the repetitions are independent of each other.

The total number of experiments where the outcome turns out to be a success is a random variable whose distribution is called binomial distribution.

The distribution has two parameters: the number n of repetitions of the experiment, and the probability p of success of an individual experiment.

Note A binomial distribution can be seen as a sum of mutually independent Bernoulli random variables

Binomial distribution

Definition:

A random variable *X* has the *binomial distribution with parameters n and p* if *X* has a discrete distribution for which the p.f. is as follows:

$$p(x|n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \ldots, n,\ 0 \le p \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

In this distribution, *n* must be a positive integer, and *p* must lie in the interval

We will denote a binomial random variable with parameters *p* and *n* as *X ~ BIN*(*n, p*).

Binomial distribution

Proof :

Non-negativity is obvious. We need to prove that the sum of f(x) over its support equals 1. This is proved as follows:

$$\sum_{x=0}^{1} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1^n = 1$$

where we have used the formula for binomial expansions

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

Binomial distribution

Example :

Find the probability of getting five heads and seven tails in 12 flips of a balanced coin.

Solution

Substituting $x = 5$, $n = 12$, and $p = \frac{1}{2}$ into the formula for the binomial distribution, we get

$$b\left(5; 12, \frac{1}{2}\right) = \binom{12}{5}\left(\frac{1}{2}\right)^5\left(1 - \frac{1}{2}\right)^{12-5}$$

and, looking up the value of $\binom{12}{5}$ in binomial table, we find that the result is

$792\left(\frac{1}{2}\right)^{12}$, or approximately 0.19.

Probabilities for various binomial distributions can be obtained from the table given at the end of this book and from many statistical software programs.

Binomial Coefficients

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66
13	1	13	78	286	715	1287	1716	1716	1287	715	286
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003
16	1	16	120	560	1820	4368	8008	11440	12870	11440	8008
17	1	17	136	680	2380	6188	12376	19448	24310	24310	19448
18	1	18	153	816	3060	8568	18564	31824	43758	48620	43758
19	1	19	171	969	3876	11628	27132	50388	75582	92378	92378
20	1	20	190	1140	4845	15504	38760	77520	125970	167960	184756

Binomial distribution

H.W:

Find the probability that 7 of 10 persons will recover from a tropical disease if we can assume independence and the probability is 0.80 that any one of them will recover from the disease.

Note:

looking up the value of $\binom{10}{7}$ in binomial table

Binomial distribution

Theorem: The mean and the variance of the binomial distribution are

$$\mu = n\theta \quad \text{and} \quad \sigma^2 = n\theta(1-\theta)$$

Here p = θ

Proof

$$\mu = \sum_{x=0}^{n} x \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \theta^x (1-\theta)^{n-x}$$

where we omitted the term corresponding to $x = 0$, which is 0, and canceled the x against the first factor of $x! = x(x-1)!$ in the denominator of $\binom{n}{x}$. Then, factoring out the factor n in $n! = n(n-1)!$ and one factor θ, we get

Binomial distribution

$$\mu = n\theta \cdot \sum_{x=1}^{n} \binom{n-1}{x-1} \theta^{x-1}(1-\theta)^{n-x}$$

and, letting $y = x-1$ and $m = n-1$, this becomes

$$\mu = n\theta \cdot \sum_{y=0}^{m} \binom{m}{y} \theta^{y}(1-\theta)^{m-y} = n\theta$$

since the last summation is the sum of all the values of a binomial distribution with the parameters m and θ, and hence equal to 1.

To find expressions for μ_2' and σ^2, let us make use of the fact that $E(X^2) = E[X(X-1)] + E(X)$ and first evaluate $E[X(X-1)]$. Duplicating for all practical purposes the steps used before, we thus get

$$\boxed{\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\binom{n}{x}\theta^{x}(1-\theta)^{n-x} \\ &= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}\theta^{x}(1-\theta)^{n-x} \\ &= n(n-1)\theta^2 \cdot \sum_{x=2}^{n} \binom{n-2}{x-2}\theta^{x-2}(1-\theta)^{n-x} \end{aligned}}$$

Binomial distribution

and, letting $y = x - 2$ and $m = n - 2$, this becomes

$$E[X(X-1)] = n(n-1)\theta^2 \cdot \sum_{y=0}^{m} \binom{m}{y} \theta^y (1-\theta)^{m-y}$$
$$= n(n-1)\theta^2$$

Therefore,

$$\mu_2' = E[X(X-1)] + E(X) = n(n-1)\theta^2 + n\theta$$

and, finally,

$$\sigma^2 = \mu_2' - \mu^2$$
$$= n(n-1)\theta^2 + n\theta - n^2\theta^2$$
$$= n\theta(1-\theta)$$

Relation to the Bernoulli distribution

Proposition 1: A random variable has a binomial distribution with parameters n and p, with n = 1, if and only if it has a Bernoulli distribution with parameter p.

Proof: We demonstrate that the two distributions are equivalent by showing that they have the same probability mass function.

The probability mass function of a binomial distribution with parameters n and p, with n = 1, is:

$$p(\mathrm{x}) = \begin{cases} \binom{1}{x} p^x (1-p)^{1-x} & \text{if } x \in \{0,1\} \\ 0 & \text{if } x \notin \{0,1\} \end{cases}$$, but,

$$p(0) = \binom{1}{0} p^0 (1-p)^{1-0} = \frac{1!}{0!1!}(1-p) = 1-p$$, and,

$$p(1) = \binom{1}{1} p^1 (1-p)^{1-1} = \frac{1!}{1!0!} p = p$$

Relation to the Bernoulli distribution

Proof:

Therefore, the probability mass function can be written as

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \longrightarrow$$

which is the probability mass function of a Bernoulli random variable.

Proposition 2 : A random variable has a binomial distribution with parameters n and p if and only if it can be written as a sum of n jointly independent Bernoulli random variables with parameter p.

Proof: We will prove that later:

Binomial distribution

Theorem :

The moment generating function of a binomial random variable X is defined for any $t \in R$ as : $M_X(t) = (1 - p + p\exp(t))^n$

Proof:

The definition of m. g. f.

X can be represented as a sum of n independent Bernoulli r.v.

Y1 ; ...; Yn are jointly independent

The definition of m. g. f. Y1,...Yn

The formula for the moment generating function of a Ber. r.v.

$$
\begin{aligned}
M_X(t) &= \mathrm{E}[\exp(tX)] \\
&= \mathrm{E}[\exp(t(Y_1 + \ldots + Y_n))] \\
&= \mathrm{E}[\exp(tY_1) \cdot \ldots \cdot \exp(tY_n)] \\
&= \mathrm{E}[\exp(tY_1)] \cdot \ldots \cdot \mathrm{E}[\exp(tY_n)] \\
&= M_{Y_1}(t) \cdot \ldots \cdot M_{Y_n}(t) \\
&= (1 - p + p\exp(t)) \cdot \ldots \cdot (1 - p + p\exp(t)) \\
&= (1 - p + p\exp(t))^n
\end{aligned}
$$

Since the m.g.f. Ber. .v. exists,so is the m.g.f. of a binomial random variable exists .

Binomial distribution

Characteristic function:

The characteristic function of a binomial random variable X is

$$\varphi_X(t) = (1 - p + p\exp(it))^n$$

Proof: Similar to the previous proof

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \mathrm{E}[\exp(it(Y_1 + \ldots + Y_n))] \\
&= \mathrm{E}[\exp(itY_1) \cdot \ldots \cdot \exp(itY_n)] \\
&= \mathrm{E}[\exp(itY_1)] \cdot \ldots \cdot \mathrm{E}[\exp(itY_n)] \\
&= \varphi_{Y_1}(t) \cdot \ldots \cdot \varphi_{Y_n}(t) \\
&= (1 - p + p\exp(it)) \cdot \ldots \cdot (1 - p + p\exp(it)) \\
&= (1 - p + p\exp(it))^n
\end{aligned}$$

Binomial distribution

Distribution function: The distribution function of a binomial random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} & \text{if } 0 \le x \le n \\ 1 & \text{if } x > n \end{cases}$$

Proof. For $x < 0$, $F_X(x) = 0$, because X cannot be smaller than 0. For $x > n$, $F_X(x) = 1$, because X is always smaller than or equal to n. For $0 \le x \le n$:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \le x) \\ &= \sum_{s=0}^{x} \mathrm{P}(X = s) \\ &= \sum_{s=0}^{x} p_X(s) \quad = \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} \end{aligned}$$

1

Solved exercises

Suppose you independently flip a coin 4 times and the outcome of each toss can be either head (with probability 1/2) or tails (also with probability 1=2). What is the probability of obtaining exactly 2 tails?

Solution

Denote by X the number of times the outcome is tails (out of the 4 tosses). X has a binomial distribution with parameters n = 4 and p = 1/2. The probability of obtaining exactly 2 tails can be computed from the probability mass function of X as follows:

$$
\begin{aligned}
p_X(2) &= \binom{n}{2} p^2 (1-p)^{n-2} = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(1-\frac{1}{2}\right)^{4-2} \\
&= \frac{4!}{2!2!} \frac{1}{4} \frac{1}{4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} \frac{1}{16} = \frac{6}{16} = \frac{3}{8}
\end{aligned}
$$

2

Solved exercises

Suppose you independently throw a dart 10 times. Each time you throw a dart, the probability of hitting the target is 3/4. What is the probability of hitting the target less than 5 times (out of the 10 total times you throw a dart)?

Solution

Denote by X the number of times you hit the target. X has a binomial distribution with parameters n = 10 and p = 3/4. The probability of hitting the target less than 5 times can be computed from the distribution function of X as follows:

$$
\begin{aligned}
\mathrm{P}(X<5) &= \mathrm{P}(X \leq 4) = F_X(4) \\
&= \sum_{s=0}^{4} \binom{n}{s} p^s (1-p)^{n-s} \\
&= \sum_{s=0}^{4} \binom{10}{s} \left(\frac{3}{4}\right)^s \left(\frac{1}{4}\right)^{10-s} \simeq 0.0197
\end{aligned}
$$

3

Exercises

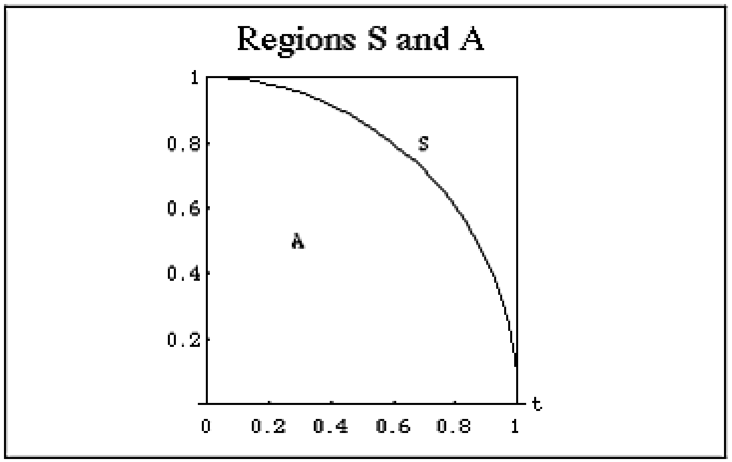
1) On a five-question multiple-choice test there are five possible answers, of which one is correct. If a student guesses randomly and independently, what is the probability that she is correct only on two questions?
2) What is the probability of rolling two sixes and three nonsixes in 5 independent casts of a fair die?
3) What is the probability of rolling at most two sixes in 5 independent casts of a fair die?
4) Suppose that 2000 points are selected independently and at random from the unit squares $S = \{(x, y) \,|\, 0 \leq x, y \leq 1\}$. Let *X* equal the number of points that fall in $A = \{(x, y) \,|\, x^2+y^2 < 1\}$. How is *X* distributed? What are the mean, variance and standard deviation of *X*?

4

Exercises

4) Hinte : If a point falls in *A*, then it is a success. If a point falls in the complement of *A*, then it is a failure. The probability of success is

$$p = \frac{\text{area of A}}{\text{area of S}} = \frac{1}{4}\pi.$$



5) Let the probability that the birth weight (in grams) of babies in America is less than 2547 grams be 0.1. If *X* equals the number of babies that weigh less than 2547 grams at birth among 20 of these babies selected at random, then what is $P(X \leq 3)$?

LECTURE 3#

➢ Outline :-

✓ Discrete distributions

3- Poisson distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

Relation to the binomial distribution

Solved exercises

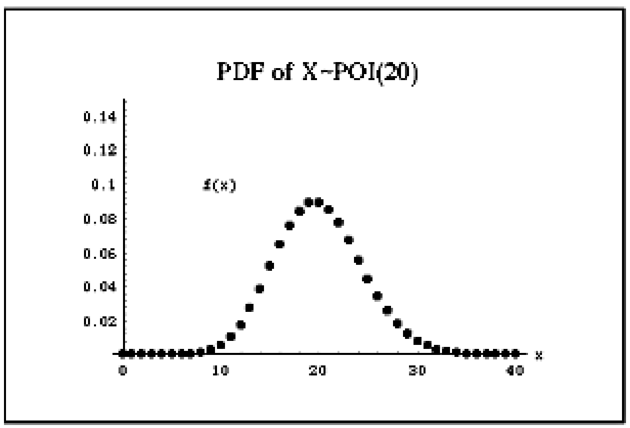
Exercises

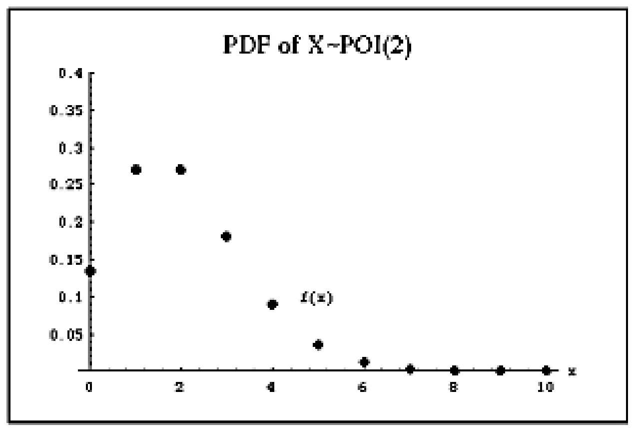
Poisson distribution

Definition ：A random variable *X* is said to have a Poisson distribution if its probability mass function is given by

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \cdots, \infty,$$

where $0 < \lambda < \infty$ is a parameter. We denote such a random variable by $X \sim POI(\lambda)$.





The probability density function *f* is called the Poisson distribution after Simeon D. Poisson (1781-1840).

Poisson distribution

Proof :

It is easy to check $f(x) \geq 0$. We show that $\sum_{x=0}^{\infty} f(x)$ is equal to one

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} e^{\lambda} = 1.$$

Poisson distribution

Theorem: The mean , the variance the m.g.f. of Poisson distribution are:

$$E(X) = \lambda$$

$$Var(X) = \lambda$$

$$M(t) = e^{\lambda\,(e^t - 1)}.$$

Proof: First, we find the moment generating function of X.

$$\begin{aligned} M(t) &= \sum_{x=0}^{\infty} e^{tx} f(x) \\ &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda\,(e^t - 1)}. \end{aligned}$$

Poisson distribution

Thus,

$$M'(t) = \lambda\, e^t\, e^{\lambda\, (e^t - 1)},$$

and

$$E(X) = M'(0) = \lambda.$$

Similarly,

$$M''(t) = \lambda\, e^t\, e^{\lambda\, (e^t - 1)} + \left(\lambda\, e^t\right)^2\, e^{\lambda\, (e^t - 1)}.$$

Hence

$$M''(0) = E(X^2) = \lambda^2 + \lambda.$$

Therefore

$$Var(X) = E(X^2) - (\, E(X)\,)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Poisson distribution

Example : A random variable X has Poisson distribution with a mean of 3. What is the probability that X is bounded by 1 and 3, that is,

$$P(1 \leq X \leq 3)?$$

Answer:

$$\mu_X = 3 = \lambda$$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Hence

$$f(x) = \frac{3^x e^{-3}}{x!}, \qquad x = 0, 1, 2, \ldots$$

Therefore

$$\begin{aligned} P(1 \leq X \leq 3) &= f(1) + f(2) + f(3) \\ &= 3\, e^{-3} + \frac{9}{2} e^{-3} + \frac{27}{6} e^{-3} \\ &= 12\, e^{-3}. \end{aligned}$$

Poisson distribution

Example : The number of tra!c accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of exactly 2 accidents occur in 2 weeks?

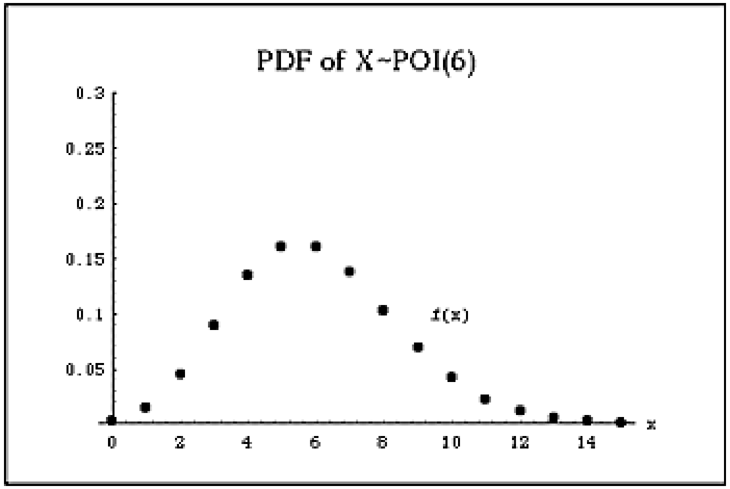
Answer: The mean tra!c accident is 3. Thus, the mean accidents in two weeks are $\lambda = (3)\,(2) = 6.$

Since

$$f(x) = \frac{\lambda^x\, e^{-\lambda}}{x!}$$

we get

$$f(2) = \frac{6^2\, e^{-6}}{2!} = 18\, e^{-6}.$$



Poisson distribution

Characteristic function:

The characteristic function of Poisson random variable X is

$$\varphi_X(t) = \exp(\lambda[\exp(it) - 1])$$

Proof:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \sum_{x \in R_X} \exp(itx)\, p_X(x) \\
&= \sum_{x \in R_X} [\exp(it)]^x \exp(-\lambda) \frac{1}{x!} \lambda^x \\
&= \exp(-\lambda) \sum_{x=0}^{\infty} \frac{(\lambda \exp(it))^x}{x!} \\
&= \exp(-\lambda) \exp(\lambda \exp(it)) \\
&= \exp(\lambda[\exp(it) - 1])
\end{aligned}$$

where:

$$\exp(\lambda \exp(it)) = \sum_{x=0}^{\infty} \frac{(\lambda \exp(it))^x}{x!}$$

is the usual Taylor series expansion of the exponential function

Poisson distribution

Distribution function: The distribution function of a Poisson random variable X is

$$F_X(x) = \begin{cases} \exp(-\lambda) \sum_{s=0}^{\lfloor x \rfloor} \frac{1}{s!} \lambda^s & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $\lfloor x \rfloor$ is the largest integer not greater than x.

Proof:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \leq x) \\ &= \sum_{s=0}^{\lfloor x \rfloor} \mathrm{P}(X = s) \\ &= \sum_{s=0}^{\lfloor x \rfloor} p_X(s) \\ &= \sum_{s=0}^{\lfloor x \rfloor} \exp(-\lambda) \frac{1}{s!} \lambda^s \\ &= \exp(-\lambda) \sum_{s=0}^{\lfloor x \rfloor} \frac{1}{s!} \lambda^s \end{aligned}$$

Solved exercises

1

Let *X* have a Poisson distribution with parameter λ = 1. What is the probability that $X \geq 2$ given that $X \leq 4$?

Solution

$$P(X \geq 2 \,/\, X \leq 4) = \frac{P(2 \leq X \leq 4)}{P(X \leq 4)}.$$

$$\begin{aligned} P(2 \leq X \leq 4) &= \sum_{x=2}^{4} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{1}{e} \sum_{x=2}^{4} \frac{1}{x!} \\ &= \frac{17}{24\, e}. \end{aligned}$$

Similarly

$$\begin{aligned} P(X \leq 4) &= \frac{1}{e} \sum_{x=0}^{4} \frac{1}{x!} \\ &= \frac{65}{24\, e}. \end{aligned}$$

Therefore, we have

$$P(X \geq 2 \,/\, X \leq 4) = \frac{17}{65}.$$

2

Solved exercises

If the moment generating function of a random variable *X* is $M(t) = e^{4.6\,(e^t-1)}$, then what are the mean and variance of *X*? What is the probability that *X* is between 3 and 6, that is $P(3 < X < 6)$?

Solution: Since the moment generating function of *X* is given by

$M(t) = e^{4.6\,(e^t-1)}$

we conclude that $X \sim POI(\lambda)$ with $\lambda = 4.6$. Thus, by

$$E(X) = 4.6 = Var(X).$$

$$\begin{aligned} P(3 < X < 6) &= f(4) + f(5) \\ &= F(5) - F(3) \\ &= 0.686 - 0.326 \\ &= 0.36. \end{aligned}$$

Table of Poisson Probabilities

$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

k	$\lambda = .1$	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	.9048	.8187	.7408	.6703	.6065	.5488	.4966	.4493	.4066	.3679
1	.0905	.1637	.2222	.2681	.3033	.3293	.3476	.3595	.3659	.3679
2	.0045	.0164	.0333	.0536	.0758	.0988	.1217	.1438	.1647	.1839
3	.0002	.0011	.0033	.0072	.0126	.0198	.0284	.0383	.0494	.0613
4	.0000	.0001	.0003	.0007	.0016	.0030	.0050	.0077	.0111	.0153
5	.0000	.0000	.0000	.0001	.0002	.0004	.0007	.0012	.0020	.0031
6	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Table of Poisson Probabilities

$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

k	$\lambda = 1.5$	2	3	4	5	6	7	8	9	10
0	.2231	.1353	.0498	.0183	.0067	.0025	.0009	.0003	.0001	.0000
1	.3347	.2707	.1494	.0733	.0337	.0149	.0064	.0027	.0011	.0005
2	.2510	.2707	.2240	.1465	.0842	.0446	.0223	.0107	.0050	.0023
3	.1255	.1804	.2240	.1954	.1404	.0892	.0521	.0286	.0150	.0076
4	.0471	.0902	.1680	.1954	.1755	.1339	.0912	.0573	.0337	.0189
5	.0141	.0361	.1008	.1563	.1755	.1606	.1277	.0916	.0607	.0378
6	.0035	.0120	.0504	.1042	.1462	.1606	.1490	.1221	.0911	.0631
7	.0008	.0034	.0216	.0595	.1044	.1377	.1490	.1396	.1171	.0901
8	.0001	.0009	.0081	.0298	.0653	.1033	.1304	.1396	.1318	.1126
9	.0000	.0002	.0027	.0132	.0363	.0688	.1014	.1241	.1318	.1251
10	.0000	.0000	.0008	.0053	.0181	.0413	.0710	.0993	.1186	.1251
11	.0000	.0000	.0002	.0019	.0082	.0225	.0452	.0722	.0970	.1137
12	.0000	.0000	.0001	.0006	.0034	.0113	.0264	.0481	.0728	.0948
13	.0000	.0000	.0000	.0002	.0013	.0052	.0142	.0296	.0504	.0729
14	.0000	.0000	.0000	.0001	.0005	.0022	.0071	.0169	.0324	.0521
15	.0000	.0000	.0000	.0000	.0002	.0009	.0033	.0090	.0194	.0347
16	.0000	.0000	.0000	.0000	.0000	.0003	.0014	.0045	.0109	.0217
17	.0000	.0000	.0000	.0000	.0000	.0001	.0006	.0021	.0058	.0128
18	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0009	.0029	.0071
19	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0014	.0037
20	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0006	.0019
21	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0009
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0004
23	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002
24	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
25	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Exercises

1- Suppose that on a given weekend the number of accidents at a certain intersection has the Poisson distribution with mean 0.7. What is the probability that there will be at least three accidents at the intersection during the weekend?

2- Let $X \sim POI(\lambda)$,if P(X=1)=2P(X=2) , find λ

LECTURE 4#

Some examples for some discrete distributions

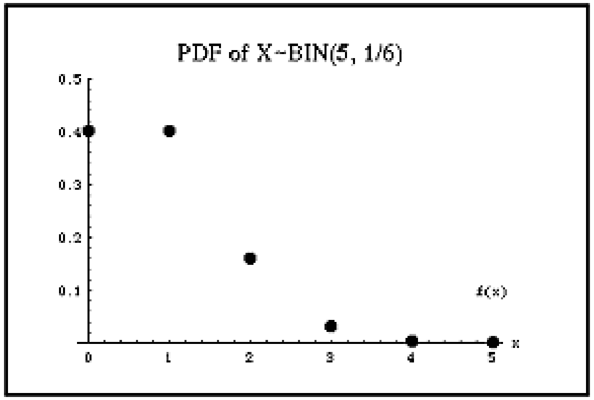
Example1

What is the probability of rolling at most two sixes in 5 independent casts of a fair die?

Sol:

Let the random variable X denote number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is p=$\frac{1}{6}$. Hence, the probability of rolling at most two sixes is:

$$
\begin{aligned}
P(X \leq 2) &= F(2) = f(0) + f(1) + f(2) \\
&= \binom{5}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{5} + \binom{5}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{4} + \binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{3} \\
&= \sum_{k=0}^{2}\binom{5}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{5-k} \\
&= \frac{1}{2}(0.9421 + 0.9734) = 0.9577 \qquad \text{(from binomial table)}
\end{aligned}
$$



Example2

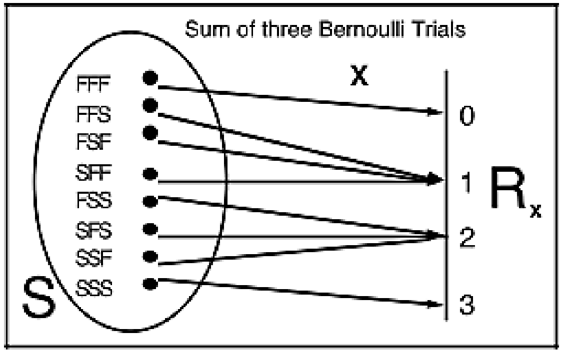
Let X_1 , X_2 , X_3 be three independent Bernoulli random variables with the same probability of success p. What is the probability density function of the random variable $X = X_1 + X_2 + X_3$? What is the mean and the variance of X?

Sol:

The sample space of the three independent Bernoulli trials is

S = { FFF, FFS, FSF, SFF, FSS, SFS, SSF, SSS } .

The random variable $X = X_1 + X_2 + X_3$ represents the number of successes in each element of S. The following diagram illustrates this.



Example2

Let p be the probability of success. Then

$$f(0) = P(X=0) = P(FFF) = (1-p)^3$$
$$f(1) = P(X=1) = P(FFS) + P(FSF) + P(SFF) = 3\,p\,(1-p)^2$$
$$f(2) = P(X=2) = P(FSS) + P(SFS) + P(SSF) = 3\,p^2\,(1-p)$$
$$f(3) = P(X=3) = P(SSS) = p^3.$$

Hence

$$f(x) = \binom{3}{x} p^x\,(1-p)^{3-x}, \qquad x = 0,\,1,\,2,\,3.$$

Thus, X~BIN(3,p). In general, if $X_i \sim BER(p)$, then $\sum_{i=1}^{n} X_i \sim BIN(n,p)$ and hence

$$E\left(\sum_{i=1}^{n} X_i\right) = n\,p \qquad , \qquad Var\left(\sum_{i=1}^{n} X_i\right) = n\,p\,(1-p).$$

Example3

If X~BER(p), What is the p.m.f. of Y=1-X?

Sol:

Since X~BER(p), then $\mathrm{P(x)}=p^{x}(1-p)^{1-x}$. Now, if x=0, then y=1 and if x=1 , then y=0. Also, Y=1-X.

Therefore, $\mathrm{P(y=1\text{-}x)}=p^{1-y}(1-p)^{y}= q^{y}(1-q)^{1-y}$, y =0,1

That is mean: Y=1-X ~BER(q).

Example4

Let X be the number of heads (successes) in n = 7 independent tosses of an unbiased coin. The pmf of X is:

$$p(x) = \begin{cases} \binom{7}{x}\left(\frac{1}{2}\right)^x\left(1-\frac{1}{2}\right)^{7-x} & x = 0, 1, 2, \ldots, 7 \\ 0 & \text{elsewhere.} \end{cases}$$

Then X has the mgf

$$M(t) = \left(\tfrac{1}{2} + \tfrac{1}{2}e^t\right)^7,$$

has mean $\mu = np = \frac{7}{2}$, and has variance $\sigma^2 = np(1-p) = \frac{7}{4}$. Furthermore, we have

$$P(0 \le X \le 1) = \sum_{x=0}^{1} p(x) = \frac{1}{128} + \frac{7}{128} = \frac{8}{128}$$

and

$$P(X = 5) = p(5) = \frac{7!}{5!2!}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 = \frac{21}{128}. \quad \blacksquare$$

Example5

The mgf of a random variable X is $(\frac{2}{3} + \frac{1}{3}e^t)^9$. Show that

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}.$$

Sol:

Since $n = 9$ and $p = 1/3$, $\mu = 3$ and $\sigma^2 = 2$. Hence, $\mu - 2\sigma = 3 - 2\sqrt{2}$ and $\mu + 2\sigma = 3 + 2\sqrt{2}$ and $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(X = 1, 2, \ldots, 5)$.

Example6

If X ~ BIN(n,p), show that: $E\left(\frac{X}{n}\right)=p$ and $E\left[\left(\frac{X}{n}-p\right)^2\right]=\frac{p(1-p)}{n}$.

Sol:

$$\begin{aligned} E\left(\frac{X}{n}\right) &= \frac{1}{n}E(X)=\frac{1}{n}(np)=p \\ E\left[\left(\frac{X}{n}-p\right)^2\right] &= \frac{1}{n^2}E[(X-np)^2]=\frac{np(1-p)}{n^2}=\frac{p(1-p)}{n}. \end{aligned}$$

Example7

Suppose that X has a Poisson distribution with $\mu = 2$. Compute $P(1 \leq X)$

Sol: The pmf of X is

$$p(x) = \begin{cases} \frac{2^x e^{-2}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

Then
$$\begin{aligned} P(1 \leq X) &= 1 - P(X = 0) \\ &= 1 - p(0) = 1 - e^{-2} = 0.865, \end{aligned}$$

Example8

If the random variable X has a Poisson distribution such that P(X = 1) = P(X = 2) , find P(X = 4).

Sol:

$$\frac{e^{-\mu}\mu}{1!} = \frac{e^{-\mu}\mu^2}{2!} \Rightarrow \mu = 2 \text{ and } P(X = 4) = \frac{e^{-2}2^4}{4!}.$$

Example9

1-The mgf of a random variable X is $e^{4(e^t-1)}$. Show that

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.931$$

Sol:

Try to solve

2- Let X have a Poisson distribution with mean 1. Compute, if it exists, the expected value E(X!).?

LECTURE 5#

➢ Outline :-

✓ Discrete distributions

4- Uniform distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

Solved exercises

Exercises

Uniform distribution

Definition : *A random variable* X *has a* ***discrete uniform distribution and it is referred to as a discrete uniform random variable if and only if its probability mass function is given by:***

$$f(x) = \frac{1}{k} \text{ for } x = 1, 2, \ldots, k,$$

We denoted by: (X~DU(k))

Proof*: *You do that.

$$\sum_{x=1}^{k} x = \frac{k(k+1)}{2}$$

Expectation and Variance:

$$E(X) = \sum_{x=1}^{k} x\, f(x) = \sum_{x=1}^{k} x \frac{1}{k} = \frac{1}{k}\sum_{x=1}^{k} x$$

, So,

$$\boxed{E(X) = \frac{(k+1)}{2}}$$

Def. of Var.

$$\boxed{Var(X) = \frac{k^2 - 1}{12}}$$

Tp proof that you have to use

$$\sum_{x=1}^{k} x^2 = \frac{k(k+1)(2k+1)}{6}$$

Uniform distribution

Moment generating and Characteristic function :

If X is a r.v. distributed as a discrete uniform dist., then the m.g.f. of X is given as follows:

Geometric series

$$M_X = E(e^{tX}) = \frac{1}{k}\sum_{x=1}^{k} e^{tX} = \frac{1}{k}\sum_{x=1}^{k} Z^x, \qquad Z = e^t$$

$$= \frac{1}{k}(Z + Z^2 + \cdots + Z^k) = \frac{Z}{k}(1 + Z + Z^2 + \cdots + Z^{k-1})$$

But $\sum_{x=0}^{k-1} Z^x = \frac{1-Z^k}{1-Z}$, then $M_X = \frac{Z}{k} \cdot \frac{1-Z^k}{1-Z} = \frac{e^t(1-e^{kt})}{k(1-e^t)} = \boxed{\frac{e^t(e^{kt}-1)}{k(e^t-1)}}$; t>0

By the same way, we can get the characteristic function as follows:

$$\varphi_X(t) = \boxed{\frac{e^{it}(e^{kit}-1)}{k(e^{it}-1)}};$$

Uniform distribution

Distribution function: The distribution function of a discrete uniform random variable X is:

$$F(X) = P(X \le x) = \sum_{u=1}^{x} f(u) = \sum_{u=1}^{x} \frac{1}{k} = \boxed{\frac{x}{k}} \quad ; \quad x = 1,2,\dots,k$$

Example1: Let X~DU(8). Find pmf, CDF, E(X), Var(X) and P(X ≤4).

Sol.: f(x)= $\frac{1}{8}$, F(x) $= \frac{x}{8}, E(X) = 4.5, Var(X) = \frac{63}{12}$

P(X $\le$ 4) $= F(4) = 0.5$. (Try to find P(X $\ge$ 3)?).

Example2: Let X~DU(k).Find the mean and the variance of Y=a+bX where a and be are two real constants.

Sol.: It will be direct by using the properties of discrete uniform distribution.

5-Hypergeometric Distribution

Consider a collection of n objects which can be classified into two classes, say class 1 and class 2. Suppose that there are n_1 objects in class 1 and n_2 objects in class 2. A collection of r objects is selected from these n objects at random and without replacement. We are interested in finding out the probability that exactly x of these r objects are from class 1. If x of these r objects are from class 1, then the remaining r − x objects must be from class 2. We can select x objects from class 1 in any one of $\binom{n_1}{x}$ways. Similarly, the remaining r − x objects can be selected in $\binom{n_2}{r-x}$ ways. Thus, the number of ways one can select a subset of r objects from a set of n objects, such that x number of objects will be from class 1 and r − x number of objects will be from class 2, is given by $\binom{n_1}{x}$ $\binom{n_2}{r-x}$ Hence,

$$P(X = x) = \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{r}},$$

where $x \leq r$, $x \leq n_1$ and $r - x \leq n_2$.

Hypergeometric Distribution

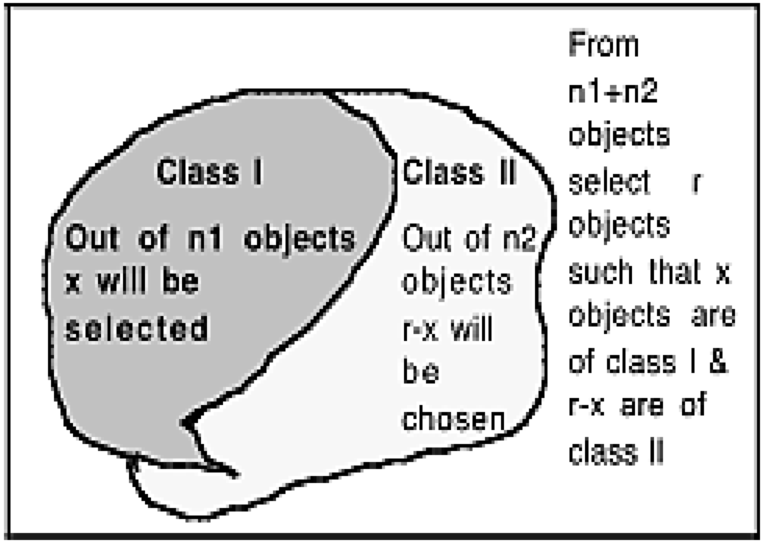
Definition : A random variable X is said to have a hypergeometric distribution if its probability mass function is of the form:

$$f(x) = \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}, \qquad x = 0, 1, 2, ..., r$$

where $x \leq n_1$ and $r - x \leq n_2$ with n_1 and n_2 being two positive integers.

We shall denote such a random variable by writing $X \sim HYP(n_1, n_2, r)$.

Example :Suppose there are 3 defective items in a lot of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective items in the sample. What is the probability that the sample contains at most one defective item?



Hypergeometric Distribution

Answer: Clearly, $X \sim HYP(3, 47, 10)$. Hence the probability that the sample contains at most one defective item is

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \frac{\binom{3}{0}\binom{47}{10}}{\binom{50}{10}} + \frac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} \\ &= 0.504 + 0.4 \\ &= 0.904. \end{aligned}$$

Theorem If $X \sim HYP(n_1, n_2, r)$, then

$$\begin{aligned} E(X) &= r\,\frac{n_1}{n_1 + n_2} \\ Var(X) &= r\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{n_1+n_2-r}{n_1+n_2-1}\right). \end{aligned}$$

Hypergeometric Distribution

Proof: Let $X \sim HYP(n_1, n_2, r)$. We compute the mean and variance of X by computing the first and the second factorial moments of the random variable X. First, we compute the first factorial moment (which is same as the expected value) of X. The expected value of X is given by

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{r} x\, f(x) \\
&= \sum_{x=0}^{r} x \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}} \\
&= n_1 \sum_{x=1}^{r} \frac{(n_1-1)!}{(x-1)!\,(n_1-x)!} \frac{\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}} \\
&= n_1 \sum_{x=1}^{r} \frac{\binom{n_1-1}{x-1}\binom{n_2}{r-x}}{\frac{n_1+n_2}{r}\binom{n_1+n_2-1}{r-1}} \\
&= r\,\frac{n_1}{n_1+n_2} \sum_{y=0}^{r-1} \frac{\binom{n_1-1}{y}\binom{n_2}{r-1-y}}{\binom{n_1+n_2-1}{r-1}}, \qquad \text{where } y = x-1 \\
&= r\,\frac{n_1}{n_1+n_2}.
\end{aligned}
$$

The last equality is obtained since $\sum_{y=0}^{r-1} \frac{\binom{n_1-1}{y}\binom{n_2}{r-1-y}}{\binom{n_1+n_2-1}{r-1}} = 1.$ where $\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$

Similarly, we find the second factorial moment of X to be

$E(X(X-1)) = \dfrac{r(r-1)\,n_1\,(n_1-1)}{(n_1+n_2)\,(n_1+n_2-1)}$. Therefore, the variance of X is

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= E(X(X-1)) + E(X) - E(X)^2 \\ &= \frac{r(r-1)\,n_1\,(n_1-1)}{(n_1+n_2)\,(n_1+n_2-1)} + r\,\frac{n_1}{n_1+n_2} - \left(r\,\frac{n_1}{n_1+n_2}\right)^2 \\ &= r\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{n_1+n_2-r}{n_1+n_2-1}\right). \end{aligned}$$

Distribution Function: The distribution function of a discrete hypergeometric random variable X is:

$F(X) = P(X \leq x) = \sum_{k=c}^{x} \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}$, where c=max(0,r-$n_1 + n_2$)

Moment generating function :

The m g. f. of a discrete hypergeometric random variable X is:

$$M_X(t) = \frac{(n_1 - r)!\,(n_1 - n_2)!}{n_1} . H(-r; -n_2; n_1 - n_2 + 1; e^t)$$

where $H(-r; -n_2; n_1 - n_2 + 1; e^t)$=$\sum_{j=0}^{\infty} \frac{(-r)^{[j]}(-n_2)^{[j]}(e^t)^j}{(n_1-n_2-r+1)^{[j]} j!}$ and in general ,

for any number a , then :

$$a^{[j]} = a(a+1)(a+2) \ldots (a+j-1).$$

Note: Let X1, X2 are r.v's distributed as Ber(p). If X2 is not independent of X1, and we should not expect X to have a binomial distribution. (why?)

See you next Lecture

LECTURE 6#

➢ Outline :-

- ✓ Discrete distributions

  6- Geometric distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Solved exercises

Exercises

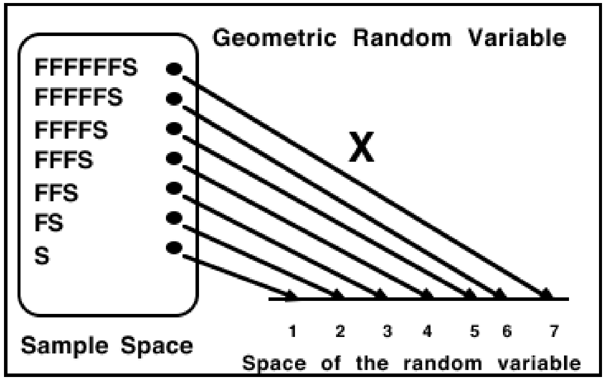
Geometric distribution

- If X represents the total number of successes in n independent Bernoulli trials, then the random variable X~BIN(n,p), where p is the probability of success of a single Bernoulli trial and the probability mass function of X is given by:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, \ldots, n.$$

Now, Let X denote the trial number on which the first success occurs. Hence the probability that the first success occurs on x *th* trial is given by:

$$f(x) = P(X = x) = (1-p)^{x-1} p.$$



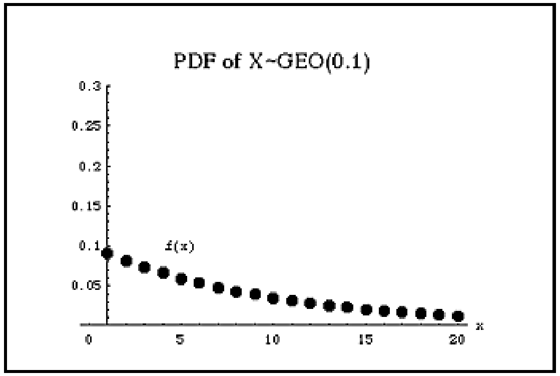
Geometric distribution

Definition: A random variable X has a geometric distribution if its probability mas function is given by :

$$\boxed{f(x) = (1-p)^{x-1}\, p \qquad x = 1,\, 2,\, 3, \ldots,\, \infty,}$$

Check that

where p denotes the probability of success in a single Bernoulli trial. If X has a geometric distribution we denote it as X ~ GEO(p).



Example: The probability that a machine produces a defective item is 0.02. Each item is checked as it is produced. Assuming that these are independent trials, what is the probability that at least 100 items must be checked to find one that is defective?

Geometric distribution

Answer: Let X denote the trial number on which the first defective item is observed. We want to find :

$$\begin{aligned} P(X \geq 100) &= \sum_{x=100}^{\infty} f(x) \\ &= \sum_{x=100}^{\infty} (1-p)^{x-1}\, p \\ &= (1-p)^{99} \sum_{y=0}^{\infty} (1-p)^{y}\, p \\ &= (1-p)^{99} \\ &= (0.98)^{99} = 0.1353. \end{aligned}$$

Hence the probability that at least 100 items must be checked to find one that is defective is 0.1353.

Geometric distribution

Theorem: If X is a geometric random variable with parameter p, then the mean, variance and moment generating functions are respectively given by:

$$\mu_X = \frac{1}{p} \quad , \quad \sigma_X^2 = \frac{1-p}{p^2} \quad , \; M_X(t) = \frac{p\, e^t}{1-(1-p)\, e^t}, \qquad \text{if } t < -ln(1-p).$$

Proof: First, we compute the moment generating function of X and then we generate all the mean and variance of X from it.

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{tx}\,(1-p)^{x-1}\,p \\ &= p\sum_{y=0}^{\infty} e^{t(y+1)}\,(1-p)^y, \qquad \text{where } y = x-1 \\ &= p\,e^t \sum_{y=0}^{\infty} \left(e^t\,(1-p)\right)^y \\ &= \frac{p\,e^t}{1-(1-p)\,e^t}, \qquad \text{if } t < -ln(1-p). \end{aligned}$$

Differentiating M(t) with respect to t, we obtain

$$\begin{aligned} M'(t) &= \frac{(1-(1-p)\,e^t)\;p\,e^t + p\,e^t\,(1-p)\,e^t}{[1-(1-p)e^t]^2} \\ &= \frac{p\,e^t\;[1-(1-p)\,e^t + (1-p)\,e^t]}{[1-(1-p)e^t]^2} \\ &= \frac{p\,e^t}{[1-(1-p)e^t]^2}. \end{aligned}$$

Hence $\boxed{\mu_X = E(X) = M'(0) = \frac{1}{p}.}$

Geometric distribution

Similarly, the second derivative of M(t) can be obtained from the first derivative as:

$$M''(t) = \frac{[1-(1-p)\,e^t]^2\,p\,e^t + p\,e^t\,2\,[1-(1-p)\,e^t]\,(1-p)\,e^t}{[1-(1-p)e^t]^4}.$$ Hence, $$M''(0) = \frac{p^3 + 2\,p^2(1-p)}{p^4} = \frac{2-p}{p^2}.$$

Therefore, the variance of X is:

$$\begin{aligned}\sigma_X^2 &= M''(0) - (\,M'(0)\,)^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2}.\end{aligned}$$

Theorem**.** The cumulative distribution function of a geometric random variable X is:

$$F(X) = P(X \le x) = 1 - (1-p)^x$$

Proof: $P(X \le k) = 1 - P(X > k)$

But $P(X > k) = P(X \ge k+1) = \sum_{x=k+1}^{\infty}(1-p)^{x-1}p$

$$= p\big[(1-p)^k\big[1 + (1-p) \quad + (1-p)^2 + \cdots\big]\big]$$

$$= p\left[(1-p)^k\left[\frac{1}{1-(1-p)}\right]\right] = (1-p)^k \longrightarrow P(X \le k) = 1 - (1-p)^k$$

Geometric distribution

Theorem: The characteristic function of a geometric random variable X is:

$$\varphi(t) = \frac{pe^{it}}{1-(1-p)e^{it}}$$

Proof: Similar to the proof of m.g.f.

Example: If the probability of engine malfunction during any one-hour period is p =.02 and Y denotes the number of one-hour intervals until the first malfunction, find the mean and standard deviation of Y .

Solution :Y has a geometric distribution with p =.02. Then: $E(Y) = 1/p = 1/(.02) = 50,$

$V(Y) = .98/.0004 = 2450,$ and the standard deviation of Y is $\sigma = \sqrt{2450} = 49.497.$

Exercises

1- Suppose that Y is a random variable with a geometric distribution. Show that

a $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p = 1$.

b $\dfrac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots$. This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

2- Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool. Find the probability that the first applicant with advanced training in programming is found on the fifth interview.

3- Suppose that X has the geometric distribution with parameter p. Show that for every nonnegative integer k,

$$\Pr(X \geq k) = (1 - p)^k.$$

SEE YOU IN THE NEXT LECTURE

LECTURE 7#

➢ Outline :-

✓ Continuous distributions

1- Uniform distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Solved exercises

Gamma function

Uniform distribution

Definition: A random variable X is said to be uniform on the interval [l,u], if its probability density function is of the form :

$$f(x) = \frac{1}{u-l} \quad , \qquad\qquad l \le x \le u$$

where a and b are constants. We denote a random variable X with the uniform distribution on the interval [l, u] as X ~ UNIF(*l*, u).

Theorem: If X is uniform on the interval [l, u] then the mean, variance and moment generating function of X are given by:

$$E(X) = \frac{u+l}{2}, \qquad Var(X) = \frac{(u-l)^2}{12}, M(t) = \frac{1}{(u-l)}[\exp(tu) - \exp(tl)]$$

Proof:

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_l^u x \frac{1}{u-l}\,dx = \frac{1}{u-l}\int_l^u x\,dx = \frac{1}{u-l}\left[\frac{1}{2}x^2\right]_l^u \\ &= \frac{1}{u-l}\frac{1}{2}\left[u^2 - l^2\right] = \frac{(u-l)(u+l)}{2(u-l)} = \frac{u+l}{2} \end{aligned}$$

Uniform distribution

Now, we want to find the variance of X:

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_l^u x^2 \frac{1}{u-l}dx = \frac{1}{u-l}\int_l^u x^2 dx = \frac{1}{u-l}\left[\frac{1}{3}x^3\right]_l^u = \frac{1}{u-l}\frac{1}{3}\left[u^3 - l^3\right]$$

$$= \frac{(u-l)\left(u^2+ul+l^2\right)}{3(u-l)} = \frac{u^2+ul+l^2}{3}$$

Also,

$$\mathrm{E}[X]^2 = \left(\frac{u+l}{2}\right)^2 = \frac{u^2+2ul+l^2}{4}$$, then

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{u^2+ul+l^2}{3} - \frac{u^2+2ul+l^2}{4} \\
&= \frac{4u^2+4ul+4l^2-3u^2-6ul-3l^2}{12} \\
&= \frac{(4-3)u^2+(4-6)ul+(4-3)l^2}{12} \\
&= \frac{u^2-2ul+l^2}{12} = \frac{(u-l)^2}{12}
\end{aligned}$$

Using the definition of m.g.f.:

$$\begin{aligned}
M_X(t) &= \mathrm{E}[\exp(tX)] = \int_{-\infty}^{\infty} \exp(tx) f_X(x)\,dx \\
&= \int_l^u \exp(tx)\frac{1}{u-l}dx = \frac{1}{u-l}\left[\frac{1}{t}\exp(tx)\right]_l^u \\
&= \frac{\exp(tu)-\exp(tl)}{(u-l)t}
\end{aligned}$$

Uniform distribution

Theorem: The characteristic function of a uniform random variable X is :

$$\varphi_X(t) = \begin{cases} \frac{1}{(u-l)it}\left[\exp(itu) - \exp(itl)\right] & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Proof: Using the definition of characteristic function:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}\left[\exp(itX)\right] = \mathrm{E}\left[\cos(tX)\right] + i\mathrm{E}\left[\sin(tX)\right] \\
&= \int_{-\infty}^{\infty} \cos(tx) f_X(x)\,dx + i\int_{-\infty}^{\infty} \sin(tx) f_X(x)\,dx \\
&= \int_l^u \cos(tx)\frac{1}{u-l}dx + i\int_l^u \sin(tx)\frac{1}{u-l}dx = \frac{1}{u-l}\left\{\int_l^u \cos(tx)\,dx + i\int_l^u \sin(tx)\,dx\right\} \\
&= \frac{1}{u-l}\left\{\left[\frac{1}{t}\sin(tx)\right]_l^u + i\left[-\frac{1}{t}\cos(tx)\right]_l^u\right\} = \frac{1}{(u-l)\,t}\left\{\sin(tu) - \sin(tl) - i\cos(tu) + i\cos(tl)\right\} \\
&= \frac{1}{(u-l)\,it}\left\{i\sin(tu) - i\sin(tl) + \cos(tu) - \cos(tl)\right\} = \frac{1}{(u-l)\,it}\left\{\left[\cos(tu) + i\sin(tu)\right] - \left[\cos(tl) + i\sin(tl)\right]\right\} \\
&= \frac{\exp(itu) - \exp(itl)}{(u-l)\,it}
\end{aligned}$$

Uniform distribution

Theorem: The Distribution function of a uniform random variable X is :

$$F_X(x) = \begin{cases} 0 & \text{if } x < l \\ (x-l)/(u-l) & \text{if } l \le x \le u \\ 1 & \text{if } x > u \end{cases}$$

Proof: If x < l, then $F_X(x) = \mathrm{P}(X \le x) = 0$ because X can not take on values smaller than l. $if\ l \le x \le u,$ then:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \le x) \\ &= \int_{-\infty}^{x} f_X(t)\,dt \\ &= \int_{l}^{x} \frac{1}{u-l}\,dt \\ &= \frac{1}{u-l}[t]_l^x \\ &= (x-l)/(u-l) \end{aligned}$$

If x > u, then $F_X(x) = \mathrm{P}(X \le x) = 1$

because X can not take on values greater than u.

Uniform distribution

1- Suppose Y ~ UNIF(0, 1) and Y $=\frac{1}{4}X^2$. What is the probability density function of X?

Sol: We shall find the probability density function of X through the cumulative distribution function of Y . The cumulative distribution function of X is given by:

$$F(x) = P(X \le x) = P(X^2 \le x^2) = P\left(\frac{1}{4}X^2 \le \frac{1}{4}x^2\right) \quad = P\left(Y \le \frac{x^2}{4}\right) = \int_0^{\frac{x^2}{4}} f(y)\,dy$$

$$= \int_0^{\frac{x^2}{4}} dy \quad = \frac{x^2}{4}.$$

Thus, $f(x) = \frac{d}{dx}F(x) = \frac{x}{2}.$ Hence the probability density function of X is given by:

$$\boxed{f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}}$$

Uniform distribution

2- If X has a uniform distribution on the interval from 0 to 10, then what is

$P\left(X + \frac{10}{X} \geq 7\right)$?

Sol: Since X ~ UNIF(0, 10), the probability density function of X is $f(x) = \frac{1}{10}$ for $0 \leq x \leq 10.$

Hence,

$$P\left(X + \frac{10}{X} \geq 7\right) = P\left(X^2 + 10 \geq 7X\right) = P\left(X^2 - 7X + 10 \geq 0\right) = P\left((X-5)(X-2) \geq 0\right)$$

$$= P\left(X \leq 2 \text{ or } X \geq 5\right) = 1 - P\left(2 \leq X \leq 5\right) = 1 - \int_2^5 f(x)\, dx = 1 - \int_2^5 \frac{1}{10}\, dx$$

$$= 1 - \frac{3}{10} = \frac{7}{10}.$$

3- A box to be constructed so that its height is 10 inches and its base is X inches by X inches. If X has a uniform distribution over the interval (2, 8), then what is the expected volume of the box in cubic inches?

Sol: Since X ~ UNIF(2, 8), $f(x) = \frac{1}{8-2} = \frac{1}{6}$ on $(2, 8)$. The volume V of the box is: $V = 10\, X^2.$

Hence,

$$E(V) = E\left(10X^2\right) = 10\, E\left(X^2\right) = 10 \int_2^8 x^2 \frac{1}{6}\, dx = \frac{10}{6}\left[\frac{x^3}{3}\right]_2^8 = \frac{10}{18}\left[8^3 - 2^3\right] = (5)\,(8)\,(7) = 280 \text{ cubic inches.}$$

Gamma distribution

The gamma distribution involves the notion of gamma function. First, we develop the notion of gamma function and study some of its well known properties. The gamma function, $\Gamma(z)$, is a generalization of the notion of factorial. The gamma function is defined as:

$$\Gamma(z) := \int_0^\infty x^{z-1}\, e^{-x}\, dx,$$

where z is positive real number (that is, $z > 0$).

Lemma 1: $\Gamma(1) = 1$.

Proof:

$$\Gamma(1) = \int_0^\infty x^0\, e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Lemma 2: The gamma function $\Gamma(z)$ satisfies the functional equation

$\Gamma(z) = (z-1)\,\Gamma(z-1)$ for all real number $z > 1$.

Proof: Let z be a real number such that $z > 1$, and consider $\Gamma(z) = \int_0^\infty x^{z-1}\, e^{-x}\, dx$

$$= \left[-x^{z-1}\, e^{-x}\right]_0^\infty + \int_0^\infty (z-1)\, x^{z-2}\, e^{-x}\, dx = (z-1)\int_0^\infty x^{z-2}\, e^{-x}\, dx = (z-1)\,\Gamma(z-1).$$

Gamma distribution

Lemma 3: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof: We want to show that $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is equal to $\sqrt{\pi}$. We substitute $y = \sqrt{x}$, hence the above integral becomes

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = 2\int_0^\infty e^{-y^2}\,dy, \qquad \text{where } y = \sqrt{x}.$$

Hence, $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du$ and also $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-v^2}\,dv.$

Multiplying the above two expressions, we get $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\,du\,dv.$

Now we change the integral into polar form by the transformation:

$u = r\cos(\theta)$ and $v = r\sin(\theta)$, The Jacobian of the transformation is

$$J(r,\theta) = det\begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{pmatrix} = det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Hence, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} J(r,\theta)\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} 2r\,dr\,d\theta$

Gamma distribution

Lemma 3: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof:

$$= 2\int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2}\, 2r\, dr\, dt = 2\int_0^{\frac{\pi}{2}} \Gamma(1)\, d\theta \quad = \pi.$$

Therefore, we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Note: If n is a natural number, then $\Gamma(n + 1) = n!$.

Lemma 4 : $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Proof: By Lemma 1 , we get: Γ (z) = (z − 1) Γ (z − 1). Letting $z = \frac{1}{2}$, we get

$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)\Gamma\left(\frac{1}{2} - 1\right)$, which is $\Gamma\left(-\frac{1}{2}\right) = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$.

Example: Evaluate $\Gamma\left(\frac{5}{2}\right)$

Answer: $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

Example: Evaluate $\Gamma\left(-\frac{7}{2}\right)$

Answer: $\Gamma\left(-\frac{1}{2}\right) = -\frac{3}{2}\,\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right)$

$$= \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\Gamma\left(-\frac{7}{2}\right).$$

Hence,

$$\Gamma\left(-\frac{7}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{7}\right)\Gamma\left(-\frac{1}{2}\right) = \frac{16}{105}\sqrt{\pi}.$$

SEE YOU IN THE NEXT LECTURE

LECTURE 8#

➢ Outline :-

✓ Discrete distributions

5- Hypergeometric distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

Solved exercises

Exercises

5-Hypergeometric Distribution

Consider a collection of n objects which can be classified into two classes, say class 1 and class 2. Suppose that there are n_1 objects in class 1 and n_2 objects in class 2. A collection of r objects is selected from these n objects at random and without replacement. We are interested in finding out the probability that exactly x of these r objects are from class 1. If x of these r objects are from class 1, then the remaining r − x objects must be from class 2. We can select x objects from class 1 in any one of $\binom{n_1}{x}$ways. Similarly, the remaining r − x objects can be selected in $\binom{n_2}{r-x}$ ways. Thus, the number of ways one can select a subset of r objects from a set of n objects, such that x number of objects will be from class 1 and r − x number of objects will be from class 2, is given by $\binom{n_1}{x}$ $\binom{n_2}{r-x}$ Hence,

$$P(X=x)=\frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{r}},$$

where $x \le r$, $x \le n_1$ and $r-x \le n_2$.

Hypergeometric Distribution

Definition : A random variable X is said to have a hypergeometric distribution if its probability mass function is of the form:

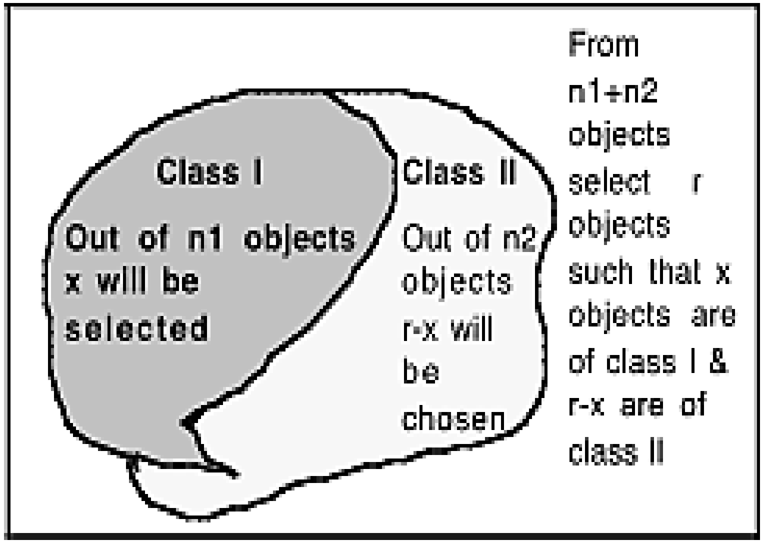
$$f(x) = \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}, \qquad x = 0, 1, 2, \ldots, r$$

where $x \leq n_1$ and $r - x \leq n_2$ with n_1 and n_2 being two positive integers.

We shall denote such a random variable by

writing $X \sim HYP(n_1, n_2, r).$

Example :Suppose there are 3 defective items in a lot of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective items in the sample. What is the probability that the sample contains at most one defective item?



Hypergeometric Distribution

Answer: Clearly, $X \sim HYP(3, 47, 10)$. Hence the probability that the sample contains at most one defective item is

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \frac{\binom{3}{0}\binom{47}{10}}{\binom{50}{10}} + \frac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} \\ &= 0.504 + 0.4 \\ &= 0.904. \end{aligned}$$

Theorem If $X \sim HYP(n_1, n_2, r)$, then

$$E(X) = r\,\frac{n_1}{n_1 + n_2}$$

$$Var(X) = r\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{n_1+n_2-r}{n_1+n_2-1}\right).$$

Hypergeometric Distribution

Proof: Let $X \sim HYP(n_1, n_2, r)$. We compute the mean and variance of X by computing the first and the second factorial moments of the random variable X. First, we compute the first factorial moment (which is same as the expected value) of X. The expected value of X is given by

$$\begin{aligned}
E(X) &= \sum_{x=0}^{r} x\, f(x) \\
&= \sum_{x=0}^{r} x \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}} \\
&= n_1 \sum_{x=1}^{r} \frac{(n_1-1)!}{(x-1)!\,(n_1-x)!} \frac{\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}} \\
&= n_1 \sum_{x=1}^{r} \frac{\binom{n_1-1}{x-1}\binom{n_2}{r-x}}{\frac{n_1+n_2}{r}\binom{n_1+n_2-1}{r-1}} \\
&= r\,\frac{n_1}{n_1+n_2} \sum_{y=0}^{r-1} \frac{\binom{n_1-1}{y}\binom{n_2}{r-1-y}}{\binom{n_1+n_2-1}{r-1}}, \qquad \text{where } y = x-1 \\
&= r\,\frac{n_1}{n_1+n_2}.
\end{aligned}$$

The last equality is obtained since $\sum_{y=0}^{r-1} \frac{\binom{n_1-1}{y}\binom{n_2}{r-1-y}}{\binom{n_1+n_2-1}{r-1}} = 1.$ where $\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$

Similarly, we find the second factorial moment of X to be

$$E(X(X-1)) = \frac{r(r-1)\,n_1\,(n_1-1)}{(n_1+n_2)\,(n_1+n_2-1)}.$$ Therefore, the variance of X is

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= E(X(X-1)) + E(X) - E(X)^2 \\ &= \frac{r(r-1)\,n_1\,(n_1-1)}{(n_1+n_2)\,(n_1+n_2-1)} + r\,\frac{n_1}{n_1+n_2} - \left(r\,\frac{n_1}{n_1+n_2}\right)^2 \\ &= r\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{n_1+n_2-r}{n_1+n_2-1}\right). \end{aligned}$$

Distribution Function: The distribution function of a discrete hypergeometric random variable X is:

$$F(X) = P(X \le x) = \sum_{k=c}^{x} \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}},$$ where c=max(0,r-$n_1 + n_2$)

Moment generating function :

The m g. f. of a discrete hypergeometric random variable X is:

$$M_X(t) = \frac{(n_1 - r)!\,(n_1 - n_2)!}{n_1} . H(-r; -n_2; n_1 - n_2 + 1; e^t)$$

where $H(-r; -n_2; n_1 - n_2 + 1; e^t)$=$\sum_{j=0}^{\infty} \frac{(-r)^{[j]}(-n_2)^{[j]}(e^t)^j}{(n_1-n_2-r+1)^{[j]} j!}$ and in general ,

for any number a , then :

$$a^{[j]} = a(a+1)(a+2)\ldots(a+j-1).$$

Note: Let X1, X2 are r.v's distributed as Ber(p). If X2 is not independent of X1, and we should not expect X to have a binomial distribution. (why?)

Hypergeometric Distribution

Example : A random sample of 5 students is drawn without replacement from among 300 seniors, and each of these 5 seniors is asked if she/he has tried a certain drug. Suppose 50% of the seniors actually have tried the drug. What is the probability that two of the students interviewed have tried the drug?

Answer: Let X denote the number of students interviewed who have tried the drug. Hence the probability that two of the students interviewed have tried the drug is

$$P(X=2) = \frac{\binom{150}{2}\binom{150}{3}}{\binom{300}{5}}$$
$$= 0.3146.$$

Hypergeometric Distribution

Example: A box contains 20 balls , 12 is red and others are black , if we select 8 ball a r.s. form this box, what is the probability of:

1- to get 3 red balls from this sample

2- At least two red balls have been got.

Sol: let X be the number of red balls selected from the sample.

So, X~HYP(20,12,8). And that means,

$$p(x) = \frac{\binom{12}{x}\binom{8}{8-x}}{\binom{20}{8}} , \qquad 0 \leq x \leq 8$$

So,

$$1 - \ p(3) = \frac{\binom{12}{3}\binom{8}{5}}{\binom{20}{8}} = 0.098801$$

2- $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)]$

$$=1- \left[\frac{\binom{12}{0}\binom{8}{8}}{\binom{20}{8}} + \frac{\binom{12}{1}\binom{8}{7}}{\binom{20}{8}} \right]=1-0.0008=0.9992$$

See you next Lecture

LECTURE 9#

➢ Outline :-

✓ Continuous distributions

2- Gamma distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Two special Distributions

Solved exercises

Gamma distribution

Let us take two parameters $\alpha > 0$ and $\beta > 0$. Gamma function $\Gamma(\alpha)$ is defined by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx. \quad \ldots\ldots\ldots(*)$$

Let $y = \beta x \longrightarrow x = \frac{y}{\beta}$ and then $dx = \frac{1}{\beta} dy$. Then,

If we divide both sides of (*) by $\Gamma(\alpha)$ we get :

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}} dy \quad \ldots.(**)$$

Then the integration in (**) will be a probability density function since it is nonnegative and it integrates to one.

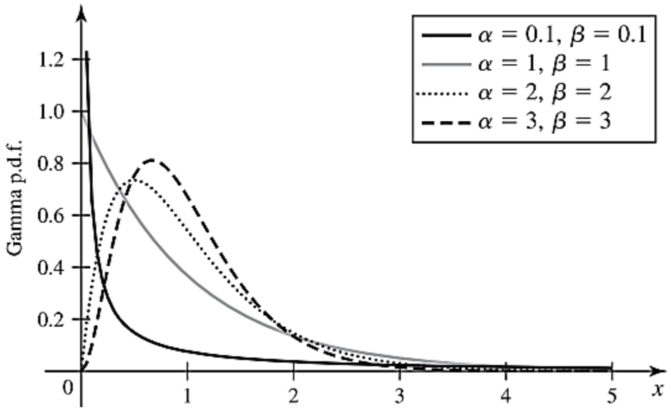
Therefore, we get the following definition:

Gamma distribution

Definition : A continuous random variable X is said to have a gamma distribution if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\theta > 0$. We denote a random variable with gamma distribution as $X \sim GAM(\theta, \alpha)$. The following diagram shows the graph of the gamma density for various values of values of the parameters θ and α.



Gamma distribution

Theorem: If X ~ GAM(θ, α), then, E(X) = θ α , Var(X) = θ^2 α and

$$M(t) = \left(\frac{1}{1-\theta\, t}\right)^{\alpha}, \qquad \text{if} \qquad t < \frac{1}{\theta}.$$

Proof: First, we derive the moment generating function of X and then we compute the mean and variance of it. The moment generating function:

$$\begin{aligned} M(t) &= E\left(e^{tX}\right) \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}}\, e^{tx}\, dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, x^{\alpha-1}\, e^{-\frac{1}{\theta}(1-\theta t)x}\, dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, \frac{\theta^\alpha}{(1-\theta t)^\alpha}\, y^{\alpha-1}\, e^{-y}\, dy, \qquad \text{where} \quad y = \frac{1}{\theta}(1-\theta t)x \\ &= \frac{1}{(1-\theta t)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)}\, y^{\alpha-1}\, e^{-y}\, dy \\ &= \frac{1}{(1-\theta t)^\alpha}, \qquad \text{since the integrand is } GAM(1,\alpha). \end{aligned}$$

$$\boxed{x = \frac{\theta\, y}{(1-\theta t)}, \quad dx = \frac{\theta}{(1-\theta t)}\, d\, y}$$

Gamma distribution

The first derivative of the moment generating function is:

$$\begin{aligned} M'(t) &= \frac{d}{dt}(1-\theta t)^{-\alpha} \\ &= (-\alpha)\,(1-\theta t)^{-\alpha-1}\,(-\theta) \\ &= \alpha\,\theta\,(1-\theta t)^{-(\alpha+1)}. \end{aligned}$$

Hence from above, we find the expected value of X to be $E(X) = M'(0) = \alpha\,\theta.$

Similarly,

$$\begin{aligned} M''(t) &= \frac{d}{dt}\left(\alpha\,\theta\,(1-\theta t)^{-(\alpha+1)}\right) \\ &= \alpha\,\theta\,(\alpha+1)\,\theta\,(1-\theta t)^{-(\alpha+2)} \\ &= \alpha\,(\alpha+1)\,\theta^2\,(1-\theta t)^{-(\alpha+2)}. \end{aligned}$$

Thus, the variance of X is

$$Var(X) = M''(0) - (M'(0))^2 = \alpha\,(\alpha+1)\,\theta^2 - \alpha^2\,\theta^2 \;= \alpha\,\theta^2$$

Gamma distribution

Theorem: The characteristic function of a Gamma random variable X is:

$$\varphi(t) = \frac{1}{(1-\theta it)^{\alpha}}.$$

Proof: By the same procedure for m.g.f.

Distribution function: The distribution function of a Gamma random variable is:

$F(X) = P(X \leq x) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha)}$, where $\Gamma_x(\alpha)$ is incomplete gamma function and it has the formula:

$$\Gamma_x(\alpha) = \int_0^x y^{\alpha-1} e^{-y} dy$$

Remark: Two special cases of gamma-distributed random variables merit particular consideration.(two special distributions)

Exponential Distribution

Definition: A continuous random variable is said to be an exponential random variable with parameter θ if its probability density function is of the form:

$$\boxed{f(x) = \begin{cases} \frac{1}{\theta}\, e^{-\frac{x}{\theta}} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}}$$

, where $\theta > 0$. If a random variable X has an exponential density function with parameter θ, then we denote it by writing $X \sim EXP(\theta)$.

Note: An exponential distribution is a special case of the gamma distribution. If the parameter $\alpha = 1$, then the gamma distribution reduces to the exponential distribution. Hence most of the information about an exponential distribution can be obtained from the gamma distribution.

Example: Let X have the density function :

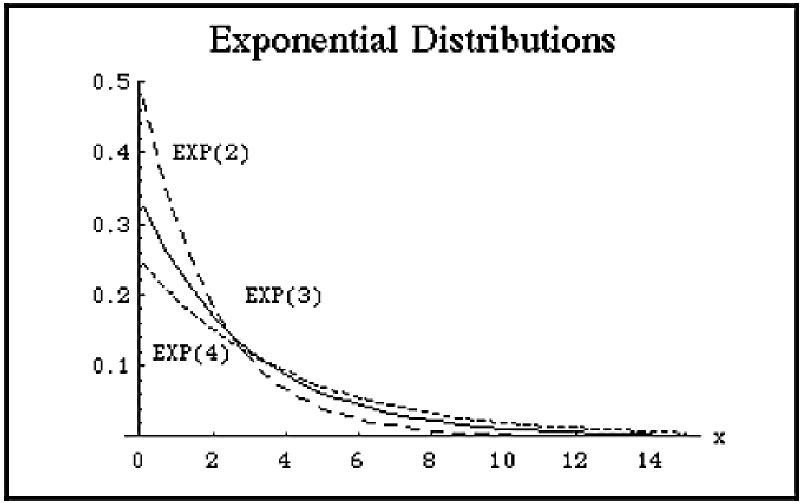
$$\boxed{f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}}$$

where $\alpha > 0$ and $\theta > 0$. If $\alpha = 4$, what is the mean of $\frac{1}{X^3}$?

Exponential Distribution

Answer:

$$
\begin{aligned}
E\left(X^{-3}\right) &= \int_0^\infty \frac{1}{x^3}\, f(x)\, dx \\
&= \int_0^\infty \frac{1}{x^3}\, \frac{1}{\Gamma(4)\,\theta^4}\, x^3\, e^{-\frac{x}{\theta}}\, dx \\
&= \frac{1}{3!\,\theta^4} \int_0^\infty e^{-\frac{x}{\theta}}\, dx \\
&= \frac{1}{3!\,\theta^3} \int_0^\infty \frac{1}{\theta}\, e^{-\frac{x}{\theta}}\, dx \\
&= \frac{1}{3!\,\theta^3} \qquad \text{since the integrand is GAM}(\theta, 1).
\end{aligned}
$$



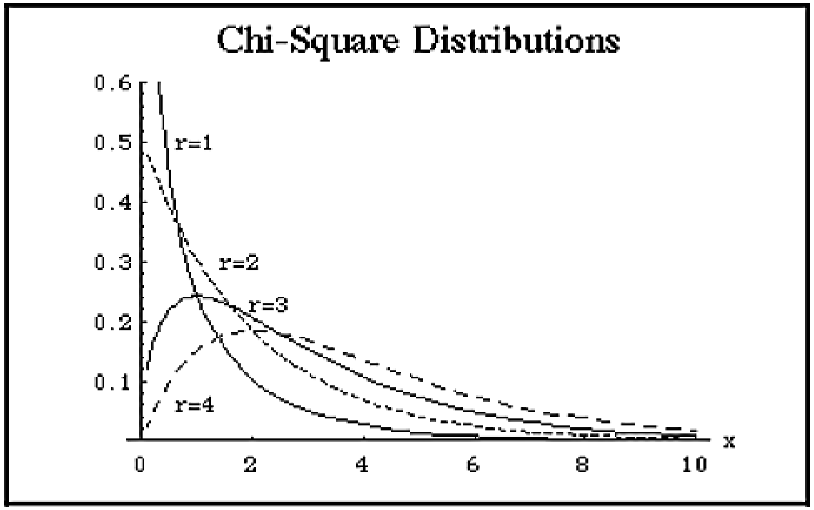
Chi-square Distribution

Definition: A continuous random variable X is said to have a chi-square distribution with r degrees of freedom if its probability density function is of the form:

$$f(x) = \begin{cases} \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where r > 0. If X has a chi-square distribution, then we denote it by writing $X \sim \chi^2(r)$.

Note: The gamma distribution reduces to the chi-square distribution if $\alpha = \frac{r}{2}$ and θ = 2. Thus, the chi-square distribution is a special case of the gamma distribution. Hence most of the information about an chi-square distribution can be obtained from the gamma distribution.



Example: If X ~ GAM(1, 1), then what is the probability density function of the random variable 2X?

Answer: We will use the moment generating method to find the distribution of 2X. The moment generating function of a gamma random variable is given by

$$M(t) = (1 - \theta\, t)^{-\alpha}, \qquad \text{if} \qquad t < \frac{1}{\theta}.$$

Since X ~ GAM(1, 1), the moment generating function of X is given by :

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1.$$

Hence, the moment generating function of 2X is :

$$M_{2X}(t) = M_X(2t) = \frac{1}{1-2t} \boxed{= \frac{1}{(1-2t)^{\frac{2}{2}}}}$$

The m.g.f. of $\chi^2(2)$.

Hence, if X is an exponential with parameter 1, then 2X is chi-square with 2 degrees of freedom.

SEE YOU IN THE NEXT LECTURE

LECTURE 10#

➢ Outline :-

-Solving exercises of Binomial Distribution

1) On a five-question multiple-choice test there are five possible answers, of which one is correct. If a student guesses randomly and independently, what is the probability that she is correct only on two questions?

Solution: Here the probability of success is $\frac{1}{5}$ and thus $1 - p = \frac{4}{5}$. There are $\binom{5}{2}$ different ways she can be correct on two questions. Therefore, the probability that she is correct on two questions is:

$$P(\text{correct on two questions}) = \binom{5}{2} p^2 (1-p)^3 = 10 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 = \frac{640}{5^5} = 0.2048.$$

2) What is the probability of rolling two sixes and three nonsixes in 5 independent casts of a fair die?

Solution : Let the random variable X denote the number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is $\frac{1}{6}$. Hence:

$$P(X = 2) = f(2) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 = 10 \left(\frac{1}{36}\right) \left(\frac{125}{216}\right) = \frac{1250}{7776} = 0.160751.$$

3) What is the probability of rolling at most two sixes in 5 independent casts of a fair die?

Answer: Let the random variable X denote number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is $\frac{1}{6}$.

Hence, the probability of rolling at most two sixes is :

$$P(X \leq 2) = F(2) = f(0) + f(1) + f(2)$$

$$= \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 + \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$= \sum_{k=0}^{2}\binom{5}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{5-k}$$

$$= \frac{1}{2}(0.9421 + 0.9734) = 0.9577$$ (from binomial table)

4) Suppose that 2000 points are selected independently and at random from the unit squares $S = \{(x, y) \mid 0 \le x, y \le 1\}$. Let X equal the number of points that fall in $A = \{(x, y) \mid x^2 + y^2 < 1\}$. How is X distributed? What are the mean, variance and standard deviation of X?

Answer: If a point falls in A, then it is a success. If a point falls in the complement of A, then it is a failure. The probability of success is

$$p = \frac{\text{area of A}}{\text{area of S}} = \frac{1}{4}\pi.$$

Since, the random variable represents the number of successes in 2000 independent trials, the random variable X is a binomial with parameters $p = \frac{\pi}{4}$ and n = 2000, that is X ~ BIN(2000, $\frac{\pi}{4}$).

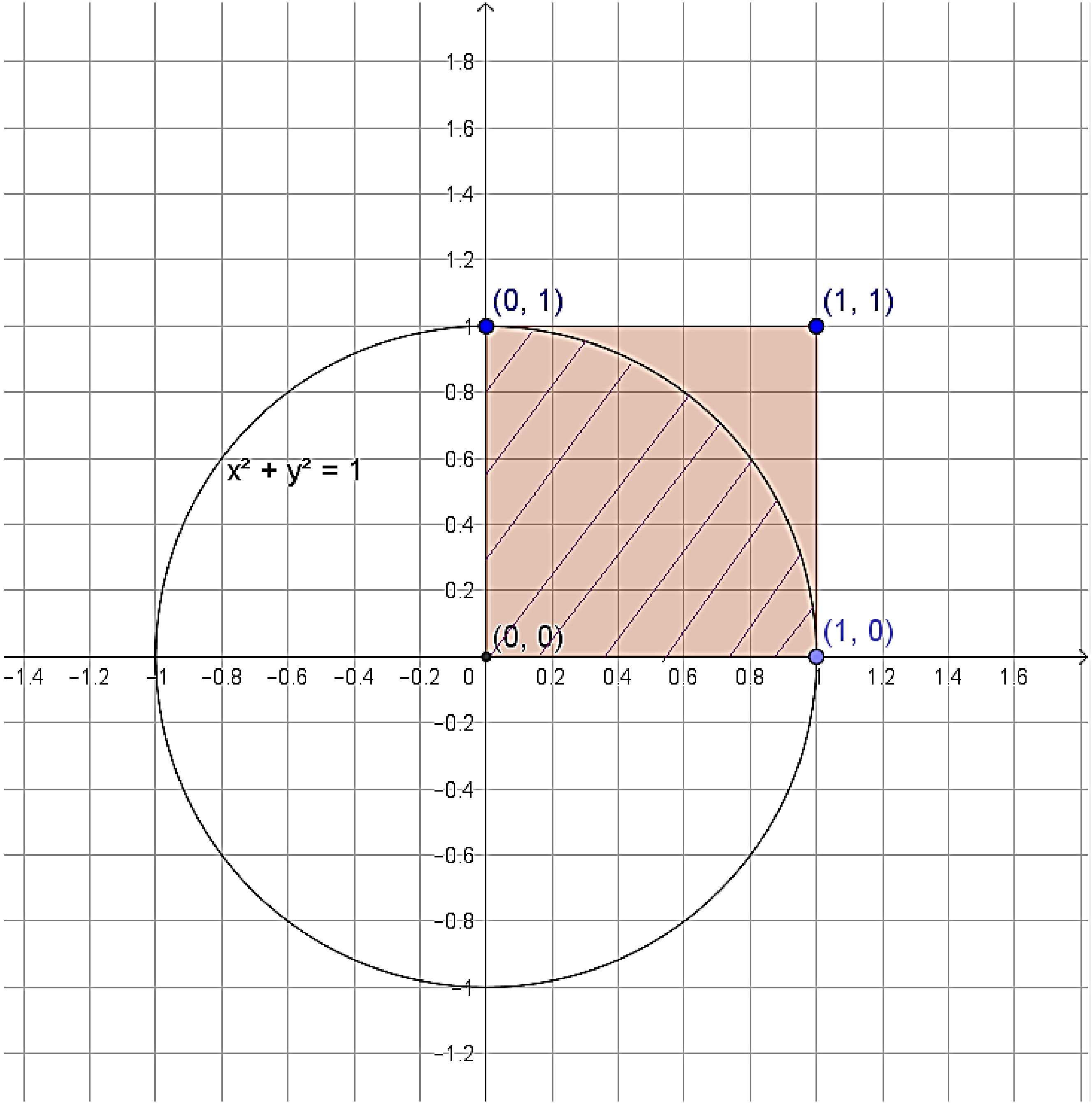
Therefore,

$$\mu_X = 2000\,\frac{\pi}{4} = 1570.8,$$

and

$$\sigma_X^2 = 2000\left(1 - \frac{\pi}{4}\right)\frac{\pi}{4} = 337.1.$$

The standard deviation of X is $\sigma_X = \sqrt{337.1} = 18.36.$



(0, 1)
(1, 1)
(0, 0)
(1, 0)
x² + y² = 1

5) Let the probability that the birth weight (in grams) of babies in America is less than 2547 grams be 0.1. If X equals the number of babies that weigh less than 2547 grams at birth among 20 of these babies selected at random, then what is $P(X \leq 3)$?

Answer: If a baby weighs less than 2547, then it is a success; otherwise it is a failure. Thus X is a binomial random variable with probability of success p and n = 20. We are given that p = 0.1. Hence

$$P(X \leq 3) = \sum_{k=0}^{3} \binom{20}{k} \left(\frac{1}{10}\right)^k \left(\frac{9}{10}\right)^{20-k}$$

$$= 0.867 \qquad \text{(from table).}$$

SEE YOU IN THE NEXT LECTURE

LECTURE 11#

➢ Outline :-

✓ Continuous distributions

3- Normal distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Solved exercises

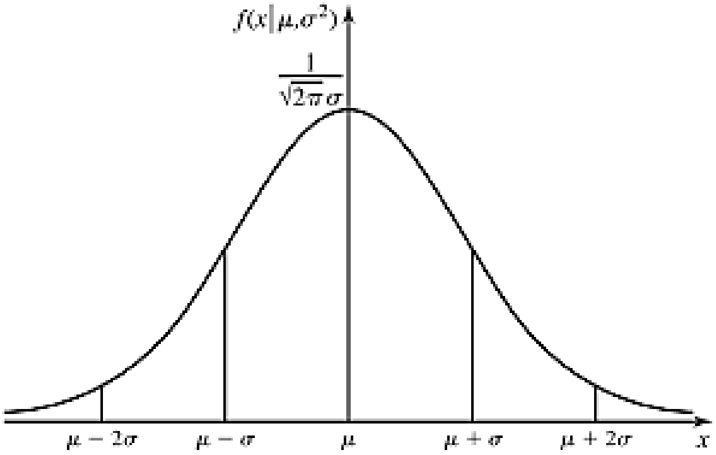
Normal distribution

Definition: A random variable X is said to have a normal distribution if its probability density function is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty,$$

where $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are arbitrary parameters. If X has a normal distribution with parameters μ and σ^2 , then we write X $\sim$N(μ, σ^2).

Proof: we must check that f is nonnegative and it integrates to 1.The nonnegative part function is always positive. Hence using property of the gamma function, we show that f integrates to 1 on IR.



Normal distribution

Proof:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \;= 2\int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} e^{-z} \frac{\sigma}{\sqrt{2z}}\,dz, \quad \text{where} \quad z = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{z}} e^{-z}\,dz$$

$x=\sigma\sqrt{2z}+\mu$

$\Rightarrow$ $dx=\frac{\sigma}{\sqrt{2z}}dz$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

Definition of Gamma function

$$= \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

From lecture 7 Lemma 3

Normal distribution

Theorem: If X ~N(μ, σ^2), then $E(X) = \mu$, $Var(X) = \sigma^2$ and $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

Proof: We prove this theorem by first computing the moment generating function and finding out the mean and variance of X from it.

$$M(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)}\, e^{-\frac{1}{2}z^2}\, dz, \qquad z = \frac{x-\mu}{\sigma}$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z - t^2\sigma^2 + t^2\sigma^2)}\, dz,$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - t\sigma)^2}\, dz}_{\boxed{Z\sim N(t\,\sigma, 1)}}$$

So, M(t)=1

Normal distribution

Proof: The first two derivatives of the m.g.f. of X is:

$M'(t) = \left(\mu + \sigma^2 t\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$, $M''(t) = \left([\mu + \sigma^2 t]^2 + \sigma^2\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

Plugging t = 0 into each of these derivatives yields:

$E(X) = M'(0) = \mu$ and $Var(X) = M''(0) - (M'(0))^2 = \sigma^2$

Characteristic function: If X~ N(μ, σ^2), then

$$\varphi_X(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

Proof: Same as the proof of m.g.f

Normal distribution

Example: If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable $Y = \frac{X-\mu}{\sigma}$?

Answer: The mean of the random variable Y is :

$$E(Y) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E\left(X-\mu\right) = \frac{1}{\sigma}\left(E(X)-\mu\right) = \frac{1}{\sigma}\left(\mu-\mu\right) = 0.$$

The variance of Y is given by:

$$Var(Y) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var\left(X-\mu\right) = \frac{1}{\sigma} Var(X) = \frac{1}{\sigma^2}\sigma^2 = 1.$$

Hence, if we define a new random variable by taking a random variable and subtracting its mean from it and then dividing the resulting by its standard deviation, then this new random variable will have zero mean and unit variance.

Normal distribution

Definition: A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by $X \sim N(0,1)$.

The probability density function of standard normal distribution is the following:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

SEE YOU IN THE NEXT LECTURE

LECTURE 12#

➢ Outline :-

Solving exercises

1- Suppose that on a given weekend the number of accidents at a certain intersection has the Poisson distribution with mean 0.7. What is the probability that there will be at least three accidents at the intersection during the weekend?

Sol: Let X be the number of accidents at a certain intersection. Then X~POI(0.7).

Form the table of the Poisson distribution that given in lecture 3, we get:

$P(X \geq 3)$ =1-P(X<3)=1-[P(X=0)+P(X=1)+P(X=2)]=1-0.9659=0.0341

2- Let X~POI(λ). If P(X=1)=2P(X=2), find λ?

Sol: P(X=1)=2P(X=2) $\longrightarrow$ $\lambda e^{-\lambda} = \frac{2\lambda^2 e^{-\lambda}}{2!}$ $\longrightarrow$ $\lambda(\lambda - 1) = 0$

$\longrightarrow$ λ=1 (λ=0 ignore)

Sampling without Replacement. Suppose that a box contains A red balls and B blue balls. Suppose also that $n \geq 0$ balls are selected at random from the box without replacement, and let X denote the number of red balls that are obtained. Clearly, we must have $n \leq A + B$ or we would run out of balls. Also, if $n = 0$, then $X = 0$ because there are no balls, red or blue, drawn. For cases with $n \geq 1$, we can let $X_i = 1$ if the ith ball drawn is red and $X_i = 0$ if not. Then each X_i has a Bernoulli distribution, but $X_1, \ldots, X_n$ are not independent in general. To see this, assume that both $A > 0$ and $B > 0$ as well as $n \geq 2$. We will now show that $\Pr(X_2 = 1|X_1 = 0) \neq \Pr(X_2 = 1|X_1 = 1)$. If $X_1 = 1$, then when the second ball is drawn there are only $A - 1$ red balls remaining out of a total of $A + B - 1$ available balls. Hence, $\Pr(X_2 = 1|X_1 = 1) = (A - 1)/(A + B - 1)$. By the same reasoning,

$$\Pr(X_2 = 1|X_1 = 0) = \frac{A}{A + B - 1} > \frac{A - 1}{A + B - 1}.$$

Hence, X_2 is not independent of X_1, and we should not expect X to have a binomial distribution. ◀

SEE YOU IN THE NEXT LECTURE

LECTURE 13#

➢ Outline :-

✓ Continuous distributions

3- Normal distribution

Distribution function

Solved exercises

Normal distribution

Example: If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable $Y = \frac{X-\mu}{\sigma}$?

Answer: The mean of the random variable Y is :

$$E(Y) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma} (E(X)-\mu) = \frac{1}{\sigma} (\mu-\mu) = 0.$$

The variance of Y is given by:

$$Var(Y) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma} Var(X) = \frac{1}{\sigma^2} \sigma^2 = 1.$$

Hence, if we define a new random variable by taking a random variable and subtracting its mean from it and then dividing the resulting by its standard deviation, then this new random variable will have zero mean and unit variance.

Normal distribution

Definition: A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by

X ~N(0,1).

The probability density function of standard normal distribution is the following:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.}$$

Distribution function: There is no simple formula for the distribution function $F_X(x)$ of a standard normal random variable X because a closed-form expression for the integral $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ does not exist; hence, its evaluation requires the use of numerical integration techniques. Probabilities and quantiles for random variables with normal distributions are easily found using any program like Matlab or R or….

Normal distribution

Note :Some values of the distribution function of X are used very frequently and people usually learn them by heart:

$$F_X(-2.576) = 0.005 \qquad F_X(2.576) = 0.995$$
$$F_X(-2.326) = 0.01 \qquad F_X(2.326) = 0.99$$
$$F_X(-1.96) = 0.025 \qquad F_X(1.96) = 0.975$$
$$F_X(-1.645) = 0.05 \qquad F_X(1.645) = 0.95$$

Note also that: $F_X(-x) = 1 - F_X(x)$ which is due to the symmetry around 0 of the standard normal density and is often used in calculations.

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \leq x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \geq 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \le x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \ge 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

Normal distribution

Therefore, if we know how to compute the values of the distribution function of a standard normal distribution (by table), we also know how to compute the values of the distribution function of a normal distribution with mean $\mu$and variance σ^2.

The following theorem is very important and allows us to find probabilities by using the standard normal table.

Theorem: If X ~ N(μ, σ^2), then the random variable $Z = \frac{X-\mu}{\sigma}$, ~N(0,1)

Proof : We will show that Z is standard normal by finding the probability density function of Z. We compute the probability density of Z by cumulative distribution function method.

X =μ+σ Z , then

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \le x) \\ &= \mathrm{P}(\mu + \sigma Z \le x) \longrightarrow Z = \frac{X-\mu}{\sigma} \longrightarrow X = \mu + \sigma Z \\ &= \mathrm{P}\left(Z \le \frac{x-\mu}{\sigma}\right) \\ &= F_Z\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

Hence,

$$f(z) = F'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Normal distribution

Example: If X ~ N(0, 1), what is the probability of the random variable X less than or equal to − 1.72?

Answer: $$\begin{aligned} P(X \le -1.72) &= 1 - P(X \le 1.72) \\ &= 1 - 0.9573 \qquad \text{(from table)} \\ &= 0.0427. \end{aligned}$$

The following example illustrates how to use standard normal table to find probability for normal random variables.

Example: If X ~ N(3, 16), then what is $P(4 \le X \le 8)$?

Answer:

$$P(4 \le X \le 8) = P\left(\frac{4-3}{4} \le \frac{X-3}{4} \le \frac{8-3}{4}\right) = P\left(\frac{1}{4} \le Z \le \frac{5}{4}\right) = P(Z \le 1.25) - P(Z \le 0.25)$$

$$= \underbrace{0.8944 - 0.5987}_{\text{(From table)}} = 0.2957.$$

Normal distribution

Example: If X ~ N(25, 36), then what is the value of the constant c such that

$$P\left(|X-25| \le c\right) = 0.9544?$$

Answer:

$$0.9544 = P\left(|X-25| \le c\right) = P\left(-c \le X-25 \le c\right) = P\left(-\frac{c}{6} \le \frac{X-25}{6} \le \frac{c}{6}\right) = P\left(-\frac{c}{6} \le Z \le \frac{c}{6}\right)$$

$$= P\left(Z \le \frac{c}{6}\right) - P\left(Z \le -\frac{c}{6}\right) = 2\,P\left(Z \le \frac{c}{6}\right) - 1.$$

Hence, $P\left(Z \le \frac{c}{6}\right) = 0.9772$

and from this, using the normal table, we get $\frac{c}{6} = 2$ or $c = 12.$

SEE YOU IN THE NEXT LECTURE

LECTURE 14#

- Outline :-

Solving exercises of Geometric distribution

1- Suppose that Y is a random variable with a geometric distribution. Show that

a $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p = 1$.

b $\dfrac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots$. This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

Solution : a) $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p$ (because Y ~ GEO(p))

Let x=y-1 $\longrightarrow$ $\sum_{y=1}^{\infty} q^{y-1} p = p \sum_{x=0}^{\infty} q^x$

But $\sum_{x=0}^{\infty} q^x$ infinite sum of a geometric series , therefore,

$$\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p = p \sum_{x=0}^{\infty} q^x = p \frac{1}{1-q} := p \frac{1}{1-(1-p)} = 1.$$

1- Suppose that Y is a random variable with a geometric distribution. Show that

a $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p = 1.$

b $\dfrac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots$. This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

Solution : b) $\dfrac{p(y)}{p(y-1)} = \dfrac{q^{y-1}p}{q^{y-2}p} = q.$ (because Y ~ GEO(p))

Also, The event Y = 1 has the highest probability for all p, 0 < p < 1, because :

P(Y=1)=p(1)=$(1-p)^{1-1}\, p = p.$

2- Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool. Find the probability that the first applicant with advanced training in programming is found on the fifth interview.

Solution : Let X be the Applicants are interviewed sequentially and are selected at random from the pool . So X ~ GEO(p=0.3) and then:

P(X=5)=p(5)=$(1-0.3)^{5-1}\,0.3=0.7^{4}\,0.3=0.07203.$

3- Suppose that X has the geometric distribution with parameter p. Show that for every positive integer a,

$$P(Y > a) = q^a.$$

Solution:

$$P(Y > a) = \sum_{y=a+1}^{\infty} q^{y-1} p = q^a \sum_{x=1}^{\infty} q^{x-1} p = q^a. \quad \text{(because } \sum_{y=a+1}^{\infty} q^{y-1} p = q^a p + q^{a+1} p + \cdots)$$

SEE YOU IN THE NEXT LECTURE

LECTURE 15#

➢ Outline :-

✓ Continuous distributions

4- Student's t-distribution

Definition

Expected value Variance

Moment generating function

Solved exercises

Student's t-distribution

Definition :A continuous random variable X is said to have a t-distribution with ν degrees of freedom if its probability density function is of the form:

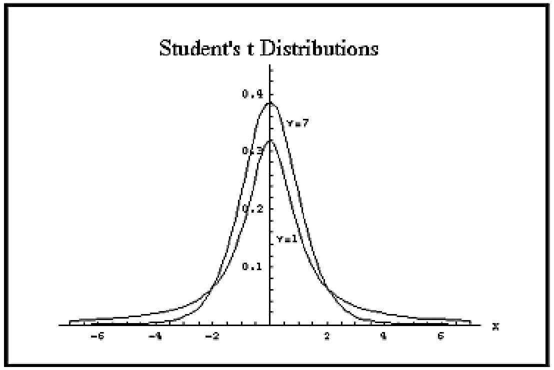
$$f(x;\,\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\,\nu}\;\Gamma\left(\frac{\nu}{2}\right)\,\left(1+\frac{x^2}{\nu}\right)^{\left(\frac{\nu+1}{2}\right)}}, \qquad -\infty < x < \infty$$

where $\nu > 0$. If X has a t-distribution with ν degrees of freedom, then we denote it by writing X~t(v). The t-distribution was discovered by W.S. Gosset (1876-1936) of England who published his work under the pseudonym of student. Therefore, this distribution is known as Student's t-distribution.

Note: if $\nu \rightarrow \infty$, then

$$\lim_{\nu\rightarrow\infty} f(x;\,\nu) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \qquad -\infty < x < \infty,$$

which is the probability density function of the standard normal distribution.



Student's t-distribution

Theorem : If the random variable X has a t-distribution with ν degrees of freedom, then:

$$E[X] = \begin{cases} 0 & \text{if} \quad \nu \geq 2 \\ DNE & \text{if} \quad \nu = 1 \end{cases} \quad \text{and} \quad Var[X] = \begin{cases} \frac{\nu}{\nu-2} & \text{if} \quad \nu \geq 3 \\ DNE & \text{if} \quad \nu = 1, 2 \end{cases}$$

where DNE means does not exist.

Theorem: If Z ~ N(0, 1) and $U \sim \chi^2(\nu)$ and in addition, Z and U are independent, then the random variable W defined by :

$$W = \frac{Z}{\sqrt{\frac{U}{\nu}}}$$

has a t-distribution with ν degrees of freedom.

Note: A standard Student's t random variable X does not possess a moment generating function.

Student's t-distribution

Example: If T ~ t(10), then what is the probability that T is at least 2.228 ?

Solution:

$$P(T \geq 2.228) = 1 - P(T < 2.228)$$
$$= 1 - 0.975 \qquad \text{(from t - table)}$$
$$= 0.025.$$

Example: If T ~ t(19), then what is the value of the constant c such that

$P\left(|T| \leq c\right) = 0.95$?

(1-α) compared to the table

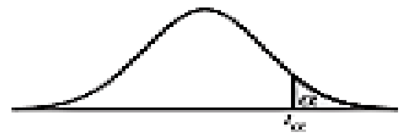
Solution:

$$0.95 = P\left(|T| \leq c\right) = P\left(-c \leq T \leq c\right) = P\left(T \leq c\right) - 1 + P\left(T \leq c\right) = 2\,P\left(T \leq c\right) - 1.$$

Hence:

$$P\left(T \leq c\right) = 0.975. \qquad \boxed{0.025}$$

Thus, using the t-table, we get for 19 degrees of freedom c = 2.093.



$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	df
3.078	6.314	12.706	31.821	63.657	1
1.886	2.920	4.303	6.965	9.925	2
1.638	2.353	3.182	4.541	5.841	3
1.533	2.132	2.776	3.747	4.604	4
1.476	2.015	2.571	3.365	4.032	5
1.440	1.943	2.447	3.143	3.707	6
1.415	1.895	2.365	2.998	3.499	7
1.397	1.860	2.306	2.896	3.355	8
1.383	1.833	2.262	2.821	3.250	9
1.372	1.812	2.228	2.764	3.169	10
1.363	1.796	2.201	2.718	3.106	11
1.356	1.782	2.179	2.681	3.055	12
1.350	1.771	2.160	2.650	3.012	13
1.345	1.761	2.145	2.624	2.977	14
1.341	1.753	2.131	2.602	2.947	15
1.337	1.746	2.120	2.583	2.921	16
1.333	1.740	2.110	2.567	2.898	17
1.330	1.734	2.101	2.552	2.878	18
1.328	1.729	2.093	2.539	2.861	19
1.325	1.725	2.086	2.528	2.845	20
1.323	1.721	2.080	2.518	2.831	21
1.321	1.717	2.074	2.508	2.819	22
1.319	1.714	2.069	2.500	2.807	23
1.318	1.711	2.064	2.492	2.797	24
1.316	1.708	2.060	2.485	2.787	25
1.315	1.706	2.056	2.479	2.779	26
1.314	1.703	2.052	2.473	2.771	27
1.313	1.701	2.048	2.467	2.763	28
1.311	1.699	2.045	2.462	2.756	29
1.282	1.645	1.960	2.326	2.576	inf.

SEE YOU IN THE NEXT LECTURE

LECTURE 16#

➢ Outline :-

Gamma function

Gamma Function

The gamma function, $\Gamma(z)$, is a generalization of the notion of factorial. The gamma function is defined as:

$$\Gamma(z) := \int_0^\infty x^{z-1}\, e^{-x}\, dx,$$

where z is positive real number (that is, $z > 0$).

Lemma 1: $\Gamma(1) = 1$.

Lemma 2: The gamma function $\Gamma(z)$ satisfies the functional equation

$\Gamma(z) = (z - 1)\,\Gamma(z - 1)$ for all real number $z > 1$.

Lemma 3: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Lemma 4 : $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Lemma 5 : If n is a natural number, then $\Gamma(n + 1) = n!$.

Lemma 6 : If $n \neq 0$, then $\Gamma(n + 1) = n\,\Gamma(n) \longrightarrow \Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

Gamma Function

Example: Evaluate $\Gamma\left(\frac{5}{2}\right)$

Answer: By Lemma 6

$$\Gamma\left(\frac{5}{2}\right) = \Gamma(\frac{3}{2}+\frac{2}{2}) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

EX) FIND

1) $\Gamma(7)$ 2) $\Gamma(2.5)$ 3) $\Gamma(-0.5)$ 4) $\Gamma(0.4)$

SOL /

1)

$$\Gamma(n+1) = n! \qquad \Gamma(7) = \Gamma(6+1) = 6! = 720$$

2)

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(2.5) = \Gamma(1.5+1) = (1.5)\Gamma(1.5) = (1.5)\Gamma(0.5+1)$$

$$= (1.5)(0.5)\Gamma(0.5) = 0.75\sqrt{\pi} = 0.866226$$

3)

$$\Gamma n = \frac{\Gamma(n+1)}{n} \Rightarrow \Gamma(-0.5) = \frac{1}{-0.5}\Gamma(0.5) = -2\sqrt{\pi}$$

4)

$$\Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(0.4) = \frac{1}{0.4}\Gamma(1.4) = \frac{5}{2}(0.8873) = 2.21825$$

قيم دالة كاما للاعداد بين 1.00 و 2.00

n	Γ(n)	n	Γ(n)	n	Γ(n)	n	Γ(n)	n	Γ(n)
1·00	1·0000	1·20	0·9182	1·40	0·8873	1·60	0·8935	1·80	0·9314
1·02	0·9888	1·22	0·9131	1·42	0·8864	1·62	0·8959	1·82	0·9368
1·04	0·9784	1·24	0·9085	1·44	0·8858	1·64	0·8986	1·84	0·9426
1·06	0·9687	1·26	0·9044	1·46	0·8856	1·66	0·9017	1·86	0·9487
1·08	0·9597	1·28	0·9007	1·48	0·8857	1·68	0·9050	1·88	0·9551
1·10	0·9514	1·30	0·8975	1·50	0·8862	1·70	0·9086	1·90	0·9618
1·12	0·9436	1·32	0·8946	1·52	0·8870	1·72	0·9126	1·92	0·9688
1·14	0·9364	1·34	0·8922	1·54	0·8882	1·74	0·9168	1·94	0·9761
1·16	0·9298	1·36	0·8902	1·56	0·8896	1·76	0·9214	1·96	0·9837
1·18	0·9237	1·38	0·8885	1·58	0·8914	1·78	0·9262	1·98	0·9917
1·20	0·9182	1·40	0·8873	1·60	0·8935	1·80	0·9314	2·00	1·0000

SEE YOU IN THE NEXT LECTURE

LECTURE 16_1#

- # Outline :-

Solved exercises for Normal distribution

Solved exercises

1- Let Z denote a normal random variable with mean 0 and standard deviation 1.

a Find $P(Z > 2)$.
b Find $P(-2 \le Z \le 2)$.
c Find $P(0 \le Z \le 1.73)$.

By SND table

Solution : a) Since μ = 0 and σ = 1, then , P (Z > 2) = 1-P(Z≤ 2) = 1 − 0.9772 =. 0228.

b) $P(-2 \le Z \le 2) =$ 2P(Z≤ 2)-1=0.9544

c) $P(0 \le Z \le 1.73) =$ P(Z≤ 1.73) -P(Z≤ 0)= 0.9582-0.5=0.5482

2- If Z is a standard normal random variable, find the value z_0 such that:

a $P(Z > z_0) = .5$.
b $P(Z < z_0) = .8643$.
c $P(-z_0 < Z < z_0) = .90$.
d $P(-z_0 < Z < z_0) = .99$.

Solved exercises

Solution : a) $P\,(\,Z > z_0\,) = 1\text{-}P(Z \leq z_0) = 0.5 \Longrightarrow P(Z \leq z_0) = 0.5 \longrightarrow z_0 = 0$

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \leq x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \geq 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

Solved exercises

Solution : b) $P(Z < z_0) = 0.8643 \Longrightarrow z_0 = 1.10$ (by table)

c) $P(-z_0 < Z < z_0) = .90 \Longrightarrow 2\,P(Z < z_0) - 1 = 0.90 \Longrightarrow P(Z < z_0) = 0.95$

Thus, $z_0 = 1.645$

3) company that manufactures and bottles apple juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce. Use Table of SND, to determine the proportion of bottles that will have more than 17 ounces dispensed into them.

Solution:

Note that the value 17 is $\underbrace{(17 - 16)/1 = 1}_{\text{Transform the value 17 to SND}}$ standard deviation above the mean.

So, $P(Z > 1) = .1587$.

SEE YOU IN THE NEXT LECTURE

LECTURE 17#

➢ Outline :-

➢ Functions of Random Variables and Their Distribution

1)Distribution Function Method

2) Moment Method for Sums of Random Variables

Functions of Random Variables and Their Distribution

In many statistical applications, given the probability distribution of a univariate random variable X, one would like to know the probability distribution of another univariate random variable Y = φ(X), where φ is some known function. For example, if we know the probability distribution of the random variable X, we would like know the distribution of Y = ln(X).

For univariate random variable X, some commonly used transformed random variable Y of X are: $Y = X^2$, $Y = |X|$, $Y = \sqrt{|X|}$, $Y = \ln(X)$, $Y = \frac{X-\mu}{\sigma}$, and $Y = \left(\frac{X-\mu}{\sigma}\right)^2$.

Similarly for a bivariate random variable (X, Y), some of the most common transformations of X and Y are X + Y , XY , $\frac{X}{Y}$, min { X, Y } , max { X, Y } or $\sqrt{X^2 + Y^2}$.

In these lectures, we examine various methods for finding the distribution of a transformed univariate or bivariate random variable, when transformation and distribution of the variable are known. First, we treat the univariate case. Then we treat the bivariate case. We begin with an example for univariate discrete random variable.

1)Distribution Function Method

If Y has probability density function f (y) and if U is some function of Y , then we can find $F_U(u) = P(U \leq u)$ directly by integrating f (y) over the region for which $U \leq u$.

The probability density function for U is found by differentiating $F_U(u)$.

The following example illustrates the method.

Example: A box is to be constructed so that the height is 4 inches and its base is X inches by X inches. If X has a standard normal distribution, what is the distribution of the volume of the box?

Answer: The volume of the box is a random variable, since X is a random variable. This random variable V is given by variable. This random variable V is given by V = 4X 2 . To find the density function of V , we first determine the form of the distribution function G(v) of V and then we differentiate G(v) to find the density function of V . The distribution function of V is given by $V = 4X^2$.

1)Distribution Function Method

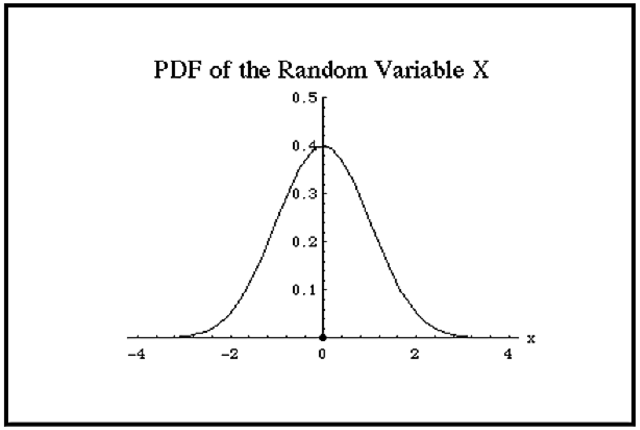
$$G(v) = P(V \le v) = P\left(4X^2 \le v\right) = P\left(-\frac{1}{2}\sqrt{v} \le X \le \frac{1}{2}\sqrt{v}\right) = \int_{-\frac{1}{2}\sqrt{v}}^{\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$

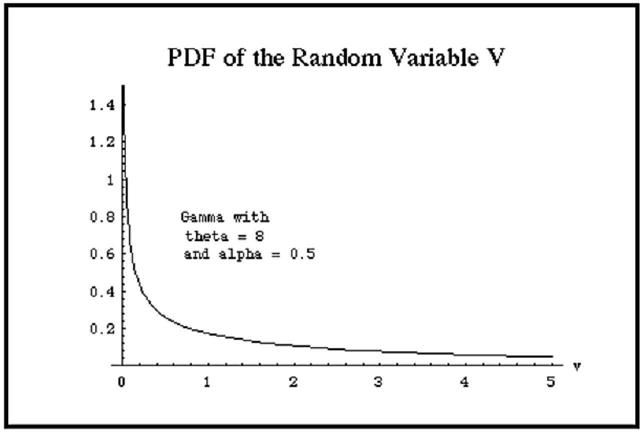
$$= 2\int_0^{\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$ (since the integrand is even).

Hence, by the Fundamental Theorem of Calculus, we get

$$g(v) = \frac{dG(v)}{dv} = \frac{d}{dv}\left(2\int_0^{\frac{1}{2}\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx\right) = 2\,\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{2}\sqrt{v}\right)^2} \left(\frac{1}{2}\right) \frac{d\sqrt{v}}{dv} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{8}v}\, \frac{1}{2\sqrt{v}}$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{8}}\, v^{\frac{1}{2}-1} e^{-\frac{v}{8}} \Longrightarrow V \sim GAM\left(8, \frac{1}{2}\right).$$





1)Distribution Function Method

Example : If the density function of X is

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } -1 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

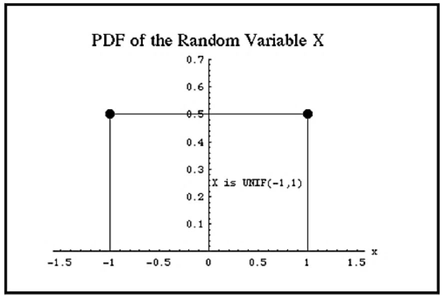
what is the probability density function of $Y = X^2$?

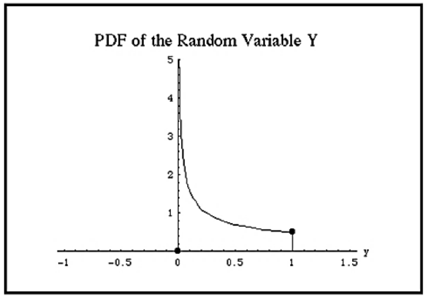
Answer: We first find the cumulative distribution function of Y and then by differentiation, we obtain the density of Y . The distribution function G(y) of Y is given by :

$$G(y) = P\left(Y \leq y\right) = P\left(X^2 \leq y\right) = P\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx = \sqrt{y}.$$

Hence, the density function of Y is given by

$$g(y) = \frac{dG(y)}{dy} = \frac{d\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}}$$ for 0 < y < 1.





2) Moment Generating Function Method

We know that if X and Y are independent random variables, then

$$M_{X+Y}(t) = M_X(t)\, M_Y(t).$$

This result can be used to find the distribution of the sum X + Y . Like the convolution method, this method can be used in finding the distribution of X + Y if X and Y are independent random variables. We briefly illustrate the method using the following example.

Example: Let $X \sim POI(\lambda_1)$ and $Y \sim POI(\lambda_2)$. What is the probability density function of X + Y if X and Y are independent?

Answer: Since , $X \sim POI(\lambda_1)$ and $Y \sim POI(\lambda_2)$, we get $M_X(t) = e^{\lambda_1 (e^t - 1)}$ and

$M_Y(t) = e^{\lambda_2 (e^t - 1)}$.

Further, since X and Y are independent, we have

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = e^{\lambda_1 (e^t-1)}\, e^{\lambda_2 (e^t-1)} = e^{\lambda_1 (e^t-1) + \lambda_2 (e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)},$$

that is, $X+Y \sim POI(\lambda_1+\lambda_2)$. Hence the density function h(z) of Z = X+Y is given by

$$h(z) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \, (\lambda_1 + \lambda_2)^z & \text{for } z = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

2) Moment Generating Function Method

Example: What is the probability density function of the sum of two independent random variable, each of which is gamma with parameters θ and α?

Answer: Let X and Y be two independent gamma random variables with parameters θ and α, that is X ~ GAM(θ, α) and Y ~ GAM(θ, α). Therefore:

$M_X(t) = (1-\theta)^{-\alpha}$ and $M_Y(t) = (1-\theta)^{-\alpha}$, respectively. Since, X and Y are independent, we have $M_{X+Y}(t) = M_X(t)\, M_Y(t) = (1-\theta)^{-\alpha}\,(1-\theta)^{-\alpha} = (1-\theta)^{-2\alpha}$.

Thus X + Y has a moment generating function of a gamma random variable with parameters θ and 2α. Therefore $X + Y \sim GAM(\theta, 2\alpha)$.

Theorem (*) Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with moment-generating functions $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \cdots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t).$$

Proof:

$$m_U(t) = E\left[e^{t(Y_1+\cdots+Y_n)}\right] = E\left(e^{tY_1}e^{tY_2}\cdots e^{tY_n}\right)$$
$$= E\left(e^{tY_1}\right) \times E\left(e^{tY_2}\right) \times \cdots \times E\left(e^{tY_n}\right).$$

Thus, by the definition of moment-generating functions,

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t).$$

2) Moment Generating Function Method

Example: Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$, and let $a_1, a_2, \ldots, a_n$ be constants. If

$U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + a_2 Y_2 + \cdots + a_n Y_n$, then U is a normally distributed random variable with

$E(U) = \sum_{i=1}^{n} a_i \mu_i = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$ and $V(U) = \sum_{i=1}^{n} a_i^2\sigma_i^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$.

Solution: Because Y_i is normally distributed with mean μ_i and variance ${\sigma_i}^2$, Y_i has moment-generating function given by $m_{Y_i}(t) = \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right)$. Therefore, $a_i Y_i$ has moment-generating function given by:

$$m_{a_i Y_i}(t) = E(e^{ta_i Y_i}) = m_{Y_i}(a_i t) = \exp\left(\mu_i a_i t + \frac{a_i^2\sigma_i^2 t^2}{2}\right)$$

Because the random variables Y_i are independent, the random variables $a_i Y_i$ are independent, for i = 1 , 2 , . . . , n, and Theorem (*) implies that:

$$\begin{aligned} m_U(t) &= m_{a_1 Y_1}(t) \times m_{a_2 Y_2}(t) \times \cdots \times m_{a_n Y_n}(t) \\ &= \exp\left(\mu_1 a_1 t + \frac{a_1^2\sigma_1^2 t^2}{2}\right) \times \cdots \times \exp\left(\mu_n a_n t + \frac{a_n^2\sigma_n^2 t^2}{2}\right) = \exp\left(t\sum_{i=1}^{n} a_i\mu_i + \frac{t^2}{2}\sum_{i=1}^{n} a_i^2\sigma_i^2\right). \end{aligned}$$

Thus, U has a normal distribution with mean $\sum_{i=1}^{n} a_i\mu_i$ and variance $\sum_{i=1}^{n} a_i^2\sigma_i^2$.

SEE YOU IN THE NEXT LECTURE

LECTURE 18#

➢ Outline :-

- ➢ Functions of Random Variables and Their Distribution

  - More applications

Functions of Random Variables and Their Distribution

Example : Let each of the independent random variables X and Y have the density function:

$$f(x) = \begin{cases} e^{-x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the joint density of U = X and V = 2X + 3Y and the domain on which this density is positive?

Solution: Since U = X , V = 2X + 3Y, we get by solving for X and Y :

$$X = U \quad , \quad Y = \frac{1}{3}\, V - \frac{2}{3}\, U.$$

Hence, the Jacobian of the transformation is given by :

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \quad = 1 \cdot \left(\frac{1}{3}\right) - 0 \cdot \left(-\frac{2}{3}\right) = \frac{1}{3}.$$

Functions of Random Variables and Their Distribution

The joint density function of U and V is:

$$g(u,v) = |J|\ f(R(u,v),\ S(u,v)) = \left|\frac{1}{3}\right|\ f\left(u,\ \frac{1}{3}v - \frac{2}{3}u\right) = \frac{1}{3}e^{-u}\ e^{-\frac{1}{3}v+\frac{2}{3}u} = \frac{1}{3}\ e^{-\left(\frac{u+v}{3}\right)}.$$

Since $0 < x < \infty$, $0 < y < \infty$, we get $0 < u < \infty$,$0 < v < \infty$,

Further, since $v = 2u + 3y$ and $3y > 0$, we have $v > 2u$.

Hence, the domain of g(u, v) where nonzero is given by $0 < 2u < v < \infty$.

The joint density g(u, v) of the random variables U and V is given by:

$$g(u,v) = \begin{cases} \frac{1}{3}\ e^{-\left(\frac{u+v}{3}\right)} & \text{for } 0 < 2u < v < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Example: Let X and Y be independent random variables, each with density function

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Let $U = X + 2Y$ and $V = 2X + Y$. What is the joint density of U and V?

Functions of Random Variables and Their Distribution

Answer: Since U = X + 2Y, V = 2X + Y, we get by solving for X and Y:

$$X = -\frac{1}{3}U + \frac{2}{3}V \quad , \quad Y = \frac{2}{3}U - \frac{1}{3}V.$$

Hence, the Jacobian of the transformation is given by:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{9} - \frac{4}{9} = -\frac{1}{3}.$$

The joint density function of U and V is:

$$g(u,v) = |J|\; f(R(u,v),\, S(u,v)) = \left|-\frac{1}{3}\right| f(R(u,v))\; f(S(u,v)) = \frac{1}{3}\lambda\, e^{\lambda R(u,v)}\,\lambda\, e^{\lambda S(u,v)} = \frac{1}{3}\lambda^2\, e^{\lambda[R(u,v)+S(u,v)]}$$

$$= \frac{1}{3}\lambda^2\, e^{-\lambda\left(\frac{u+v}{3}\right)}.$$

Hence, the joint density g(u, v) of the random variables U and V is given by

$$g(u,v) = \begin{cases} \frac{1}{3}\lambda^2\, e^{-\lambda\left(\frac{u+v}{3}\right)} & \text{for } 0 < u < \infty;\; 0 < v < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Functions of Random Variables and Their Distribution

Example: Let X and Y be independent random variables, each with density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}, \qquad -\infty < x < \infty.$$

Let $U = \frac{X}{Y}$ and $V = Y$. What is the joint density of U and V? Also, what is the density of U?

Answer: Since $U = \dfrac{X}{Y}$, $V = Y$, we get by solving for X and Y : X = UV, Y = V.

Hence, the Jacobian of the transformation is given by:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = v \cdot (1) - u \cdot (0) \quad = v.$$

The joint density function of U and V is

$$g(u,v) = |J|\; f(R(u,v),\, S(u,v)) = |v|\; f(R(u,v))\; f(S(u,v)) = |v|\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}R^2(u,v)}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}S^2(u,v)}$$

$$= |v|\, \frac{1}{2\pi}\, e^{-\frac{1}{2}\left[R^2(u,v)+S^2(u,v)\right]} = |v|\, \frac{1}{2\pi}\, e^{-\frac{1}{2}\left[u^2v^2+v^2\right]} = |v|\, \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2(u^2+1)}.$$

Hence, the joint density g(u, v) of the random variables U and V is given by

$$g(u,v) = |v|\, \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2(u^2+1)},$$ where $-\infty < u < \infty$ and $-\infty < v < \infty$.

Functions of Random Variables and Their Distribution

Next, we want to find the density of U. We can obtain this by finding the marginal of U from the joint density of U and V . Hence, the marginal $g_1(u)$ of U is given by

$$g_1(u) = \int_{-\infty}^{\infty} g(u, v)\, dv \quad = \int_{-\infty}^{\infty} |v|\, \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2\left(u^2+1\right)}\, dv$$

$$= \int_{-\infty}^{0} -v\, \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2\left(u^2+1\right)}\, dv + \int_{0}^{\infty} v\, \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2\left(u^2+1\right)}\, dv$$

$$= \frac{1}{2\pi}\, \left(\frac{1}{2}\right) \left[\frac{2}{u^2+1}\, e^{-\frac{1}{2}\, v^2\left(u^2+1\right)}\right]_{-\infty}^{0} \quad + \frac{1}{2\pi}\, \left(\frac{1}{2}\right) \left[\frac{-2}{u^2+1}\, e^{-\frac{1}{2}\, v^2\left(u^2+1\right)}\right]_{0}^{\infty}$$

$$= \frac{1}{2\pi}\, \frac{1}{u^2+1} + \frac{1}{2\pi}\, \frac{1}{u^2+1} \quad = \frac{1}{\pi\left(u^2+1\right)}.$$

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