Lecture Notes in Foundations of Mathematics

Department of Mathematics

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Chapter

1 Logic and Proofs

Section 1.1: Propositions and Connectives

Definition 1.1.1

A **proposition P** is a sentence which is either true **T** or false **F**. That is, the truth values of propositions are **T** or **F**.

Example 1.1.1

Consider the following sentences:

- Propositions:

  a) $\frac{1}{2}$ is a rational number. [**T**].

  b) $2 + 4 = 1$. [**F**].

- Not propositions:

  c) How are you doing? [not a proposition].

  d) $x^2 = 36$. [where is x coming from?].

  e) This sentence is false. [depends on the given sentence!].

The previous propositions studied in a and b are called **simple** propositions. **Compound** propositions can be formed by **connectives** with simple propositions. For example,

Compound proposition: $1 + 2 = 5$ "and" the sun is made of an orange.

Definition 1.1.2

Let **P** and **Q** be two propositions. Then,

1. the **conjunction** of **P** and **Q**, denoted by $\mathbf{P} \wedge \mathbf{Q}$, is the proposition "**P** and **Q**". $\mathbf{P} \wedge \mathbf{Q}$ is true exactly when both **P** and **Q** are true.

2. the **disjunction** of $\mathbf{P}$ and $\mathbf{Q}$, denoted by $\mathbf{P} \vee \mathbf{Q}$, is the proposition "$\mathbf{P}$ or $\mathbf{Q}$". $\mathbf{P} \vee \mathbf{Q}$ is true exactly when at least one of $\mathbf{P}$ or $\mathbf{Q}$ is true.
3. the **negation** of $\mathbf{P}$, denoted by $\sim \mathbf{P}$, is the proposition "not $\mathbf{P}$". $\sim \mathbf{P}$ is true exactly when $\mathbf{P}$ is false.

Example 1.1.2

Let $\mathbf{P}$ be "Kuwait is an island" and let $\mathbf{Q}$ be "Sea water contains salt". Discuss $\mathbf{P} \wedge \mathbf{Q}$, $\mathbf{P} \vee \mathbf{Q}$, and $\sim \mathbf{P}$.

Solution:

It is clear the $\mathbf{P}$ is false and $\mathbf{Q}$ is true. Thus,

1. $\mathbf{P} \wedge \mathbf{Q}$: Kuwait is an island and sea water contains salt. [**F**].
2. $\mathbf{P} \vee \mathbf{Q}$: Kuwait is an island or sea water contains salt. [**T**].
3. $\sim \mathbf{P}$: It is not the case that Kuwait is an island. [**T**].

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{P} \wedge \mathbf{Q}$	$\mathbf{P} \vee \mathbf{Q}$	$\sim \mathbf{P}$	$\sim \mathbf{Q}$
T	**T**	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**	**T**
F	**T**	**F**	**T**	**T**	**F**
F	**F**	**F**	**F**	**T**	**T**

Definition 1.1.3

A **propositional form** is an expression involving finitely many propositions connected by connectives such as $\wedge$, $\vee$, and $\sim$.

Example 1.1.3

Let $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ be propositions. Write down the truth table of the propositional form $\big((\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \vee (\sim \mathbf{R}))\big)$.

Solution:

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{R}$	$\sim\mathbf{R}$	$\mathbf{P}\wedge\mathbf{Q}$	$\mathbf{P}\vee(\sim\mathbf{R})$	$\big((\mathbf{P}\wedge\mathbf{Q})\vee(\mathbf{P}\vee(\sim\mathbf{R}))\big)$
T	**T**	**T**	**F**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**
F	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**T**	**F**	**T**	**T**

Definition 1.1.4

Two propositional forms **P** and **Q** are called **equivalent** if and only if their truth tables are identical. In that case, we write $\mathbf{P}\equiv\mathbf{Q}$.

Definition 1.1.5

A **denial** of a proposition **P** is any proposition equivalent to $\sim\mathbf{P}$.

A proposition **P** has only one negation "$\sim\mathbf{P}$", but it has many denials. For instance, $\sim\mathbf{P}$, $\sim\sim\sim\mathbf{P}$, and $\sim\sim\sim\sim\sim\mathbf{P}$ are all examples of denials. Note that $\sim(\sim\mathbf{P})$ is simply **P**.

Example 1.1.4

Let **P** be "π is an irrational number". Find the negation of **P**, and give some examples of denials of **P**.

Solution:

- negation $\sim\mathbf{P}$: It is not the case that π is irrational.
- denials of **P**: a. π is rational. b. π is the quotient of two integers r/s. c. π has a finite decimal expansion.

Note that since **P** is true, all of its denials are false.

Definition 1.1.6

A propositional form is called a **tautology** if it is true for all possible truth values of its components. It is called a **contradiction** if it is the negation of a tautology.

Example 1.1.5

Show that $\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$ is a tautology for any propositions **P** and **Q**.

Solution:

$\mathbf{P}$	$\mathbf{Q}$	$\sim \mathbf{P}$	$\sim \mathbf{Q}$	$\mathbf{P} \vee \mathbf{Q}$	$(\sim \mathbf{P}) \wedge (\sim \mathbf{Q})$	$\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**	**T**
F	**T**	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**T**	**T**

Moreover, it can be seen that the negation of $\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$ is a contradiction.

Remark 1.1.1

The negation of a tautology is a contradiction, and the negation of a contradiction is a tautology.

Section 1.2: Conditionals and Biconditionals

Definition 1.2.1

Given two propositions $\mathbf{P}$ and $\mathbf{Q}$, the conditional sentence $\mathbf{P} \Rightarrow \mathbf{Q}$ (reads "$\mathbf{P}$ implies $\mathbf{Q}$") is the proposition "if $\mathbf{P}$, then $\mathbf{Q}$". In that case, $\mathbf{P}$ is called **antecedent** and $\mathbf{Q}$ is called **consequent**.

Remark 1.2.1

The proposition $\mathbf{P} \Rightarrow \mathbf{Q}$ is true whenever $\mathbf{P}$ is false or $\mathbf{Q}$ is true. In general, $\mathbf{P} \Rightarrow \mathbf{Q}$ is equivalent to $(\sim \mathbf{P}) \vee \mathbf{Q}$.

Example 1.2.1

Consider the following propositions:

a) if "x is an odd integer", then "$x+1$ is an even integer". [**T**].
b) if "$2+1=0$", then "$1+1=0$". [**T**].
c) if "$1-1=0$", then "$2+9=1$". [**F**].

Definition 1.2.2

For propositions $\mathbf{P}$ and $\mathbf{Q}$, the **converse** of $\mathbf{P} \Rightarrow \mathbf{Q}$ is $\mathbf{Q} \Rightarrow \mathbf{P}$, and the **contrapositive** of $\mathbf{P} \Rightarrow \mathbf{Q}$ is $(\sim \mathbf{Q}) \Rightarrow (\sim \mathbf{P})$.

Theorem 1.2.1

For any propositions $\mathbf{P}$ and $\mathbf{Q}$, we have
(i) $\mathbf{P} \Rightarrow \mathbf{Q}$ is equivalent to $(\sim \mathbf{Q}) \Rightarrow (\sim \mathbf{P})$, and (ii) $\mathbf{P} \Rightarrow \mathbf{Q}$ is not equivalent to $\mathbf{Q} \Rightarrow \mathbf{P}$.

Proof:

We prove both results in the following truth table.

$\mathbf{P}$	$\mathbf{Q}$	$\sim\mathbf{P}$	$\sim\mathbf{Q}$	$\mathbf{P}\Rightarrow\mathbf{Q}$	$\mathbf{Q}\Rightarrow\mathbf{P}$	$\sim\mathbf{Q}\Rightarrow\sim\mathbf{P}$
T	**T**	**F**	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Definition 1.2.3

Let $\mathbf{P}$ and $\mathbf{Q}$ be two propositions. The **biconditional** sentence $\mathbf{P}\Leftrightarrow\mathbf{Q}$ is "$\mathbf{P}$ if and only if (iff.) $\mathbf{Q}$". $\mathbf{P}\Leftrightarrow\mathbf{Q}$ is true exactly when both $\mathbf{P}$ and $\mathbf{Q}$ have the same truth value.

Remark 1.2.2

The following phrases are translated as $\mathbf{P}\Rightarrow\mathbf{Q}$ for any propositions $\mathbf{P}$ and $\mathbf{Q}$:

• if $\mathbf{P}$, then $\mathbf{Q}$.	• if $a>5$, then $a>3$.
• $\mathbf{P}$ implies $\mathbf{Q}$.	• $a>5$ implies $a>3$.
• $\mathbf{P}$ is sufficient for $\mathbf{Q}$.	• $a>5$ is sufficient for $a>3$.
• $\mathbf{P}$ only if $\mathbf{Q}$.	• $a>5$ only if $a>3$
• $\mathbf{Q}$, if $\mathbf{P}$.	• $a>3$, if $a>5$.
• $\mathbf{Q}$ whenever $\mathbf{P}$.	• $a>3$ whenever $a>5$.
• $\mathbf{Q}$ is necessary for $\mathbf{P}$.	• $a>3$ is necessary for $a>5$.
• $\mathbf{Q}$, when $\mathbf{P}$.	• $a>3$, when $a>5$.

Remark 1.2.3

Moreover, the following phrases are translated as $\mathbf{P}\Leftrightarrow\mathbf{Q}$ for any propositions $\mathbf{P}$ and $\mathbf{Q}$:

• $\mathbf{P}$ if and only if $\mathbf{Q}$.	• $\lvert x\rvert=2$ iff $x^2=4$.
• $\mathbf{P}$ if, but only if, $\mathbf{Q}$.	• $\lvert x\rvert=2$ if, but only if, $x^2=4$.
• $\mathbf{P}$ is equivalent to $\mathbf{Q}$.	• $\lvert x\rvert=2$ is equivalent to $x^2=4$.
• $\mathbf{P}$ is necessary and sufficient for $\mathbf{Q}$.	• $\lvert x\rvert=2$ is necessary and sufficient for $x^2=4$.

Theorem 1.2.2

Let **P**, **Q**, and **R** be propositions. Then,

a. $\mathbf{P} \Rightarrow \mathbf{Q} \equiv (\sim \mathbf{P}) \vee \mathbf{Q}$.

b. $\mathbf{P} \Leftrightarrow \mathbf{Q} \equiv (\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$.

c. $\sim (\mathbf{P} \wedge \mathbf{Q}) \equiv (\sim \mathbf{P}) \vee (\sim \mathbf{Q})$.

d. $\sim (\mathbf{P} \vee \mathbf{Q}) \equiv (\sim \mathbf{P}) \wedge (\sim \mathbf{Q})$.

e. $\sim (\mathbf{P} \Rightarrow \mathbf{Q}) \equiv \mathbf{P} \wedge (\sim \mathbf{Q})$.

f. $\sim (\mathbf{P} \wedge \mathbf{Q}) \equiv \mathbf{P} \Rightarrow (\sim \mathbf{Q})$.

g. $\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) \equiv (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})$.

h. $\mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R}) \equiv (\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R})$.

Proof:

b.

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{P} \Leftrightarrow \mathbf{Q}$	$\mathbf{P} \Rightarrow \mathbf{Q}$	$\mathbf{Q} \Rightarrow \mathbf{P}$	$(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$
T	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**

g.

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{R}$	$\mathbf{Q} \vee \mathbf{R}$	$\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R})$	$\mathbf{P} \wedge \mathbf{Q}$	$\mathbf{P} \wedge \mathbf{R}$	$(\mathbf{P} \vee \mathbf{Q}) \vee (\mathbf{P} \vee \mathbf{R})$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

Section 1.3: Quantifiers

⋆ NOTATIONS:

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ is the set of **natural numbers**.
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is the set of **integer numbers**.
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ is the set of **rational numbers**.
- $\mathbb{R}$ is the set of **real numbers**.

The sentence $x \geq 5$ is not a proposition, unless we assign a value to x. It is an open sentence. In general, an open sentence with n variables is denoted by $P(x_1, x_2, \cdots, x_n)$. For example, the open sentence $P(x_1, x_2, x_3)$: "x_1 equals to $x_2 + x_3$" is an open sentence. On the other hand, $P(7, 3, 4)$ and $P(7, 2, 3)$ are propositions with true and false values, respectively.

Definition 1.3.1

The set of objects for which an open sentence is true is called the **truth set**, and is denoted by $\mathcal{T}$.
On the other hand, the set from where the objects can be taken from is called the **universe**, and is denoted by $\mathcal{U}$. In particular, two open sentences are said to be equivalent for a particular universe if and only if their truth sets are equal.

Example 1.3.1

Let $\mathcal{U} = \mathbb{N}$. Then, $P(x) : x + 3 > 7$ is equivalent to $Q(x) : x > 4$, since $\mathcal{T} = \{5, 6, 7, \cdots\}$ for both P and Q.
Also, $P(x) : x^2 = 4$ is equivalent to $Q(x) : x = 2$. However, if $\mathcal{U}$ was the set of all integers, then $P(x) : x^2 = 4$ with truth set $\{-2, 2\}$ is not equivalent to $Q(x) : x = 2$ with truth set $\{2\}$.

Definition 1.3.2

Let $\mathbf{P}(x)$ be an open sentence with variable $x \in \mathcal{U}$. Then,

a) The sentence "$(\forall x)\mathbf{P}(x)$" reads as "for all x, $\mathbf{P}(x)$". It is true iff $\mathcal{T} = \mathcal{U}$ for $\mathbf{P}(x)$. "$\forall$" is called the **universal quantifiers**.

b) The sentence "$(\exists x)\mathbf{P}(x)$" reads as "there exists x such that $\mathbf{P}(x)$". It is true iff $\mathcal{T} \neq \emptyset$ (the empty set). "$\exists$ is called the **existential quantifiers**.

c) The sentence "$(\exists! x)\mathbf{P}(x)$" reads as "there exists a unique x such that $\mathbf{P}(x)$". It is true iff $\mathcal{T}$ contains only one element. "$\exists!$ is called the **unique existential quantifiers**.

Example 1.3.2

Let $\mathcal{U} = \mathbb{R}$. Decide the truth value and the truth set for each of the following.

Solution:

Consider the following table where we different sentences along with its truth value as true or false and the corresponding truth set.

sentence	**T** or **F**	$\mathcal{T}$
a. $(\forall x)(x \geq 3)$	**F**	$[3, \infty)$.
b. $(\forall x)(\lvert x\rvert > 0)$	**F**	$\mathbb{R}\backslash\{0\}$.
c. $(\forall x)(x - 1 < x)$	**T**	$\mathbb{R}$.
d. $(\exists x)(x \geq 3)$	**T**	$[3, \infty)$.
e. $(\exists! x)(\lvert x\rvert = 0)$	**T**	$\{0\}$.
f. $(\exists! x)(\lvert x\rvert = 2)$	**F**	$\{-2, 2\}$.
g. $(\exists x)(x^2 = -4)$	**F**	$\emptyset$.
h. $(\exists x)(\exists y)(2x + y = 0 \wedge x - y = 1)$	**T**	$\{x = \frac{1}{3}, y = -\frac{2}{3}\}$.
i. $(\exists! x)(\exists! y)(2x + y = 0 \vee x - y = 1)$	**F**	$(x, y) \in \big\{(0,0), (1,0), (3,2), \cdots\big\}$.
j. $(\forall x)(\forall y)(x^2 + y^2 > 0)$	**F**	$\mathbb{R}^2\backslash(0,0)$.

Definition 1.3.3

Two quantified sentences are equivalent for a particular universe $\mathcal{U}$ iff they have the same truth set in $\mathcal{U}$. Two quantified sentences are equivalent iff they are equivalent in every universe.

For instance, $(\forall x)(\mathbf{P}(x) \wedge \mathbf{Q}(x))$ is equivalent to $(\forall x)(\mathbf{Q}(x) \wedge \mathbf{P}(x))$ and $(\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)]$ is equivalent to $(\forall x)[\sim \mathbf{Q}(x) \Rightarrow \sim \mathbf{P}(x)]$.

Theorem 1.3.1

Let $\mathbf{P}(x)$ be an open sentence with a variable $x \in \mathcal{U}$ for some $\mathcal{U}$. Then,

a. $\sim (\forall x)[\mathbf{P}(x)]$ is equivalent to $(\exists x)[\sim \mathbf{P}(x)]$.

b. $\sim (\exists x)[\mathbf{P}(x)]$ is equivalent to $(\forall x)[\sim \mathbf{P}(x)]$.

Proof:

(a.) The sentence $\sim (\forall x)[\mathbf{P}(x)]$ is true iff $(\forall x)[\mathbf{P}(x)]$ is false iff the truth set for $\mathbf{P}(x)$ is not the entire universe, i.e. $\mathcal{T} \neq \mathcal{U}$ iff there exists an $x \in \mathcal{U}$ such that $\mathbf{P}(x)$ is false iff $(\exists x)[\sim \mathbf{P}(x)]$ is true.

(b.) The sentence $\sim (\exists x)[\mathbf{P}(x)]$ is true iff $(\exists x)[\mathbf{P}(x)]$ is false iff the truth set of $\mathbf{P}(x)$ is empty iff $(\forall x)[\sim \mathbf{P}(x)]$ is true.

Remark 1.3.1

Let $\mathbf{P}(x)$ be an open sentence with a variable $x \in \mathcal{U}$ for some $\mathcal{U}$. Then,

$$(\exists! x)\mathbf{P}(x) \equiv (\exists x)\Big[\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big].$$

Example 1.3.3

Find a denial (or the negation) for "$(\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)]$".

Solution:

Using Theorem 1.3.1 and Theorem 1.2.2 (part e), we conclude

$$\sim (\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)] \equiv (\exists x)[\sim (\mathbf{P}(x) \Rightarrow \mathbf{Q}(x))] \equiv (\exists x)[\mathbf{P}(x) \wedge (\sim \mathbf{Q}(x))].$$

Example 1.3.4

Find a denial (or the negation) for "$(\exists! x)\mathbf{P}(x)$".

Solution:

Using Remark 1.3.1 and Theorem 1.2.2, we conclude

$$\begin{aligned}
\sim (\exists! x)\mathbf{P}(x) &\equiv \sim (\exists x)\Big[\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \Big(\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big)\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee \sim (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y) \sim [\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y)[\mathbf{P}(y) \wedge \sim (x = y)]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y)[\mathbf{P}(y) \wedge x \neq y]\Big]
\end{aligned}$$

Example 1.3.5

Find a denial (or the negation) for

$$(\forall z)(\exists x)(\exists y)\Big[\big((x > z) \wedge (y > z)\big) \wedge \sim (\exists w)\big(x + y < w < xz\big)\Big]. \tag{1.3.1}$$

Solution:

Using Theorem1.3.1 and Theorem 1.2.2, we conclude

$$\begin{aligned}
\sim \text{Equation}(1.3.5) &\equiv \sim (\forall z)(\exists x)(\exists y)\Big[\big((x > z) \wedge (y > z)\big) \wedge \sim (\exists w)\big(x + y < w < xz\big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y) \sim \Big[\big((x > z) \wedge (y > z)\big) \wedge \sim (\exists w)\big(x + y < w < xz\big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y)\Big[\big((x > z) \wedge (y > z)\big) \Rightarrow \sim\sim (\exists w)\big(x + y < w < xz\big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y)\Big[\big((x > z) \wedge (y > z)\big) \Rightarrow (\exists w)\big(x + y < w < xz\big)\Big].
\end{aligned}$$

Example 1.3.6

Let $\mathcal{U} = \mathbb{R}$. Decide the truth value and the truth set for each of the following.

Solution:

sentence	**T** or **F**	$\mathcal{T}$
a. $(\forall y)(\exists x)[x + y = 0]$	**T**	for any y, $x = -y$ is a solution.
b. $(\exists x)(\forall y)[x + y = 0]$	**F**	given $x = 0$ not all $y \in \mathbb{R}$ is a solution.
c. $(\exists x)(\exists y)[x^2 + y^2 = 10]$	**T**	for $x \in \mathbb{R}$ there is $y = \sqrt{10 - x^2} \in \mathbb{R}$.
d. $(\forall y)(\exists x)(\forall z)[xy = xz]$	**T**	for any $y \in \mathbb{R}$, $x = 0$ for any $z \in \mathbb{R}$.
e. $(\forall y)(\exists! x)[x = y^2]$	**T**	for any $y \in \mathbb{R}$, $x = y^2$ is a solution.

Section 1.4: Mathematical Proofs

Definition 1.4.1

A **proof** is a justification of the truth of a given statement called theorem, proposition, claim, or lemma.

Remark 1.4.1

Tools of proofs: We may use any of the following:

- Axioms: Initial statements which are assumed to be true.
- Theorems: Some previously proved statement can be use.
- Assumptions: Assumed fact about the problem at hand.
- Tautologies: Examples follow:

a. $\mathbf{P} \vee (\sim \mathbf{P})$ .... (Excluded Middle).

b. $(\mathbf{P} \Rightarrow \mathbf{Q}) \Leftrightarrow (\sim \mathbf{Q} \Rightarrow \sim \mathbf{P})$ .... (Contrapositive).

c. $\left.\begin{array}{l} \mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R}) \Leftrightarrow (\mathbf{P} \vee \mathbf{Q}) \vee \mathbf{R} \\ \mathbf{P} \wedge (\mathbf{Q} \wedge \mathbf{R}) \Leftrightarrow (\mathbf{P} \wedge \mathbf{Q}) \wedge \mathbf{R} \end{array}\right\}$ .... (Associativity).

d. $\left.\begin{array}{l} \mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) \Leftrightarrow (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R}) \\ \mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R}) \Leftrightarrow (\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R}) \end{array}\right\}$ .... (Distributivity).

e. $(\mathbf{P} \Leftrightarrow \mathbf{Q}) \Leftrightarrow [(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})]$ .... (Biconditional).

f. $\sim (\mathbf{P} \Rightarrow \mathbf{Q}) \Leftrightarrow (\mathbf{P} \wedge \sim \mathbf{Q})$ .... (Denial of Implication).

g. $\left.\begin{array}{l} \sim (\mathbf{P} \wedge \mathbf{Q}) \Leftrightarrow (\sim \mathbf{P} \vee \sim \mathbf{Q}) \\ \sim (\mathbf{P} \vee \mathbf{Q}) \Leftrightarrow (\sim \mathbf{P} \wedge \sim \mathbf{Q}) \end{array}\right\}$ .... (De Morgan's Laws).

h. $\mathbf{P} \Leftrightarrow [\sim \mathbf{P} \Rightarrow (\mathbf{Q} \wedge \sim \mathbf{Q})]$ .... (Contradiction).

i. $[(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{R})] \Leftrightarrow (\mathbf{P} \Rightarrow \mathbf{R})$ .... (Transitivity).

j. $[\mathbf{P} \wedge (\mathbf{P} \Rightarrow \mathbf{Q})] \Rightarrow \mathbf{Q}$ .... (Modus Ponens).

In what follows, we consdier different types of proof.

1.4.1 Type 1: Direct Proof

Direct proof $\mathbf{P} \Rightarrow \mathbf{Q}$: Assume $\mathbf{P}$, then $\cdots\ \cdots$. Therefore, $\mathbf{Q}$.

Example 1.4.1

Let n be an integer. Show that if n is odd, then $n+1$ is even.

Solution:

Assume that $n = 2k+1$ for some integer k. Then, $n+1 = (2k+1)+1$. That is $n+1 = 2k+2 = 2(k+1)$. Therefore, $n+1$ is even.

Example 1.4.2

Assume that $\sin(x)$ is an odd funtion, i.e. $\sin(-x) = -\sin(x)$. Show that $f(x) = \sin^2(x)$ for any $x \in \mathbb{R}$ is an even function, i.e. $f(-x) = f(x)$.

Solution:

$f(-x) = (\sin(-x))^2 = (-\sin(x))^2 = \sin(x) = f(x)$. Therefore, $f(x)$ is an even function.

Theorem 1.4.1

Suppose that a, b, and c are integers. If a divides b and b divides c, then a divides c.

Proof:

Since a divides b $(a \mid b)$, then there is an integer k such that $b = ka$. Also, since $b \mid c$ there is an integer h such that $c = hb$. Thus, $c = hb = h(ka) = (hk)a$, and therefore $a \mid c$.

Theorem 1.4.2

Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $a \mid c$, then $a \mid b \pm c$.

Proof:

Since $a \mid b$, $\exists k \in \mathbb{Z}$ such that $b = ka$, and since $a \mid c$, $\exists h \in \mathbb{Z}$ such that $c = ha$. Thus,

$$b \pm c = ka \pm ha = (k \pm h)a.$$

Therefore, $a \mid b \pm c$.

1.4.2 Type 2: Proof By Contradiction

Contradiction to proof $\mathbf{P}$: Suppose $\sim \mathbf{P}$, then $\cdots\cdots$. Thus $\mathbf{Q}$. Then, $\cdots\cdots$. Therefore, $\sim \mathbf{Q}$, contradiction.

This technique uses the tautology $\mathbf{P} \Leftrightarrow [\sim \mathbf{P} \Rightarrow (\mathbf{Q} \wedge \sim \mathbf{Q})]$.

Example 1.4.3

The equation $x^3 + x - 1 = 0$ has at most one real root.

Solution:

Let $f(x) = x^3 + x - 1$. Suppose that $f(x)$ has two real roots a and b, then $f(a) = f(b) = 0$. f is continuouse on $[a, b]$ and is differentiable on (a, b) since it is a polynomial. Then, by Rolle's Theorem, there is a $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 3c^2 + 1 \neq 0$ for all $c \in \mathbb{R}$. This is a contradiction. Therefore, f has at most one real root.

Remark 1.4.2

- Any square integer has an even number of 2's as prime factors.
- All natural number greater than 1 has a prime divisor $q > 1$.

Example 1.4.4

Prove that $\sqrt{2}$ is an irrational number.

Solution:

Recall the fact that any square integer number has an even number of 2's as prime factors. Suppose that $\sqrt{2}$ is rational number. Then, $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Thus, $2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. Since p^2 and q^2 are both square numbers, p^2 contains an even number of 2's as prime factors (might be 0 times for odd numbers) and q^2 contains an even number of 2's as prime factors. But then $2q^2$ has an odd number of 2's as prime factors and thus p^2 has an odd number of 2's as prime factors because $p^2 = 2q^2$. This is a contradiction. Thus, $\sqrt{2}$ is an irrational number.

Theorem 1.4.3

The set of primes in $\mathbb{N}$ is infinite.

Proof:

Suppose that the set of primes $W = \{p_1, p_2, \cdots, p_k\}$ is finite for some $k \in \mathbb{N}$. Let $n = p_1 p_2 \cdots p_k + 1 \in \mathbb{N}$. (fact) All natural number has a prime divisor $q > 1$. So, $q \mid n$, and since q is a prime, then $q \in W$ and $q \mid p_1 p_2 \cdots p_k$ (because $q = p_i$ for some $1 \le i \le k$). Also, $q \mid n$. Therefore, $q \mid (n - p_1 p_2 \cdots p_k)$, but $n - p_1 p_2 \cdots p_k = 1$. Thus $q = 1$, Contradition. Thus W is infinite.

1.4.3 Type 3: Contrapositive Proofs

Contraposition to show $\mathbf{P} \Rightarrow \mathbf{Q}$: Suppose $\sim \mathbf{Q}$, then $\cdots\cdots$. Thus $\sim \mathbf{P}$.
Therefore, $\mathbf{P} \Rightarrow \mathbf{Q}$. This technique uses the tautology $(\mathbf{P} \Rightarrow \mathbf{Q}) \Leftrightarrow (\sim \mathbf{Q} \Rightarrow \sim \mathbf{P})$.

Example 1.4.5

Let $m \in \mathbb{Z}$. If m^2 is odd, then m is odd.

Solution:

Assume that m is even. Then $m = 2k$ for some $k \in \mathbb{Z}$ and $m^2 = 4k^2 = 2(2k^2)$ which is even. By contraposition, the result is proved.

Example 1.4.6

Let $x, y \in \mathbb{R}$ such that $x < 2y$. Show that if $7xy \le 3x^2 + 2y^2$, then $3x \le y$.

Solution:

Assume that $x < 2y$. By contraposition, assume that $3x > y$. Then, $2y - x > 0$ and $3x - y > 0$, but

$$(2y - x)(3x - y) = 7xy - 3x^2 - 2y^2 > 0 \quad \Rightarrow \quad 7xy > 3x^2 + 2y^2.$$

Therefore, if $7xy \le 3x^2 + 2y^2$, then $3x \le y$.

1.4.4 Type 4: Two-Directions Proofs

Two directions to show $\mathbf{P} \Leftrightarrow \mathbf{Q}$: By any method, (i) Show that $\mathbf{P} \Rightarrow \mathbf{Q}$. (ii) Show that $\mathbf{Q} \Rightarrow \mathbf{P}$. Therefore, $\mathbf{P} \Leftrightarrow \mathbf{Q}$.

Theorem 1.4.4

Let a be a prime number, and let b and c be positive integers. Prove that $a \mid bc$ if and only if $a \mid b$ or $a \mid c$.

Proof:

We show the result by two direction: „ $\Rightarrow$ " and „ $\Leftarrow$ ".

„ $\Rightarrow$ ": Assume that $a \mid bc$. By Fundamental Theorem of Arithmetic, b and c can be written uniquely as products of primes. Assume $b = p_1p_2\cdots p_k$ and $c = q_1q_2\cdots q_h$ for some $h, k \in \mathbb{N}$. But then $bc = p_1p_2\cdots p_kq_1q_2\cdots q_h$. Since $a \mid bc$ and a is a prime, a is one of the prime factors. If $a = p_i$ for some $1 \leq i \leq k$, then $a \mid b$ or if $a = q_i$ for some $1 \leq i \leq h$, then $a \mid c$. Thus, either $a \mid b$ or $a \mid c$.

„ $\Leftarrow$ ": Assume that $a \mid b$ or $a \mid c$. Thus,

Case 1: $a \mid b$ then $b = ka$ for some $k \in \mathbb{Z}$ and hence $bc = (ka)c = (kc)a$. Thus $a \mid bc$.

Case 2: $a \mid c$ then $c = ha$ for some $h \in \mathbb{Z}$ and hence $bc = b(ha) = (bh)a$. Thus $a \mid bc$.

In either cases, $a \mid bc$.

1.4.5 Type 5: Proofs By Cases (Exhaustion)

Contradiction to show $(\mathbf{P}_1 \vee \mathbf{P}_2) \Rightarrow \mathbf{Q}$: By any method, (i) Show that $\mathbf{P}_1 \Rightarrow \mathbf{Q}$ and (ii) show that $\mathbf{P}_2 \Rightarrow \mathbf{Q}$. Using the tautology $[(\mathbf{P}_1 \vee \mathbf{P}_2) \Rightarrow \mathbf{Q}] \Leftrightarrow [(\mathbf{P}_1 \Rightarrow \mathbf{Q}) \wedge (\mathbf{P}_2 \Rightarrow \mathbf{Q})]$.

Example 1.4.7

Show that for any $x, y \in \mathbb{Z}$, if either x or y is even, then xy is even.

Solution:

We have two cases:

Case 1: Assume x-even. Then $x = 2k$ for some $k \in \mathbb{Z}$. That is $xy = 2(ky)$ which is even.

Case 2: Assume y-even. Then $y = 2h$ for some $h \in \mathbb{Z}$. That is $xy = 2(xh)$ which is even.

Thus, in both cases, xy is even.

Example 1.4.8

Let $x, y \in \mathbb{Z}$. If x and y are both odd, then xy is odd.

Solution:

a. Direct Proof: Assume x and y are odd integers. Then, there are m and n in $\mathbb{Z}$ such that $x = 2m+1$ and $y = 2n+1$. Thus, $xy = (2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1$. Therefore, xy is odd as well.

b1. Proof by Contradiction: Assume that xy is even. Thus $2 \mid xy$ which implies that $2 \mid x$ or $2 \mid y$ (since 2 is a prime number) which is a contradiction both ways since both of x and y are odd.

b2. Another Proof by Contradiction: Assume that xy is even. Since x and y are odd, there are m and n in $\mathbb{Z}$ such that $x = 2m+1$ and $y = 2n+1$. Thus, $xy = (2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1$ which is odd, contradiction. Therefore, xy is odd.

c. Proof by Contraposition: We use $\sim$ (xy is odd) $\Rightarrow\sim$ (x is odd and y is odd) which is equivalent to (xy is even) $\Rightarrow$ [(x is even) or (y is even)].

   Assume that xy is even. Thus, $2 \mid xy$. Since 2 is a prime number, we have either $2 \mid x$ or $2 \mid y$. Thus, either x is even or y is even. Therefore, if x and y are odd, then xy is odd.

Exercise 1.4.1

Let $a, b \in \mathbb{Z}$. Use a contrapositive proof to show that if ab-odd, then a - odd and b-odd.

Section 1.6: Proofs Involving Quantifiers

1.6.1 Type 1: Proof of $(\exists x)\mathbf{P}(x)$

- Direct proof: Name or construct an element $x \in \mathcal{U}$ which has the property $\mathbf{P}(x)$.
- Proof by contradiction: Suppose $\sim (\exists x)\mathbf{P}(x)$. Then $(\forall x)(\sim \mathbf{P}(x))$ ... ... .Therefore, $\mathbf{Q}(x) \wedge \sim \mathbf{Q}(x)$, contradiction. Hence, $\sim (\exists x)\mathbf{P}(x)$ is false, then $(\exists x)\mathbf{P}(x)$ is true.

Example 1.6.1

Show that there is an even prime number.

Solution:

2 is a prime even number.

Example 1.6.2

Let $\mathcal{U} = \mathbb{R}$. Show that $(\exists x)[x^3 + 3x^2 + x - 1 = 0]$.

Solution:

Using direct proof: $x = -1$ is a solution. On the other hand, using a proof by contradiction:

Assume $\sim (\exists x)[x^3 + 3x^2 + x - 1 = 0] \equiv (\forall x)[x^3 + 3x^2 + x - 1 \neq 0]$. Therefore, either:

Case 1: $(\forall x)[x^3 + 3x^2 + x - 1 > 0]$ which is false for if $x = -10$, or

Case 2: $(\forall x)[x^3 + 3x^2 + x - 1 < 0]$ which is false for if $x = 10$.

Therefore, $(\exists x)[x^3 + 3x^2 + x - 1 = 0]$.

1.6.2 Type 2: Proof of $(\forall x)\mathbf{P}(x)$

- Direct proof: Let $x \in \mathcal{U}$ be arbitrary, then ... . Hence, $\mathbf{P}(x)$ is true. Since x was arbitrary chosen, $(\forall x)\mathbf{P}(x)$ is true.
- Proof by contradiction: Suppose $\sim (\forall x)\mathbf{P}(x)$. Then $(\exists x)(\sim \mathbf{P}(x))$ ... ... .Therefore, $\mathbf{Q}(x) \wedge \sim \mathbf{Q}(x)$, contradiction. Hence, $\sim (\forall x)\mathbf{P}(x)$ is false, then $(\forall x)\mathbf{P}(x)$ is true.

Example 1.6.3

Let $\mathcal{U} = \mathbb{Z}$. Show that $(\forall x)$, if x is even, then x^2 is even.

Solution:

Assume that $x \in \mathbb{Z}$ so that $x = 2k$ for some integer k. Then $x^2 = (2k)^2 = 2(2k^2)$ which is even.

Example 1.6.4

Show that for all rational numbers p and q, $\dfrac{p+q}{2}$ is rational.

Solution:

Assume that $p = \frac{x}{y}$ and $q = \frac{u}{v}$ where $x, y, u, v \in \mathbb{Z}$ with $y, v \neq 0$. Then,

$$\frac{p+q}{2} = \frac{1}{2}\left(\frac{x}{y} + \frac{u}{v}\right) = \frac{1}{2}\left(\frac{xv + yu}{yv}\right) = \frac{xv + yu}{2yv},$$

which is rational.

1.6.3 Type 3: Proof of $(\exists! x)\mathbf{P}(x)$

1. Prove that $(\exists x)\mathbf{P}(x)$ by any method.
2. Assume that $x, y \in \mathcal{U}$ such that $\mathbf{P}(x)$ and $\mathbf{P}(y)$ are true ... . Thus, $x = y$. Therefore, $(\exists! x)\mathbf{P}(x)$.

Example 1.6.5

Prove that every nonzero real number has a unique multiplicative inverse.

Solution:

Let x be any nonzero real number. We want to show that $xy = 1$ for exactly one real number y. Let $y = \frac{1}{x}$, then y is a real number. Since $x \neq 0$, then $xy = x\frac{1}{x} = 1$. Thus, x has a multiplicative inverse.

Assume that y and z are two real numbers such that $xy = xz = 1$. Since $x \neq 0$, $xy = xz$ implies that $y = z$. Therefore, every nonzero real number has a unique multiplicative inverse.

Exercise 1.6.1

Prove that every nonsingular matrix has a unique inverse.

Chapter

2 Set Theory

Section 2.1: Basic Notations of Set Theory

Definition 2.1.1

A **set** is a collection of objects called elements. Sets are usually denoted by capital letters $A, B, C, \cdots$ while elements are usually denoted by small letters $a, b, c, \cdots$.

- If a is an element of a set A, then we write $a \in A$. Otherwise, we write $a \notin A$.
- The empty set $\phi := \{x : x \neq x\}$. That is, ϕ is a set with no elements.
- A set B is a **subset** of A, denoted by $B \subseteq A$, if and only if every elements of B is also an element of A. That is, $\forall b \in B \Rightarrow b \in A$.
- A set B is called a **proper subset** of set A, if $B \subseteq A$ and $B \neq \phi$, but $B \neq A$. In this case, we write $B \subset A$.
- Two subsets A and B are equal , denoted by $A = B$, if and only of $A \subseteq B$ and $B \subseteq A$.
- If a set A contains n elements, we say that $|A| = n$.

Theorem 2.1.1

For any sets A, B, and C, we have:

1) $\phi \subseteq A$,
2) $A \subseteq A$, and
3) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof:

The first two results are trivial so we leave those. For part 3) let a be any element of A. Since $A \subseteq B$, $a \in B$. But since $B \subseteq C$, $a \in C$. Thus, if $a \in A \Rightarrow a \in C$. Thus, $A \subseteq C$.

Definition 2.1.2

Let A be a set. The **power set** of A is the set whose elements are all the subsets of A and is denoted by $\mathcal{P}(A)$. Thus,

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Example 2.1.1

Let $A = \{a, b, c\}$. Find $\mathcal{P}(A)$.

Solution:

$\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$.

Remark 2.1.1

Let A be any given set. Then,

a. Theorem: If $|\,A\,| = n$, then $|\,\mathcal{P}(A)\,| = 2^n$.

b. $A \nsubseteq \mathcal{P}(A)$, but $A \in \mathcal{P}(A)$.

Example 2.1.2

Let $A = \{1, \{1, 3\}, \{2, 3\}\}$. Find $\mathcal{P}(A)$.

Solution:

$$\mathcal{P}(A) = \{\,\phi, \{1\}, \{\,\{\,1, 3\,\}\,\}, \{\,\{\,2, 3\,\}\,\}, \{\,1, \{\,1, 3\,\}\,\}, \{\,1, \{\,2, 3\,\}\,\}, \{\,\{\,1, 3\,\}, \{\,2, 3\,\}\,\}, A\,\}.$$

Note that, $1 \in A$, while $2 \notin A$ and $3 \notin A$. Also, $\{1\} \notin A$ where $\{2, 3\} \in A$ and $\{\{2, 3\}\} \subseteq A$ hence $\{\{2, 3\}\} \in \mathcal{P}(A)$. Moreover, $1 \notin \mathcal{P}(A)$, $\{1\} \in \mathcal{P}(A)$, and $\{\{1\}\} \subseteq \mathcal{P}(A)$. Also, $\phi \subseteq A$, $\phi \in \mathcal{P}(A)$ and $\{\phi\} \subseteq \mathcal{P}(A)$. Finally, $\{1, 3\} \notin \mathcal{P}(A)$, but $\{\{1, 3\}\} \in \mathcal{P}(A)$ and $\{\{\{1, 3\}\}\} \subseteq \mathcal{P}(A)$.

Theorem 2.1.2

Let A and B be two sets. Then, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof:

„ $\Rightarrow$ ": Assume that $A \subseteq B$. Let $X \in \mathcal{P}(A)$. Then, $X \subseteq A \subseteq B$. That is, $X \in \mathcal{P}(B)$. Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

„ $\Leftarrow$ ": Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have $A \in \mathcal{P}(B) \Rightarrow A \subseteq B$.

Exercise 2.1.1

Let $A = \{ 9^n : n \in \mathbb{Z} \}$ and $B = \{ 3^n : n \in \mathbb{Z} \}$. Show that $A \subsetneqq B$.

Exercise 2.1.2

Let $A = \{ 9^n : n \in \mathbb{Q} \}$ and $B = \{ 3^n : n \in \mathbb{Q} \}$. Show that $A = B$.

Exercise 2.1.3

Find $\mathcal{P}(\phi)$, $\mathcal{P}(\mathcal{P}(\phi))$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\phi)))$.

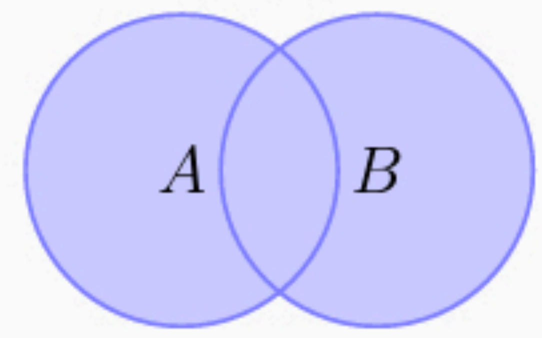
Section 2.2: Set Operations

Definition 2.2.1

Let A and B be two sets. Then,

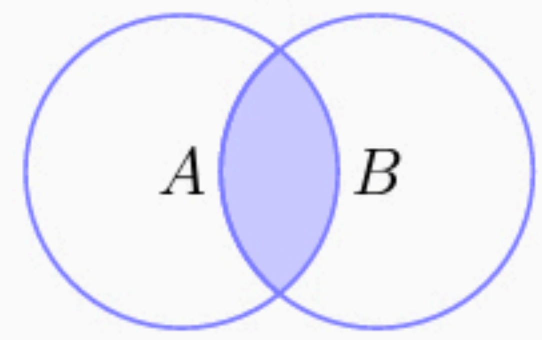
1. **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
   What is the meaning of $x \notin A \cup B$?

$A \cup B$



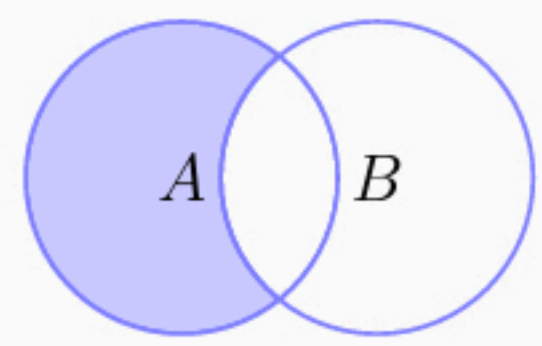
2. **intersection**: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
   What is the meaning of $x \notin A \cap B$?

$A \cap B$



3. **Difference**: $A - B = \{x : x \in A \text{ and } x \notin B\}$.
   What is the meaning of $x \notin A - B$?

$A - B$

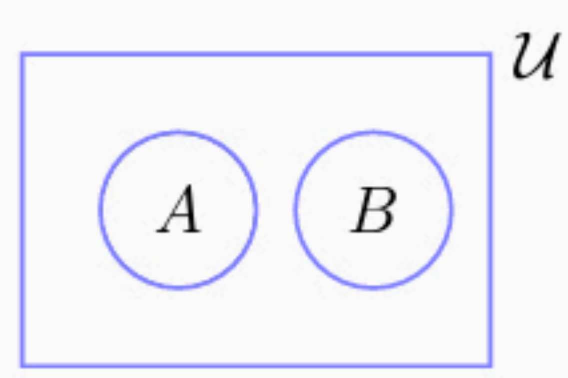


4. **Complement**: If $\mathcal{U}$ is the universal, then
   $\tilde{A} = \{x : x \notin A\} = \{x : x \in \mathcal{U} - A\}$.

$\tilde{A} = \mathcal{U} - A$

5. **Disjoint**: A and B are called **disjoint** if $A \cap B = \phi$.

$A \cap B = \phi$



Theorem 2.2.1

Let A, B, and C be sets. Then,

1. $A \subseteq A \cup B$.
2. $A \cap B \subseteq A$.
3. $A \cap \phi = \phi$.
4. $A \cup \phi = A$.

5. $A \cap A = A$.
6. $A \cup A = A$.
7. $A \cup B = B \cup A$.
8. $A \cap B = B \cap A$.
9. $A - \phi = A$.
10. $\phi - A = \phi$.
11. $A \cup (B \cup C) = (A \cup B) \cup C$.
12. $A \cap (B \cap C) = (A \cap B) \cap C$.
13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
15. $A \subseteq B$ if and only if $A \cup B = B$.
16. $A \subseteq B$ if and only if $A \cap B = A$.
17. if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.
18. if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Proof:

Proof of (13): Using the fact "$\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) = (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})$" as follows.

$$\begin{aligned} x \in A \cap (B \cup C) \quad & \text{iff} \quad x \in A \text{ and } x \in B \cup C \\ & \text{iff} \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \\ & \text{iff} \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ & \text{iff} \quad x \in A \cap B \text{ or } x \in A \cap C \\ & \text{iff} \quad x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Proof of (15): „ $\Rightarrow$ ": Assume that $A \subseteq B$. By part (1), $B \subseteq A \cup B$ so we only show that $A \cup B \subseteq B$. Let $x \in A \cup B$, then $x \in A \subseteq B$ or $x \in B$. In both cases, $x \in B$. Thus, $A \cup B \subseteq B$. Therefore, $B = A \cup B$.

„ $\Leftarrow$ ": Assume that $A \cup B = B$. By part (1) $A \subseteq A \cup B = B$. Thus, $A \subseteq B$.

Proof of (18): Assume that $A \subseteq B$. Let $x \in A \cap C$, then $x \in A \subseteq B$ and $x \in C$. Thus, $x \in B$ and $x \in C$, which implies that $x \in B \cap C$. Therefore, $A \cap C \subseteq B \cap C$.

Theorem 2.2.2

Let A and B be two subsets of the universe $\mathcal{U}$. Then:

1. $\tilde{\tilde{A}} = A$.
2. $A \cup \tilde{A} = \mathcal{U}$.
3. $A \cap \tilde{A} = \phi$.
4. $A - B = A \cap \tilde{B}$.
5. $A \subseteq B$ iff $\tilde{B} \subseteq \tilde{A}$.
6. $A \cap B = \phi$ iff $A \subseteq \tilde{B}$.
7. a. $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$.
   b. $\widetilde{A \cap B} = \tilde{A} \cup \tilde{B}$. .... (De Morgan's Laws).

Proof:

Proof of (2): If $x \in A \cup \tilde{A}$ then $x \in A \subseteq \mathcal{U}$ or $x \in \tilde{A} = \mathcal{U} - A$. In either cases, $x \in \mathcal{U}$. Thus, $A \cup \tilde{A} \subseteq \mathcal{U}$.

Assume now that $x \in \mathcal{U}$. Thus, $x \in A$ or $x \in \mathcal{U} - A = \tilde{A}$ which implies $x \in A \cup \tilde{A}$. Thus $\mathcal{U} \subseteq A \cup \tilde{A}$. Therefore, $\mathcal{U} = A \cup \tilde{A}$.

Proof of (5): Using a contrapositive proof as follows:

$$\begin{aligned} A \subseteq B \quad & \text{iff} \quad (\forall x)(x \in A \Rightarrow x \in B) \\ & \text{iff} \quad (\forall x)(x \notin B \Rightarrow x \notin A) \\ & \text{iff} \quad (\forall x)(x \in \tilde{B} \Rightarrow x \in \tilde{A}) \\ & \text{iff} \quad \tilde{B} \subseteq \tilde{A}. \end{aligned}$$

Proof of (7.b): Recall that $\sim (\mathbf{P} \wedge \mathbf{Q}) = \sim \mathbf{P} \vee \sim \mathbf{Q}$:

$$\begin{aligned} x \in \widetilde{A \cap B} \quad & \text{iff} \quad x \notin A \cap B \\ & \text{iff} \quad \sim (x \in A \text{ and } x \in B) \\ & \text{iff} \quad x \notin A \text{ or } x \notin B \\ & \text{iff} \quad x \in \tilde{A} \text{ or } x \in \tilde{B} \\ & \text{iff} \quad x \in \tilde{A} \cup \tilde{B}. \end{aligned}$$

Example 2.2.1

Let $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$ be the universe and let $A = \{1,5,7\}$, $B = \{2,5,8\}$, and $C = \{3,4,5,6,7\}$ Answer Each of the following:

1. $A \cap B = \{5\}$.
2. $B \cup C = \{2,3,4,5,6,7,8\}$.
3. $(A \cap B) \cup (A \cap C) = \{5\} \cup \{5,7\} = \{5,7\}$.
4. $A - C = \{1\}$.
5. $(A \cup C) - (B \cap C) = \{1,3,4,5,6,7\} - \{5\} = \{1,3,4,6,7\}$.
6. $\widetilde{A} = \mathcal{U} - A = \{2,3,4,6,8\}$.
7. $\widetilde{A} \cap \widetilde{B} = \{2,3,4,6,8\} \cap \{1,3,4,6,7\} = \{3,4,6\}$.

Example 2.2.2

Let $A \subseteq B \cup C$ and $A \cap B = \phi$. Show that $A \subseteq C$.

Solution:

Let $x \in A$. Since $A \subseteq B \cup C$, $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$, contradiction. Thus, $x \in C$ and therefore, $A \subseteq C$.

Example 2.2.3

Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Solution:

$$\begin{aligned} \text{Let } X \in \mathcal{P}(A \cap B) &\text{ iff } X \subseteq A \cap B \\ &\text{ iff } X \subseteq A \text{ and } X \subseteq B \\ &\text{ iff } X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\ &\text{ iff } X \in \mathcal{P}(A) \cap \mathcal{P}(B). \end{aligned}$$

Example 2.2.4

Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Is $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ in general? Explain.

Solution:

$$\begin{aligned} \text{Let } X \in \mathcal{P}(A) \cup \mathcal{P}(B) \quad &\Rightarrow \quad X \in \mathcal{P}(A) \text{ or } X \in \mathcal{P}(B) \\ &\Rightarrow \quad X \subseteq A \text{ or } X \subseteq B \\ &\Rightarrow \quad X \subseteq A \cup B \\ &\Rightarrow \quad X \in \mathcal{P}(A \cup B). \end{aligned}$$

In general, $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ and thus $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

For instance, consider $A = \{a\}$ and $B = \{b\}$. Then $A \cup B = \{a, b\}$, $\mathcal{P}(A) = \{\phi, \{a\}\}$ and $\mathcal{P}(B) = \{\phi, \{b\}\}$. Therefore,

$$\mathcal{P}(A \cup B) = \{\phi, \{a\}, \{b\}, \{a, b\}\} \neq \mathcal{P}(A) \cup \mathcal{P}(B) = \{\phi, \{a\}, \{b\}\}.$$

Remark 2.2.1

If $A \subseteq B$, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

Exercise 2.2.1

Suppose that A, B, and C are three nonempty sets. Show that if $A \subseteq B$, then $A - C \subseteq B - C$.

Exercise 2.2.2

Suppose that A, and B are two nonempty sets. Show that $A - B = \phi$ iff $A \cap B = A$.

Section 2.3: Extended Set Operations

Definition 2.3.1

Let $\mathcal{I}$ be a nonempty set. Suppose that for each $i \in \mathcal{I}$, there is a corresponding set A_i. Then, the family of sets $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ is called an **indexed family of sets**. Each $i \in \mathcal{I}$ is called an **index** and $\mathcal{I}$ is called an **indexing set**. Then

1. The **union over** $\mathcal{A}$ is defined by

$$\bigcup_{i \in \mathcal{I}} A_i = \{\, x : (\exists A_i \in \mathcal{A})\,[x \in A_i]\,\} = \{\, x : (\,\exists A_i\,)\,[A_i \in \mathcal{A} \wedge x \in A_i]\,\}.$$

2. the **intersection over** $\mathcal{A}$ is defined by

$$\bigcap_{i \in \mathcal{I}} A_i = \{\, x : (\forall A_i \in \mathcal{A})\,[x \in A_i]\,\} = \{\, x : (\,\forall A_i\,)\,[A_i \in \mathcal{A} \Rightarrow x \in A_i]\,\}.$$

3. The indexed family $\mathcal{A}$ of sets is said to be **pairwise disjoint** if and only if for all i and j in $\mathcal{I}$, either $A_i = A_j$ or $A_i \cap A_j = \phi$.

Example 2.3.1

Let $\mathcal{I} = \{\,1, 2, 3\,\}$, and define $A_i = \{\,i, i+1\,\}$ for each $i \in \mathcal{I}$. Find $\bigcup\limits_{i \in \mathcal{I}} A_i$ and $\bigcap\limits_{i \in \mathcal{I}} A_i$.

Solution:

Note that $A_1 = \{\,1, 2\,\}$, $A_2 = \{\,2, 3\,\}$, and $A_3 = \{\,3, 4\,\}$. Thus, $\bigcup\limits_{i \in \mathcal{I}} A_i = \{\,1, 2, 3, 4\,\}$, and $\bigcap\limits_{i \in \mathcal{I}} A_i = \phi$.

Example 2.3.2

For each $i \in \mathbb{N}$, let $A_i = \{j \in \mathbb{N} : j \leq i\}$. Find $\bigcup\limits_{i \in \mathbb{N}} A_i$ and $\bigcap\limits_{i \in \mathbb{N}} A_i$.

Solution:

Note that $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $\cdots$, $A_n = \{1, 2, \cdots, n\}$ and so on. Thus, $\bigcup\limits_{i \in \mathbb{N}} A_i = \mathbb{N}$ while $\bigcap\limits_{i \in \mathbb{N}} A_i = \{1\}$.

Theorem 2.3.1

Let $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ be an indexed family of sets. Then,

1. For each $k \in \mathcal{I}$, $A_k \subseteq \bigcup_{i\in\mathcal{I}} A_i$.
2. For each $k \in \mathcal{I}$, $\bigcap_{i\in\mathcal{I}} A_i \subseteq A_k$.
3. a. $\widetilde{\bigcup_{i\in\mathcal{I}} A_i} = \bigcap_{i\in\mathcal{I}} \widetilde{A_i}$.
   b. $\widetilde{\bigcap_{i\in\mathcal{I}} A_i} = \bigcup_{i\in\mathcal{I}} \widetilde{A_i}$.
   .... (De Morgan's Laws).

Proof:

Proof of (1): Let $x \in A_k$. Since $A_k \in \mathcal{A}$, $x \in \bigcup_{i\in\mathcal{I}} A_i$. Thus, $A_k \subseteq \bigcup_{i\in\mathcal{I}} A_i$.

Proof of (2): Let $x \in \bigcap_{i\in\mathcal{I}} A_i$. Then, $x \in A_i$ for every $i \in \mathcal{I}$. Since $k \in \mathcal{I}$, $x \in A_k$. Thus, $\bigcap_{i\in\mathcal{I}} A_i \subseteq A_k$.

Proof of (3.a):

$$\begin{aligned} x \in \widetilde{\bigcup_{i\in\mathcal{I}} A_i} &\Leftrightarrow x \notin \bigcup_{i\in\mathcal{I}} A_i \\ &\Leftrightarrow x \notin A_i \text{ for all } i \in \mathcal{I} \\ &\Leftrightarrow x \in \widetilde{A_i} \text{ for all } i \in \mathcal{I} \\ &\Leftrightarrow x \in \bigcap_{i\in\mathcal{I}} \widetilde{A_i}. \end{aligned}$$

Proof of (3.b): A similar proof as that in part (3.a) can be shown in this part as well. However, we use a different style as follows: Using $A_i = \widetilde{\widetilde{A_i}}$ together with part (3.a) of this theorem, we get

$$\widetilde{\bigcap_{i\in\mathcal{I}} A_i} = \widetilde{\bigcap_{i\in\mathcal{I}} \widetilde{\widetilde{A_i}}} = \widetilde{\widetilde{\bigcup_{i\in\mathcal{I}} \widetilde{A_i}}} = \bigcup_{i\in\mathcal{I}} \widetilde{A_i}.$$

Example 2.3.3

Let $\mathcal{I} = \{1,2,3,4\}$ so that $A_1 = \{1,2,7\}$, $A_2 = \{3,4,8\}$, $A_3 = \{1,4,8\}$, and $A_4 = \{1,3,4,7\}$. If $\mathcal{U} = \{1,2,3,\cdots,10\}$, answer each of the following:

a. $\bigcup_{i\in\mathcal{I}} A_i = \{1,2,3,4,7,8\}$.

b. $\bigcap_{i \in \mathcal{I}} A_i = \phi$.

c. $\bigcup_{i \in \mathcal{I}} \widetilde{A_i} = \widetilde{\bigcap_{i \in \mathcal{I}} A_i} = \mathcal{U}$.

d. $\bigcap_{i \in \mathcal{I}} \widetilde{A_i} = \widetilde{\bigcup_{i \in \mathcal{I}} A_i} = \{5, 6, 9, 10\}$.

e. Is $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ a pairwise disjoint? Explain. Answer: No, $A_3 \cap A_4 = \{1, 4\} \neq \phi$.

Example 2.3.4

Let $\mathcal{U} = \mathbb{N}$ and $\mathcal{I} = \mathbb{N}$. Define $A_i = \mathbb{N} - \{1, 2, \cdots, i\}$ for all $i \in \mathcal{I}$. Find:

a. $A_{10} = \{11, 12, 13, \cdots\}$.

b. $\bigcup_{i \in \mathcal{I}} A_i = \{2, 3, 4, 5, \cdots\}$.

c. $\bigcap_{i \in \mathcal{I}} A_i = \phi$.

Example 2.3.5

If $\mathcal{U} = \mathbb{R}$, let $A_n = [-\frac{1}{n}, 2 + \frac{1}{n})$ for all $n \in \mathbb{N}$. Find:

a. $\bigcup_{n \in \mathbb{N}} A_n = [-1, 3) =: A_1$.

b. $\bigcap_{n \in \mathbb{N}} A_n = [0, 2]$.

c. $\bigcap_{n \in \mathbb{N}} \widetilde{A_n} = \widetilde{\bigcup_{n \in \mathbb{N}} A_n} = \mathbb{R} - [-1, 3)$.

d. $\bigcup_{n \in \mathbb{N}} \widetilde{A_n} = \widetilde{\bigcap_{n \in \mathbb{N}} A_n} = \mathbb{R} - [0, 2]$.

Example 2.3.6

Let $\mathcal{U} = \mathbb{R}$ and define $S_a = (-a, a)$ for all $a \in \mathbb{N}$. Find

a. $\bigcup_{a \in \mathbb{N}} S_a = \mathbb{R}$.

b. $\bigcap_{a\in\mathbb{N}} S_a = (-1, 1)$.

Exercise 2.3.1

Let $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ be an indexed family of sets for a nonempty set $\mathcal{I}$. Show that if $B \subseteq A_i$ for every $i \in \mathcal{I}$, then $B \subseteq \bigcap_{i\in\mathcal{I}} A_i$.

Exercise 2.3.2

For each natural number $n \geq 3$, let $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n}\right]$, and $\mathcal{A} = \{A_n : n \geq 3\}$. Find $\bigcap_{n\geq 3} A_n$ and $\bigcup_{n\geq 3} A_n$.

Section 2.4: Proof by Induction

Definition 2.4.1: Principle of Mathematical Induction (PMI)

If S is a subset of $\mathbb{N}$ so that:

1. $1 \in S$, and
2. for all $n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$,

then $S = \mathbb{N}$.

2.4.1 Proof of $(\forall n \in \mathbb{N})\mathbf{P}(n)$ using PMI

- **Basic Step**: Show that $\mathbf{P}(1)$ is true.
- **Induction Step**: Show that for all $n \in \mathbb{N}$, if $\mathbf{P}(n)$ is true, then $\mathbf{P}(n+1)$ is true.
- **Conclusion**: By step 1 and step 2 and using the PMI, $\mathbf{P}(n)$ is true for all $n \in \mathbb{N}$.

Example 2.4.1

Show that for all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution:

For $n = 1$, clearly $1 = \frac{1(1+1)}{2}$ is true. Assume that for some $n \in \mathbb{N}$, we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, we want to show that $1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$.

$$\begin{aligned} \overbrace{1 + 2 + 3 + \cdots + n}^{\text{use our assumption}} + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Example 2.4.2

Show that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$.

Solution:

For $n = 1$, $2(1) - 1 = 1 = 1^2$, which is true. Assume that for some $n \in \mathbb{N}$, we have $\sum_{i=1}^{n}(2i-1) = n^2$. We want to show that $\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$. Thus,

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$

Example 2.4.3

Show that for all $n \in \mathbb{N}$, $n + 3 < 5n^2$.

Solution:

For $n = 1$ we have $1 + 3 = 4 < 5$ which is true. So, assume that for n, $n + 3 < 5n^2$ is true. For $n + 1$, we want to show that $(n+1) + 3 < 5(n+1)^2 = 5n^2 + 10n + 5$. Then,

$$(n+1) + 3 = (n+3) + 1 < 5n^2 + 1 < 5n^2 + (10n + 4) + 1 = 5(n+1)^2.$$

Therefore, for all $n \in \mathbb{N}$, $n + 3 < 5n^2$.

Definition 2.4.2

For $n \in \mathbb{N}$, define $0! = 1$ and $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots 2 \cdot 1$. Then, the **bionomial coefficient "n choose k"**, where $0 \leq k \leq n$, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)(n-3)\cdots(n-k+2)(n-k+1)}{k!}.$$

Moreover, the **bionomial expansion** of any $a, b \in \mathbb{R}$ is given by

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k \, b^{n-k}.$$

Remark 2.4.1: Pascal's Triangle

Let $a, b \in \mathbb{R}$. Then, the coefficients of the bionomial expansion $(a+b)^n$ can be computed by the Pascal's Triangle for each n.

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Example 2.4.4

Show that for all $n \in \mathbb{N}$, $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer.

Solution:

$\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15} = \dfrac{5n^3 + 3n^5 + 7n}{15}$ is an integer iff $15 \mid 5n^3 + 3n^5 + 7n$ iff $\exists k \in \mathbb{N}$ such that $5n^3 + 3n^5 + 7n = 15k$.

For $n = 1$, we have $5 + 3 + 7 = 15$ which is true. So assume that there $k \in \mathbb{N}$ such that $5n^3 + 3n^5 + 7n = 15k$. Then, we want to show that

$$5(n+1)^3 + 3(n+1)^5 + 7(n+1) = 15h \tag{2.4.1}$$

for some $h \in \mathbb{N}$. Thus, using the Pascal's Triangle we get

$$
\begin{aligned}
\text{Eqn.}(2.4.1) &= 5(n^3 + 3n^2 + 3n + 1) + 3(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) + 7n + 7 \\
&= \underbrace{(5n^3 + 3n^5 + 7n)}_{=15k} + \textcircled{15}n^2 + \textcircled{15}n + 5 + \textcircled{15}n^4 \\
&+ \textcircled{30}n^3 + \textcircled{30}n^2 + \textcircled{15}n + 3 + 7 \\
&= 15k + 15\left[n^2 + n + n^4 + 2n^3 + 2n^2 + n + 1\right]
\end{aligned}
$$

Thus $15 \mid 5(n+1)^3 + 3(n+1)^5 + 7(n+1)$ and $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer for all $n \in \mathbb{N}$.

Example 2.4.5

Express the terms of $(2x - 4yz^2)^5$ for $x, y, z \in \mathbb{R}$.

Solution:

Let $a = 2x$, $b = -4yz^2$, and $n = 5$. Using the bionomial expansion form, we get

$$\begin{aligned}(2x - 4yz^2)^5 &= (2x)^5 + 5(2x)^4(-4yz^2) + 10(2x)^3(-4yz^2)^2 + 10(2x)^2(-4yz^2)^3 \\ &\quad + 5(2x)(-4yz^2)^4 + (-4yz^2)^5.\end{aligned}$$

Definition 2.4.3: Generalized Principle of Mathematical Induction (GPMI)

Let k be a natural number. If S is a subset of $\mathbb{N}$ so that:

1. $k \in S$, and
2. for all $n \in \mathbb{N}$ with $n \geq k$, if $n \in S$, then $n + 1 \in S$,

then S contains all natural number greater than or equal to k.

Example 2.4.6

Show that for all $n \geq 5$, $n^2 - n - 20 \geq 0$.

Solution:

For $n = 5$, we have $25 - 5 - 20 = 0 \geq 0$ which is true. Assume that for some $n \geq 5$, $n^2 - n - 20 \geq 0$ is true. For $n + 1$, we have

$$(n+1)^2 - (n+1) - 20 = n^2 + 2n + \cancel{1} - n - \cancel{1} - 20 = (n^2 - n - 20) + \underbrace{2n}_{\text{positive}} \geq 0.$$

Thus, $n^2 - n - 20 \geq 0$ for all $n \geq 5$.

Example 2.4.7

Let $n \in \mathbb{N}$. Show that $(n + 1)! > 2^{n+3}$ for all $n \geq 5$.

Solution:

For $n = 5$, we have $6! = 720 \geq 2^8 = 256$ which is true. Assume that for some $n \geq 5$, $(n + 1)! > 2^{n+3}$ is true.

For $n+1$, we want to show that $(n+2)! > 2^{n+4}$ for all $n+1 \geq 5$. Since $n+2 > 2$ for all $n \geq 4$, we get

$$(n+2)! = (n+2)(n+1)! > (n+2)2^{n+3} > 2 \cdot 2^{n+3} = 2^{n+4}.$$

Thus, $(n+1)! > 2^{n+3}$ for all $n \geq 5$.