Chapter 3

Chapter

3 Relations

Section 3.1: Cartesian Products and Relations

Definition 3.1.1

Let A and B be two sets. An **ordered pair** is $(a, b) \neq \{a, b\}$ for $a \in A$ and $b \in B$. We say that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Definition 3.1.2

Let A and B be two sets. The (**Cartesian** or **cross**) **product** of A and B, denoted by $A \times B$, is defined by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Moreover, if $(a, b) \in A \times B$, then $a \in A$ and $b \in B$. If $(a, b) \notin A \times B$, then either $a \notin A$ or $b \notin B$.

Remark 3.1.1

Let A and B be two given sets. Then,

1. if A has m elements and B has n elements, then $A \times B$ has mn elements.
2. In general, $A \times B \neq B \times A$.

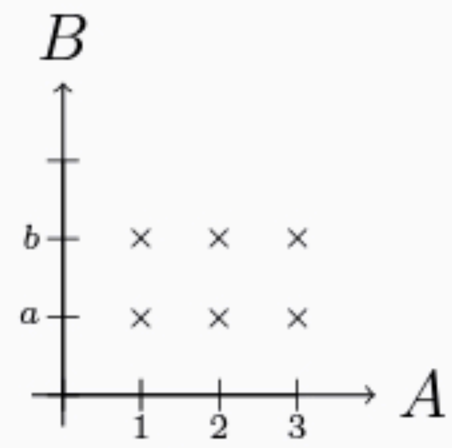
Example 3.1.1

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find $A \times B$ and $B \times A$.

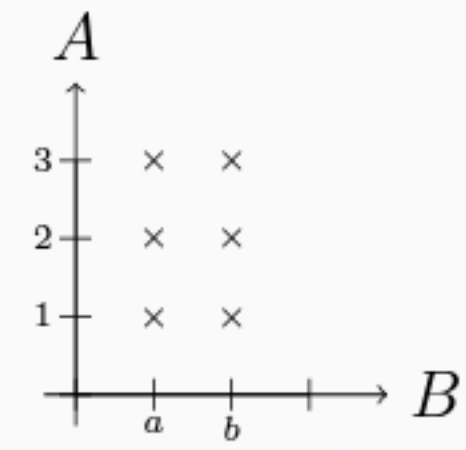
Solution:

Note that, in general $A \times B \neq B \times A$ as this example shows.

$$\begin{aligned} A \times B &= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}, \text{ and} \\ B \times A &= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}. \end{aligned}$$



$A \times B$



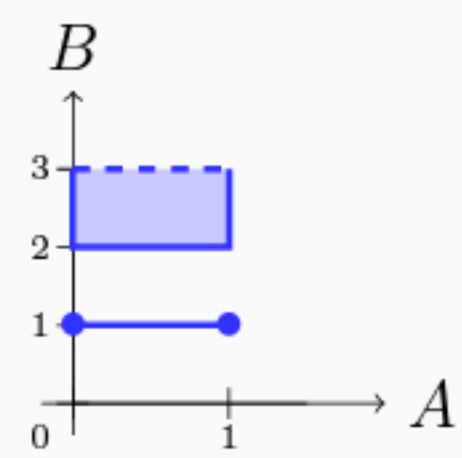
$B \times A$

Example 3.1.2

Let $A = [0, 1]$ and $B = \{1\} \cup [2, 3)$. Find $A \times B$.

Solution:

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$



$A \times B$

Theorem 3.1.1

If A and B are nonempty set, then $A \times B = B \times A$ iff $A = B$.

Proof:

" $\Rightarrow$ ": Assume that $A \neq \phi$, $B \neq \phi$ and $A \times B = B \times A$. Let $a \in A$, then there is $b \in B$ such that $(a, b) \in A \times B = B \times A$ which implies that $a \in B$ Thus, $A \subseteq B$.

Let $b \in B$, then there is $a \in A$ such that $(b, a) \in B \times A = A \times B$ which implies that $b \in A$. Thus, $B \subseteq A$ and therefore $A = B$.

" $\Leftarrow$ ": if $A = B$, then $A \times B = A \times A = B \times A$.

Theorem 3.1.2

Let A, B, C, and D be sets. Then

$$1. \begin{cases} \text{a. } A \times (B \cup C) &= (A \times B) \cup (A \times C). \\ \text{b. } (A \cup B) \times C &= (A \times C) \cup (B \times C). \\ \text{c. } A \times (B \cap C) &= (A \times B) \cap (A \times C). \\ \text{d. } (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{cases}$$

2. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

3. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof:

Proof of (1.a):

$$\begin{aligned}(x,y) \in A \times (B \cup C) \quad &\text{iff} \quad x \in A \wedge y \in B \cup C \\ &\text{iff} \quad x \in A \wedge (y \in B \vee y \in C) \\ &\text{iff} \quad (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\ &\text{iff} \quad ((x,y) \in A \times B) \vee ((x,y) \in A \times C) \\ &\text{iff} \quad (x,y) \in (A \times B) \vee (A \times C).\end{aligned}$$

Proof of (2):

$$\begin{aligned}(x,y) \in (A \times B) \cap (C \times D) \quad &\text{iff} \quad (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) \\ &\text{iff} \quad (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\ &\text{iff} \quad (x \in A \cap C) \wedge (y \in B \cap D) \\ &\text{iff} \quad (x,y) \in (A \cap C) \times (B \cap D).\end{aligned}$$

Proof of (3): Let $(x,y) \in (A \times B) \cup (C \times D)$, then $(x,y) \in A \times B$ or $(x,y) \in C \times D$.

Case(i): $(x,y) \in A \times B$ implies that $x \in A$ and $y \in B$. Then, $x \in A \cup C$ and $y \in B \cup D$. Thus, $(x,y) \in (A \cup C) \times (B \cup D)$.

Case(ii): $(x,y) \in C \times D$ implies that $x \in C$ and $y \in D$. Then again $x \in A \cup C$ and $y \in B \cup D$. Thus, $(x,y) \in (A \cup C) \times (B \cup D)$.

Therefore, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Remark 3.1.2

Note that $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$: For instance, Let $A = B = \{0\}$, and $C = D = \{1\}$. Then, $(0,1) \in (A \cup C) \times (B \cup D)$ while $(0,1) \notin (A \times B) \cup (C \times D)$. Therefore, $(A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$.

Definition 3.1.3

Let A and B be sets. A **relation** $\mathcal{R}$ from A to B is a subset of $A \times B$. In this case, we write $a\mathcal{R}b$ for $(a, b) \in \mathcal{R}$ and say that "a is related to b". Also, $a\not\mathcal{R}b$ means that $(a, b) \notin \mathcal{R} \subseteq A \times B$. Moreover, if $A = B$, then subsets of $A \times A$ are called relations on A.

Definition 3.1.4

If $\mathcal{R} \subseteq A \times B$ is a relation, then the **domain** of $\mathcal{R}$ is $\text{Dom}(\mathcal{R}) = \{a \in A : (a, b) \in \mathcal{R}\}$. Moreover, the **range** of $\mathcal{R}$ is $\text{Rng}(\mathcal{R}) = \{b \in B : (a, b) \in \mathcal{R}\}$.

Example 3.1.3

Let $A = \{1, 2, \{3\}, 4\}$ and $B = \{a, b, c, d\}$. Find the domain and range of $\mathcal{R}$, where

$$\mathcal{R} = \{(1, c), (\{3\}, a), (1, d), (2, d)\} \subseteq A \times B.$$

Solution:

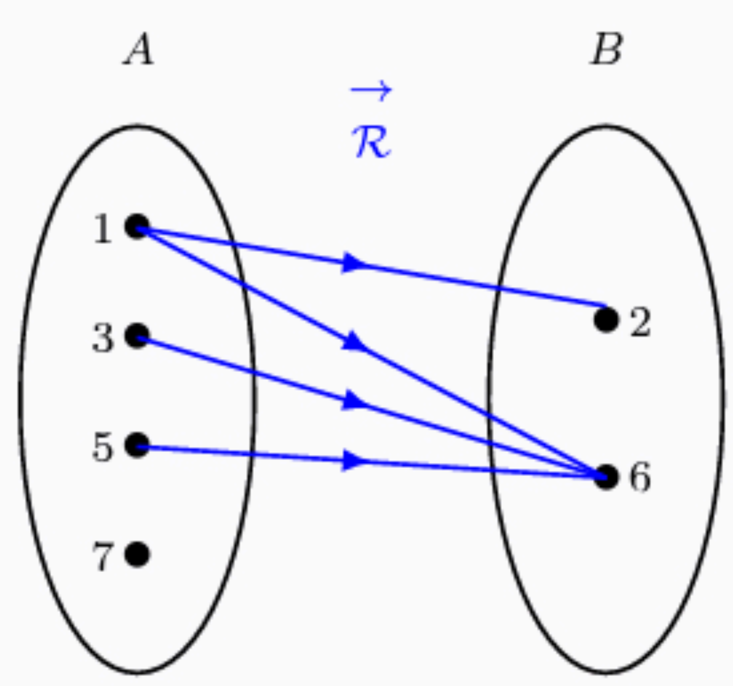
The $\text{Dom}(\mathcal{R}) = \{1, 2, \{3\}\} \subseteq A$ and the $\text{Rng}(\mathcal{R}) = \{a, c, d\} \subseteq B$. Note that $\text{Dom}(\mathcal{R}) \neq A$ and $\text{Rng}(\mathcal{R}) \neq B$.

Example 3.1.4

Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 6\}$. Let $\mathcal{R} \subseteq A \times B$ defined by $\mathcal{R} = \{(a, b) \in A \times B : a < b\}$. Find $\mathcal{R}$ along with its domain and range.

Solution:

$$\mathcal{R} = \{(1, 2), (1, 6), (3, 6), (5, 6)\}$$
$$\text{Dom}(\mathcal{R}) = \{1, 3, 5\}$$
$$\text{Rng}(\mathcal{R}) = \{2, 6\}.$$



Example 3.1.5

Let $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 3\}$. Find the domain and the range of the relation $\mathcal{R}$.

Solution:

Domain: $x \in \text{Dom}(\mathcal{R})$ iff $\exists y \in \mathbb{R}$ with $y = x^2 + 3$ which is true for all $x \in \mathbb{R}$. Thus, $\text{Dom}(\mathcal{R}) = \mathbb{R}$. Range: $y \in \text{Rng}(\mathcal{R})$ iff $\exists x \in \mathbb{R}$ with $y = x^2 + 3$ and since $x^2 \geq 0$, we have $y \geq 3$. Therefore, $\text{Rng}(\mathcal{R}) = [3, \infty)$.

Definition 3.1.5

For any set A, the relation $\mathcal{I}_A$ is the **identity relation** on A and is defined by

$$\mathcal{I}_A = \{(a, a) : a \in A\},$$

with $\text{Dom}(\mathcal{I}_A) = A = \text{Rng}(\mathcal{I}_A)$.

Definition 3.1.6

For any sets A and B, if $\mathcal{R} \subseteq A \times B$ is a relation, then the **inverse relation** is

$$\mathcal{R}^{-1} = \{(b, a) : (a, b) \in \mathcal{R}\} \subseteq B \times A,$$

with $\text{Dom}(\mathcal{R}^{-1}) = \text{Rng}(\mathcal{R})$ and $\text{Rng}(\mathcal{R}^{-1}) = \text{Dom}(\mathcal{R})$.

Definition 3.1.7

Let $\mathcal{R} \subseteq A \times B$ be a relation and let $\mathcal{S} \subseteq B \times C$ be a relation. The **composition relation** $\mathcal{S} \circ \mathcal{R}$ is defined by

$$\mathcal{S} \circ \mathcal{R} = \{(a, c) : (\exists b \in B)\big((a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{S}\big)\} \subseteq A \times C.$$

Moreover, $\text{Dom}(\mathcal{S} \circ \mathcal{R}) \subseteq \text{Dom}(\mathcal{R})$.

Example 3.1.6

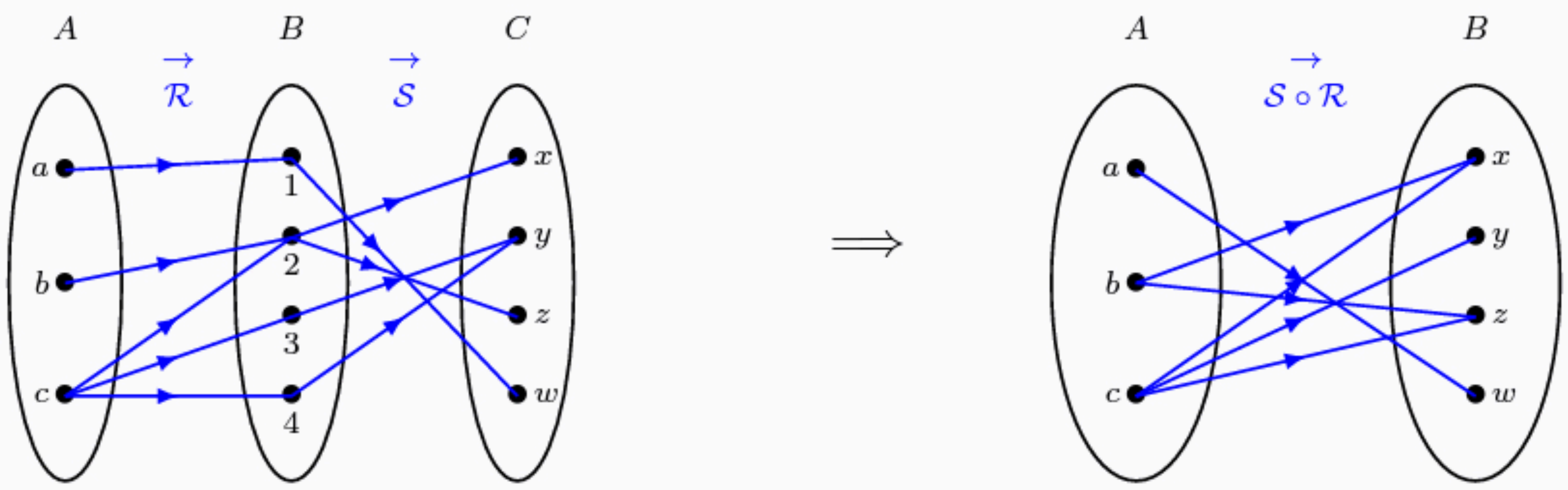
Let $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $C = \{x, y, z, w\}$. Let

$$\mathcal{R} = \{(a,1),(b,2),(c,2),(c,3),(c,4)\} \subseteq A \times B, \text{ and}$$
$$\mathcal{S} = \{(1,w),(2,x),(2,z),(3,y),(4,y)\} \subseteq B \times C.$$

Find $\mathcal{R}^{-1}$, and $\mathcal{S} \circ \mathcal{R}$.

Solution:

$$\mathcal{R}^{-1} = \{(1,a),(2,b),(2,c),(3,c),(4,c)\} \subseteq B \times A.$$
$$\mathcal{S} \circ \mathcal{R} = \{(a,w),(b,x),(b,z),(c,x),(c,z),(c,y)\} \subseteq A \times C.$$



Example 3.1.7

Let $\mathcal{R} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x < y\}$. Find $\mathcal{R}^{-1}$.

Solution:

Note that

$$\begin{aligned}(x,y) \in \mathcal{R}^{-1} \quad &\text{iff} \quad (y,x) \in \mathcal{R} \\ &\text{iff} \quad y < x \\ &\text{iff} \quad x > y.\end{aligned}$$

That is $\mathcal{R}^{-1} = \{\,(x,y) \in \mathbb{R} \times \mathbb{R} : x > y\,\}$.

Example 3.1.8

Let $\mathcal{R} = \{\,(x,y) \in \mathbb{R}\times\mathbb{R} : y = x-1\,\}$ and let $\mathcal{S} = \{\,(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2\,\}$. Find $\mathcal{S}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{S}$.

Solution:

$$\begin{aligned}
\mathcal{S}\circ\mathcal{R} &= \{\,(x,y) : (\,\exists z\in\mathbb{R}\,)(\,(x,z)\in\mathcal{R} \text{ and } (z,y)\in\mathcal{S}\,)\,\} \\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,z = x-1 \text{ and } y = z^2\,\Big)\,\Big\} \\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,y = (x-1)^2\,\Big)\,\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}\circ\mathcal{S} &= \{\,(x,y) : (\,\exists z\in\mathbb{R}\,)(\,(x,z)\in\mathcal{S} \text{ and } (z,y)\in\mathcal{R}\,)\,\} \\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,z = x^2 \text{ and } y = z-1\,\Big)\,\Big\} \\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,y = x^1 - 1\,\Big)\,\Big\}
\end{aligned}$$

Theorem 3.1.3

Let A, B, C, and D be sets. Let $\mathcal{R}\subseteq A\times B$, $\mathcal{S}\subseteq B\times C$, and $\mathcal{T}\subseteq C\times D$. Then,

1. $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$.
2. $\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R}) = (\mathcal{T}\circ\mathcal{S})\circ\mathcal{R}$.
3. $(\mathcal{S}\circ\mathcal{R})^{-1} = \mathcal{R}^{-1}\circ\mathcal{S}^{-1}$.

Proof:

Proof of part(2): Let $a\in A$ and $d\in D$ so that

$$\begin{aligned}
(a,d)\in\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R}) \quad &\text{iff} \quad (\exists c\in C)\Big[(a,c)\in\mathcal{S}\circ\mathcal{R} \text{ and } (c,d)\in\mathcal{T}\Big] \\
&\text{iff} \quad (\exists c\in C)\Big[(\exists b\in B)\Big((a,b)\in\mathcal{R} \text{ and } (b,c)\in\mathcal{S}\Big) \text{ and } (c,d)\in\mathcal{T}\Big] \\
&\text{iff} \quad (\exists c\in C)(\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (b,c)\in\mathcal{S} \text{ and } (c,d)\in\mathcal{T}\Big] \\
&\text{iff} \quad (\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (\exists c\in C)\Big((b,c)\in\mathcal{S} \text{ and } (c,d)\in\mathcal{T}\Big)\Big] \\
&\text{iff} \quad (\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (b,d)\in\mathcal{T}\circ\mathcal{S}\Big] \\
&\text{iff} \quad (a,d)\in(\mathcal{T}\circ\mathcal{S})\circ\mathcal{R}.
\end{aligned}$$

Proof of part (3): Let $a \in A$ and $c \in C$ so that

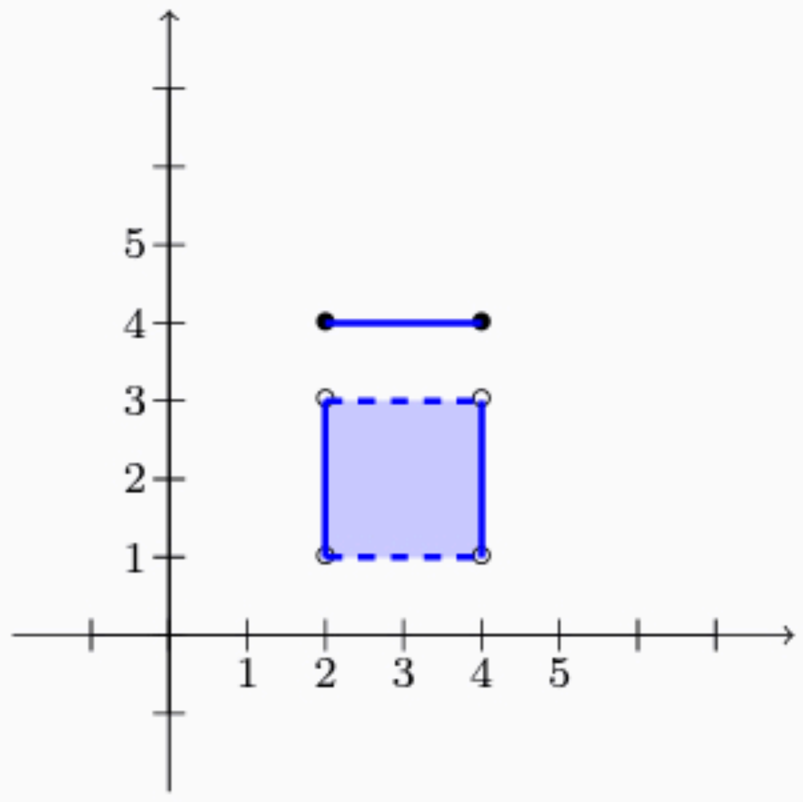
$$\begin{aligned}(c,a) \in (\mathcal{S} \circ \mathcal{R})^{-1} \quad &\text{iff} \quad (a,c) \in \mathcal{S} \circ \mathcal{R} \\ &\text{iff} \quad (\exists b \in B)\big[(a,b) \in \mathcal{R} \text{ and } (b,c) \in \mathcal{S}\big] \\ &\text{iff} \quad (\exists b \in B)\big[(b,a) \in \mathcal{R}^{-1} \text{ and } (c,b) \in \mathcal{S}^{-1}\big] \\ &\text{iff} \quad (\exists b \in B)\big[(c,b) \in \mathcal{S}^{-1} \text{ and } (b,a) \in \mathcal{R}^{-1}\big] \\ &\text{iff} \quad (c,a) \in \mathcal{R}^{-1} \circ \mathcal{S}^{-1}.\end{aligned}$$

Example 3.1.9

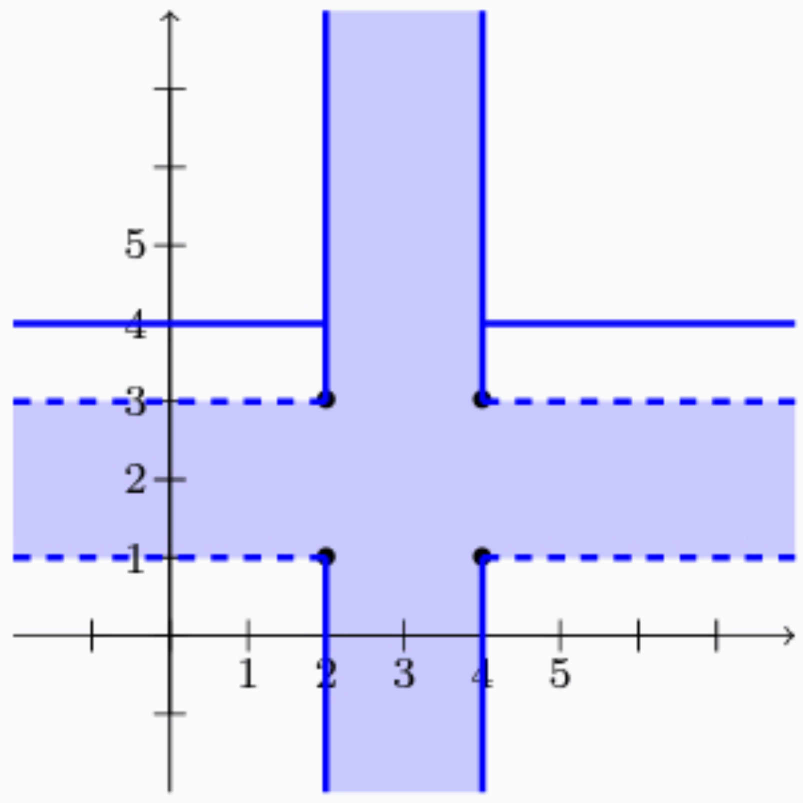
Let $A = [2,4]$ and $B = (1,3) \cup \{4\}$. Let $\mathcal{R}$ be the relation on $A \times \mathbb{R}$ with $x\mathcal{R}y$ iff $x \in A$ and let $\mathcal{S}$ be the relation on $\mathbb{R} \times B$ with $x\mathcal{S}y$ iff $y \in B$. Find $\mathcal{R} \cap \mathcal{S}$ and $\mathcal{R} \cup \mathcal{S}$.

Solution:

By Theorem 3.1.2 part(2), $\mathcal{R} \cap \mathcal{S} = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) = (A \cap \mathbb{R}) \times (\mathbb{R} \cap B) = A \times B$. Therefore, $\mathcal{R} \cap \mathcal{S} = A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$. On the other hand, $\mathcal{R} \cup \mathcal{S} = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a \in A \text{ or } b \in B\}$.



$\mathcal{R} \cap \mathcal{S}$



$\mathcal{R} \cup \mathcal{S}$

Exercise 3.1.1

Let A and B be two nonempty sets. Show that if $A \times B \subseteq B \times C$, then $A \subseteq C$.

Exercise 3.1.2

Let $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times C$ be two relations. Show that $\operatorname{Dom}(\mathcal{S} \circ \mathcal{R}) \subseteq \operatorname{Dom}(\mathcal{R})$.

Section 3.2: Equivalence Relations

Definition 3.2.1

Let A be a set and $\mathcal{R}$ be a relation on A. Then $\mathcal{R}$ is called an **equivalence relation** if and only if:

1. $\mathcal{R}$ is **reflexive** on A: $(\forall x \in A)\ x\mathcal{R}x$.
2. $\mathcal{R}$ is **symmetric** on A: $(\forall x, y \in A)$ if $x\mathcal{R}y$, then $y\mathcal{R}x$.
3. $\mathcal{R}$ is **transitive** on A: $(\forall x, y, z \in A)$ if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Example 3.2.1

Let $A = \{1, 2, 3, 4\}$ and $\mathcal{R}_1 = \{(1, 2), (2, 3), (1, 3)\}$, $\mathcal{R}_2 = \{(1, 1), (1, 2)\}$, $\mathcal{R}_3 = \{(3, 4)\}$, $\mathcal{R}_4 = \{(1, 2), (2, 1)\}$, and $\mathcal{R}_5 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. Decide which relation is reflexive, symmetric, transitive.

Solution:

$\mathcal{R}_5$ is reflexive. $\mathcal{R}_4$, and $\mathcal{R}_5$ are symmetric. $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_5$ are transitive. Therefore, $\mathcal{R}_5$ is an equivalence relation on A.

Example 3.2.2

Let $\mathcal{R} = \{(x, y) : xy > 0\}$ be a relation on $\mathbb{Z}$. Discuss whether $\mathcal{R}$ reflexive, symmetric, transitive, and equivalence relation.

Solution:

Clearly, $x\mathcal{R}x$ for all $x \in \mathbb{Z}$ except for $x = 0$, thus $\mathcal{R}$ is not reflexive. If $x\mathcal{R}y$, then $xy > 0$ or $yx > 0$ which imples that $y\mathcal{R}x$. Thus, $\mathcal{R}$ is symmetric. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $xy > 0$ and $yz > 0$. Considering the cases of $y \in \mathbb{Z} - \{0\}$, we have

1. case 1: $y > 0$, then $x > 0$ and $z > 0$ which implies that $xz > 0$ and thus $x\mathcal{R}z$.
2. case 1: $y < 0$, then $x < 0$ and $z < 0$ which implies that $xz > 0$ and thus $x\mathcal{R}z$.

In either cases, $\mathcal{R}$ is transitive on $\mathbb{Z}$. Note that $\mathcal{R}$ is not reflexive and thus it is not an equivalence relation on $\mathbb{Z}$.

Example 3.2.3

Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ given by $x\mathcal{R}y$ iff $x - y$ is even. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Solution:

Reflexive: Since $x - x = 0$ is even, $x\mathcal{R}x$ for all $x \in \mathbb{Z}$. Thus, $\mathcal{R}$ is reflexive.

Symmetric: Assume that $x\mathcal{R}y$, then there is $k \in \mathbb{Z}$ such that $x-y = 2k$. Thus, $y-x = 2(-k)$ which implies that $y\mathcal{R}x$. Thus, $\mathcal{R}$ is symmetric.

Transitive: Let $x\mathcal{R}y$ and $y\mathcal{R}z$. Then, there are $h, k \in \mathbb{Z}$ such that $x-y = 2h$ and $y-z = 2k$. Adding these two equations, we get $x - z = 2(h + k)$ which is even. Therefore, $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

Therefore, $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Definition 3.2.2

Let $\mathcal{R}$ be an equivalence relation on a set A. For $x \in A$, define the **equivalence class** of x determined by $\mathcal{R}$ as

$$x/\mathcal{R} = \{y \in A : x\mathcal{R}y\},$$

which reads "the class of x modulo $\mathcal{R}$" or "x mod $\mathcal{R}$. The set of all equivalence classes is called A modulo $\mathcal{R}$ and is defined by

$$A/\mathcal{R} = \{x/\mathcal{R} : x \in A\}.$$

Example 3.2.4

Let $\mathcal{R} = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ be an equivalence relation on $A = \{1,2,3\}$. Find:

- $1/\mathcal{R} = \{1,2\}$.
- $2/\mathcal{R} = \{1,2\}$.
- $3/\mathcal{R} = \{3\}$.
- $A/\mathcal{R} = \{\{1,2\},\{3\}\}$.

Example 3.2.5

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y \Leftrightarrow 2 \mid x+y$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$. Calculate all the equivalence classes of $\mathcal{R}$.

Solution:

reflexive: Since $x+x=2x$, $2 \mid x+x$ and thus $x\mathcal{R}x$. So, $\mathcal{R}$ is reflexive.

symmetric: if $x\mathcal{R}y$, then $2 \mid x+y$. Thus, $2 \mid y+x$ as well and $y\mathcal{R}x$. Therefore, $\mathcal{R}$ is symmetric.

transitive: Assume that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then $2 \mid x+y$ and $2 \mid y+z$. Thus, $2 \mid x+z+2y$. But because $2 \mid 2y$, we have $2 \mid x+z$. Thus, $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

Therefore, $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$.

For $x \in \mathbb{N}$, $x/\mathcal{R} = \{y \in \mathbb{N} : 2 \mid x+y\}$. Thus,

$$\overline{1} = \{1,3,5,7,9,\cdots\} = \overline{3} = \overline{5} = \cdots, \text{ and } \overline{2} = \{2,4,6,8,10,\cdots\} = \overline{2} = \overline{4} = \cdots.$$

Therefore, $\mathbb{N} = \overline{1} \cup \overline{2}$.

Theorem 3.2.1

Let $\mathcal{R}$ be an equivalence relation on a nonempty set A. For all $x, y \in A$,

1. $x/\mathcal{R} \subseteq A$ and $x \in x/\mathcal{R} \neq \phi$.
2. $x\,\mathcal{R}\,y$ iff. $x/\mathcal{R} = y/\mathcal{R}$.
3. $x \not\mathcal{R} y$ iff. $x/\mathcal{R} \cap y/\mathcal{R} = \phi$.

Proof:

1. Clearly, $x/\mathcal{R} \subseteq A$ by the definition. Since$\mathcal{R}$ is reflexive, $x\,\mathcal{R}\,x$ and hence $x \in x/\mathcal{R}$.
2. " $\Rightarrow$ ": Suppose $x\,\mathcal{R}\,y$. Then $y\,\mathcal{R}\,x$ (since $\mathcal{R}$ is symmetric). To show that $x/\mathcal{R} = y/\mathcal{R}$, we first show that $x/\mathcal{R} \subseteq y/\mathcal{R}$: Let $z \in x/\mathcal{R} \Rightarrow x\,\mathcal{R}\,z$ and $y\,\mathcal{R}\,x$. Hence, $y\,\mathcal{R}\,z$. Hence, $x/\mathcal{R} \subseteq y/\mathcal{R}$. The proof of $y/\mathcal{R} \subseteq x/\mathcal{R}$ is similar.

   " $\Leftarrow$ ": Suppose $x/\mathcal{R} = y/\mathcal{R}$. Then $x \in x/\mathcal{R} = y/\mathcal{R}$. That is $x\,\mathcal{R}\,y$.
3. " $\Rightarrow$ ": Suppose $x \not\mathcal{R} y$. We proof by contradiction: Assume that there is $z \in x/\mathcal{R} \cap y/\mathcal{R}$. Then, $z \in x/\mathcal{R}$ and $z \in y/\mathcal{R}$ and hence $x\,\mathcal{R}\,z$ and $z\,\mathcal{R}\,y$. Thus, $x\,\mathcal{R}\,y$, contradiction.

   " $\Leftarrow$ ": Suppose $x/\mathcal{R} \cap y/\mathcal{R} = \phi$. Then, $x \in x/\mathcal{R}$. Thus, $x \notin y/\mathcal{R}$ and hence $x \not\mathcal{R} y$.

Definition 3.2.3

Let $m \neq 0$ be a fixed integer. Then "$\equiv_m$" denotes the relation on $\mathbb{Z}$ and is defined by

$$\left(x \equiv y \mod m \text{ or } x \equiv_m y\right) \Leftrightarrow m \mid x - y,$$

which reads "x is congruent to y modulo m". That is $\overline{x} = \{y \in \mathbb{Z} : x \equiv_m y \Leftrightarrow m \mid x - y\}$, and the set of equivalence classes for $\equiv_m$ is $\mathbb{Z}$ mod m (denoted $\mathbb{Z}_m$) and is defined by

$$\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{m-1}\}.$$

Example 3.2.6

Find all the equivalence classes of $\mathbb{Z}_3$.

Solution:

Note that $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, where $\overline{x} = \{y \in \mathbb{Z} : x \equiv y \mod 3 \text{ or } 3 \mid x - y\}$. Therefore,

- $\overline{0} = 0/\equiv_3 = \{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\}$,
- $\overline{1} = 1/\equiv_3 = \{\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots\}$,
- $\overline{2} = 2/\equiv_3 = \{\cdots, -7, -4, -1, 2, 5, 8, 11, \cdots\}$,

Therefore, $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$.

Theorem 3.2.2

Let $m \neq 0$ be a fixed integer. The relation $\equiv_m$ is an equivalence relation on $\mathbb{Z}$. Moreover, $\mathbb{Z}_m$ has m distinct elements: $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \cdots, \overline{m-1}\}$.

Proof:

We only show that $\equiv_m$ is an equivalence relation. reflexive: Since $x - x = 0$ which is divisible by m, $x \equiv_m x$. Thus $\equiv_m$ is reflexive.

symmetric: Assume that $x \equiv_m y$, then $m \mid x - y$ which implies that $m \mid y - x$. Thus, $y \equiv_m x$ and $\equiv_m$ is symmetric.

transitive: Assume that $x \equiv_m y$ and $y \equiv_m z$, then $m \mid x - y$ and $m \mid y - z$. Thus, $m \mid (x - y) + (y - z)$ which implies $m \mid x - z$. Therefore, $x \equiv_m z$ and $\equiv_m$ is transitive. That shows that $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.

Exercise 3.2.1

Let $m \neq 0$. For $x, y \in \mathbb{Z}$: Show that $x \equiv_m y$ if and only if $\overline{x} = \overline{y}$.

Exercise 3.2.2

Let $\mathcal{R}$ be a relation on the set A. Prove that $\mathcal{R} \cup \mathcal{R}^{-1}$ is symmetric.

Exercise 3.2.3

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $3 \mid x + y$. Determine whether $\mathcal{R}$ an equivalence relation. Explain.

Exercise 3.2.4

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $3 \mid x + 2y$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$. Find the equivalence class of 1.

Exercise 3.2.5

Let $\mathcal{R}$ be a relation on $\mathbb{R}$ so that $x \, \mathcal{R} \, y$ iff $x = y$ or $xy = 1$. Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{R}$. Find the equivalence classes for 2; 0; and $-\frac{1}{5}$.

Section 3.3: Partitions

Definition 3.3.1

Let A be a set and $\mathcal{A}$ be a family of subsets of A. $\mathcal{A}$ is called a **partition** of A if and only if:

1. if $X \in \mathcal{A}$, then $X \neq \phi$.
2. if $X, Y \in \mathcal{A}$, then either $X = Y$ or $X \cap Y = \phi$.
3. $\bigcup_{X \in \mathcal{A}} X = A$.

Example 3.3.1

1. The set of even natural numbers and odd natural numbers is a partition of $\mathbb{N}$.
2. Let $A_0 = \{0\}$ and $A_i = \{-i, i\}$ for all $i \in \mathbb{N}$. Then $\mathcal{A} = \{A_0, A_1, A_2, A_3, \cdots\}$ is a partition of $\mathbb{Z}$.
3. The set $\{0/\equiv_3, 1/\equiv_3, 2/\equiv_3\}$ is a partition of $\mathbb{Z}$.
4. The set $\{\{$ male students, female students $\}\}$ is a partition for the set of all students in Kuwait University.
5. The collection $\{\, B_i : i \in \mathbb{Z} \,\}$, where $B_i = [i, i+1)$ is a partition of $\mathbb{R}$.

Theorem 3.3.1

Let $A \neq \phi$ and let $\mathcal{R}$ be an equivalence relation on A. Then, the family $A/\mathcal{R} = \{x/\mathcal{R} : x \in A\}$ is a partition of A.

Proof:

Do it your self!

Section 3.4: Ordering Relations

Definition 3.4.1

A relation $\mathcal{R}$ on a set A is called **antisymmetric** if for all $x, y \in A$, if $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x = y$.

Definition 3.4.2

A relation $\mathcal{R}$ on a set A is called a **partial order** (or **partial ordering**) for A if $\mathcal{R}$ is reflexive, antisymmetric, and transitive. In that case, A is called a **partially ordered set** or a **poset**.

Example 3.4.1

Show that "$\subseteq$" is a partial order relation on $\mathcal{P}(A)$ for any set A.

Solution:

reflexive: if $X \in \mathcal{P}(A)$, then $X \subseteq A$ and hence $X \subseteq X$ and hence $x\mathcal{R}x$.

antisymmetric: Let $X, Y \in \mathcal{P}(A)$ with $X\mathcal{R}Y$ and $Y\mathcal{R}X$. Then, $X \subseteq Y$ and $Y \subseteq X$. Therefore, $X = Y$ and $\mathcal{R}$ is antisymmetric.

transitive: Assume that $X, Y, Z \in \mathcal{P}(A)$ with $X \subseteq Y$ and $Y \subseteq Z$. Then $X \subseteq Z$ and hence $X\mathcal{R}Z$.

Therefore, $\mathcal{R}$ is a partial order relation on $\mathcal{P}(A)$.

Example 3.4.2

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $a\mathcal{R}b \Leftrightarrow a \mid b$ for all $a, b \in \mathbb{N}$. Show that $\mathcal{R}$ is a partial order on $\mathbb{N}$.

Solution:

reflexive: Since $a = 1 \cdot a$ for all $a \in \mathbb{N}$, then $a \mid a$ and $a\mathcal{R}a$. Hence, $\mathcal{R}$ is reflexive.

antisymmetric: Assume that $a \mid b$ and $b \mid a$. Then, there are $h, k \in \mathbb{N}$ such that $b = ha$ and $a = kb$. Thus, $b = ha = h(kb) = (hk)b$. Then, $hk = 1$ which implies that $h = k = 1$. Therefore, $a = b$ and $\mathcal{R}$ is antisymmetric.

transitive: Assume that $a \mid b$ and $b \mid c$. Then, Theorem 1.4.1 implies that $a \mid c$. Thus, $a\mathcal{R}c$

and $\mathcal{R}$ is transitive. Therefore, $\mathcal{R}$ is a partial order on $\mathbb{N}$.

Example 3.4.3

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $a\mathcal{R}b$ iff $2 \mid a+b$ with $a \leq b$ for all $a, b \in \mathbb{N}$. Show that $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.

Solution:

reflexive: Since $2 \mid a+a = 2a$ with $a \leq a$, $a\mathcal{R}a$ and $\mathcal{R}$ is reflexive.

antisymmetric: Assume that $a\mathcal{R}b$ and $b\mathcal{R}a$. Then, $2 \mid a+b$ with $a \leq b$ and $2 \mid b+a$ with $b \leq a$. Thus, $a \leq b \leq a$ which implies that $a = b$. Thus, $\mathcal{R}$ is antisymmetric.

transitive: Assume that $a\mathcal{R}b$ and $b\mathcal{R}c$. Then, $2 \mid a+b$ with $a \leq b$ and $2 \mid b+c$ with $b \leq c$. Therefore, by Theorem 1.4.1, $2 \mid a+2b+c$ which implies that $2 \mid a+c$ with $a \leq b \leq c$. Thus, $a\mathcal{R}c$ and $\mathcal{R}$ is transitive. Therefore, $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.

3.4.1 Upper and Lower Bounds

Definition 3.4.3

Let $\mathcal{R}$ be a partial order for A and let B be any subset of A. Then,

- $a \in A$ is an **upper bound** for B if for every $b \in B$, $b\mathcal{R}a$. Also, a is called a "**least upper bound**" or "**supremum** for B, denoted by $\sup(B)$, if:
  1. a is an upper bound for B, and
  2. $a\mathcal{R}x$ for every upper bound x for B.
- $a \in A$ is a **lower bound** for B if for every $b \in B$, $a\mathcal{R}b$. Also, a is called a "**greatest upper bound**" or "**infimum** for B, denoted by $\inf(B)$, if:
  1. a is a lower bound for B, and
  2. $x\mathcal{R}a$ for every lower bound x for B.

Theorem 3.4.1

If $\mathcal{R}$ is a partial order for a set A and $B \subseteq A$, then if the least upper bound (or greatest lower bound) for B exists, then it is unique.

Proof:

Assume that x and y are both least upper bound for B. Since x is an upper bound and y is the least upper bound, thus $y\mathcal{R}x$. Similarly, since y is an upper bound and x is the least upper bound, thus $x\mathcal{R}y$. Since $\mathcal{R}$ is antisymmetric, $x\mathcal{R}y$ and $y\mathcal{R}x$, implies $x = y$.

Example 3.4.4

Let $A = [0, 6) \subset \mathbb{R}$ be a poset with respect to "$\leq$", and let $B = \{\frac{1}{2}, 3, 5\}$ and $C = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ be two subsets of A. Find $\sup(B)$, $\inf(B)$, $\sup(C)$, and $\inf(C)$.

Solution:

$\underline{\sup(B)}$: Note that $5, 5.1, 5.35, 5.9$, and so on are all considered upper bounds for B since for example $b \leq 5$ for all $b \in B$. Then, $\sup(B) = 5$ since $5 \leq x$ for all upper bounds for B.

$\underline{\inf(B)}$: $0, \frac{1}{2}, \frac{1}{4}, \frac{1}{45}$ and so on are all considered lower bounds for B since for example $\frac{1}{4} \leq b$ for all $b \in B$. Then, $\inf(B) = \frac{1}{2}$ since $\frac{1}{2} \leq x$ for all lower bounds x for B.

$\underline{\sup(C)}$: The set of upper bounds for C consists of $\{1, 2, 1.5, 3, 5, 5.5, \cdots\}$ while the $\sup(C) = 1$.

$\underline{\inf(C)}$: The set of upper bounds for C consists of $\{0\}$ and the $\inf(C) = 0$.

Note that, if $A = (0, 6)$, then C would has no $\inf(C)$.

Example 3.4.5

Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider $\mathcal{P}(A)$ with the partial ordering "$\subseteq$". Let $B = \big\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 6\}\big\}$. Find $\sup(B)$ and $\inf(B)$.

Solution:

Upper bound for B are like $\{1, 2, 3, 6\}$, $\{1, 2, 3, 4, 6\}$, $\{1, 2, 3, 5, 6\}$, and A it self. Therefore, $\sup(B) = \{1, 2, 3, 6\} = \bigcup_{X \in B} X$. On the other hand, ϕ, $\{1\}$, $\{2\}$, and $\{1, 2\}$ are all lower bounds for B while the $\inf(B) = \{1, 2\} = \bigcap_{X \in B} X$.

Exercise 3.4.1

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $y = 2^k x$ for some integer $k \geq 0$. Show that $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.

Chapter

4 Functions

Section 4.1: Functions as Relations

Definition 4.1.1

A **function** f from A to B is a relation from A to B that satisfies

1. $\text{Dom}(f) = A$,
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Moreover, if $A = B$, we say that f is a function on A.

Remark 4.1.1: Notations

A function (mapping) f from A to B is denoted by $f : A \to B$. The **domain** of f is A and the **codomain** of f is B.

If $(x, y) \in f$, then $y = f(x)$ where we say that y is the **image** of x and that x is the **preimage** of y. The **range** of f is a subset of B and is defined as

$$\text{Rng}(f) = \{y \in B : \exists x \in A \text{ with } y = f(x)\}.$$

Example 4.1.1

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let $\mathcal{R}_1 = \{(1, a), (2, b), (2, c), (3, c)\}$, $\mathcal{R}_2 = \{(1, a), (2, c), (3, b)\}$, and $\mathcal{R}_3 = \{(1, a), (2, c)\}$ be three relations on $A \times B$. Decide whether $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ a function.

Solution:

$\mathcal{R}_1$ is clearly not a function since $(2, b)$ and $(2, c)$ both are in $\mathcal{R}_1$ where $b \neq c$. $\mathcal{R}_2$ satisfies the conditions of Definition 4.1.1 and so it is a function from A to B.

$\mathcal{R}_3$ is not a function from A to B; however, it is a function from $\{1, 2\}$ to $\{a, c\}$.

Example 4.1.2

Let $\mathcal{S} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ be a relation on $\mathbb{R}$. Is $\mathcal{S}$ a function? Explain.

Solution:

Note that for $x = 0$, we have $y = -1$ or $y = 1$ and so $\mathcal{S}$ is not a function. Another reason is that for $x = 5$, $y^2 = -24 \notin \mathbb{R}$.

Example 4.1.3

Let $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = x^2\}$. Determine whether f a function on $\mathbb{Z}$.

Solution:

$f : \mathbb{Z} \to \mathbb{Z}$ is a function with $\operatorname{Rng}(f) = \{0, 1, 4, 9, 16, \cdots\}$. That is $f(x) = x^2$ is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$\star$ **Constant Function**: $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = c$ (c is a constant) for all $x \in \mathbb{R}$.

Example 4.1.4

Let $f = \{\,(\,x, y\,) \in \mathbb{R} \times \mathbb{R} : y = 2x + 5\,\}$. Show that f is a function from $\mathbb{R}$ to $\mathbb{R}$.

Solution:

We first show that $\operatorname{Dom}(f) = \mathbb{R}$. Clearly, $\operatorname{Dom}(f) \subseteq \mathbb{R}$ by the definition of f. Next, let $x \in \mathbb{R}$. Then there is $y = 2x + 5 \in \mathbb{R}$ and hence $(x, y) \in f$. That is $x \in \operatorname{Dom}(f)$.
Now assume that $(x, y), (x, z) \in f$, we want to show that $y = z$. But since $y = 2x + 5$ and $z = 2x + 5$, we have $y = z$. Therefore, f is a function from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 4.1.1

Two functions f and g are equal iff (i) $\operatorname{Dom}(f) = \operatorname{Dom}(g)$, and (ii) for all $x \in \operatorname{Dom}(f)$, $f(x) = g(x)$.

Proof:

„ $\Rightarrow$ ": Assume that $f = g$. Proof of (i): If $x \in \operatorname{Dom}(f)$, then $(x, y) \in f = g$ for some y and hence $x \in \operatorname{Dom}(g)$. Thus, $\operatorname{Dom}(f) \subseteq \operatorname{Dom}(g)$. Similarly, if $x \in \operatorname{Dom}(g)$, then $(x, y) \in g = f$

for some y and hence $x \in \mathrm{Dom}(f)$. Thus, $\mathrm{Dom}(g) \subseteq \mathrm{Dom}(f)$. Therefore, $\mathrm{Dom}(f) = \mathrm{Dom}(g)$.
Proof of (ii): Let $x \in \mathrm{Dom}(f)$. Then for some y, $(x, y) \in f = g$. Thus, $f(x) = y = g(x)$.
„ $\Leftarrow$ ": Assume that $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ and that for all $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$. Suppose that $(x, y) \in f$, then there is y such that $y = f(x)$ and $x \in \mathrm{Dom}(f) = \mathrm{Dom}(g)$. Thus, $y = f(x) = g(x)$ which implies that $(x, y) \in g$ and hence $f \subseteq g$. Now suppose that $(x, y) \in g$. Then there is y such that $y = g(x) = f(x)$ for $x \in \mathrm{Dom}(f)$. Thus, $y = f(x)$ and $(x, y) \in f$. Hence $g \subseteq f$. Therefore, $f = g$.

Section 4.2: Constructions of Functions

Definition 4.2.1

Let $f : A \to B$ and $g : B \to C$ be two given functions. The **composition function** $g \circ f$ is defined by $g \circ f : A \to C$ where $(g \circ f)(x) = g(f(x))$ for every $x \in A$. Note that $f \circ g \neq g \circ f$, while $(f \circ g) \circ h = f \circ (g \circ h)$ for any three (appropriate) functions f, g, and h.

Example 4.2.1

Let $f(x) = \sin(x)$ and $g(x) = 2x + 1$ for $x \in \mathbb{R}$. Find $f \circ g$ and $g \circ f$.

Solution:

For any $x \in \mathbb{R}$, we have

1. $(f \circ g)(x) = f(g(x)) = f(2x + 1) = \sin(2x + 1)$.
2. $(g \circ f)(x) = g(f(x)) = g(\sin(x)) = 2\sin(x) + 1$.

Definition 4.2.2

Let $f : A \to B$ and let $D \subseteq A$. The "**restriction of f to D**", denoted by $f|_D$, is a function with domain D and is defined as

$$f|_D = \{(x, y) : (x, y) \in f \text{ and } x \in D\}.$$

In that case, we say that f is an **extension** of $f|_D$.

Example 4.2.2

Let $f : A \to B$ be a function where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $f = \{(1, a), (2, a), (3, b), (4, c)\}$. Find $f|_A$, $f|_{\{1\}}$, and $f|_{\{2,4\}}$.

Solution:

Clearly, $f|_A = f$, $f|_{\{1\}} = \{(1, a)\}$, and $f|_{\{2,4\}} = \{(2, a), (4, c)\}$.

Remark 4.2.1

Let $f : A \to B$ and $g : C \to D$ be two functions. Then,

1. $f \cap g$ is a function with $\text{Dom}(f \cap g) = \{x \in A \cap C : f(x) = y = g(x) \in B \cap D\}$.
2. If $A \cap C = \phi$, then $f \cup g$ is a function with domain $A \cup B$.

Example 4.2.3

Let $f = \{(1, 2), (3, 5), (4, 2)\}$ and $g = \{(1, 2), (3, 6), (5, -10)\}$. Find $f \cap g$ and $f \cup g$ and decide whether either of those relation is a function.

Solution:

Clearly, f is a function from $A = \{1, 3, 4\}$ to $B = \{2, 5\}$ while g is a function from $C = \{1, 3, 5\}$ to $D = \{2, 6, -10\}$. So,

- $f \cap g = \{(1, 2)\}$ which is clearly a function from $\text{Dom}(f \cap g) = \{1\}$ to $\{2\}$.
- $f \cup g = \{(1, 2), (3, 5), (4, 2), (3, 6), (5, -10)\}$ which is not a function (by the definition) since 3 maps to two different values, namely 5 and 6.

Section 4.3: Functions That are Onto; One-to-One Functions

Definition 4.3.1

A function $f : A \to B$ is **onto (surjective mapping)** B iff $\text{Rng}(f) = B$. Also, f is called a **surjection**. In that case, we write $f : A \xrightarrow{onto} B$.

Remark 4.3.1

Since $\text{Rng}(f) \subseteq B$ is always true, f is a surjection iff $B \subseteq \text{Rng}(f)$. Thus,

$$f : A \xrightarrow{onto} B \iff (\forall b \in B)(\exists a \in A)(f(a) = b).$$

Example 4.3.1

Let $f(x) = x + 2$ and $g(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Determine whether f and g are onto $\mathbb{R}$.

Solution:

- f is onto: Let $y \in \mathbb{R}$ (in the range of f), then there exists $x \in \mathbb{R}$ such that $y = x + 2$ or $x = y - 2$. Thus, $f(x) = f(y - 2) = (y - 2) + 2 = y$. Thus, f is onto $\mathbb{R}$.
- g is not onto: Let $y \in \mathbb{R}$, then $y = x^2 + 1$ so $x = \pm\sqrt{y - 1}$. So, y must be greater than or equal to 1. If we choose $y = 0$, then $x \notin \mathbb{R}$ and hence g is not onto $\mathbb{R}$.

Example 4.3.2

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by $f(m, n) = 2^{m-1}(2n - 1)$. Show that f is onto $\mathbb{N}$.

Solution:

We show that $\mathbb{N} \subseteq \text{Rng}(f)$. That is, for all $s \in \mathbb{N}$, there exists $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $f(m, n) = s$. We consider the following two cases of s.

(*i*) if s is even: s can be written as $2^k t$, where $k \geq 1$ and t is odd. Since t is odd, $t = 2n - 1$ or $n = \frac{t+1}{2}$ for some $n \in \mathbb{N}$. Choosing $m = k + 1$, we have

$$f(m, n) = 2^{m-1}(2n - 1) = 2^k t = s.$$

Thus, $\mathbb{N} \subseteq \text{Rng}(f)$.

(ii) if s is odd: $s = 2n-1$ for some $n \in \mathbb{N}$. Choosing $m = 1$, we have $f(m,n) = 2^0(2n-1) = s$. Thus, $\mathbb{N} \subseteq \text{Rng}(f)$.

Therefore, f is onto $\mathbb{N}$.

Theorem 4.3.1

Let A, B, and C be three sets. Then,

1. If $f : A \xrightarrow{onto} B$ and $g : B \xrightarrow{onto} C$, then $g \circ f : A \xrightarrow{onto} C$. That is, the composite of surjective functions is a surjection.

2. If $f : A \to B$, $g : B \to C$, and $g \circ f : A \xrightarrow{onto} C$, then g is onto C.

Proof:

1. We show that for every $c \in C$, $c \in \text{Rng}(g \circ f)$. Since g is onto C, there exists $b \in B$ such that $g(b) = c$. but since f is onto B, there exists $a \in A$ such that $f(a) = b$. Thus, $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus, $c \in \text{Rng}(g \circ f)$.

2. We show that again $C \subseteq \text{Rng}(g \circ f)$. Let $c \in C$. Since $g \circ f$ is onto C, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a) \in B$. Then, $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus, there exists $b \in B$ such that $g(b) = c$ and hence g is onto.

Definition 4.3.2

A function $f : A \to B$ is said to be "**one-to-one**" (**injective mapping**) iff $(a_1, b) \in f$ and $(a_2, b) \in f$ imply that $a_1 = a_2$. Also, f is called an **injection**. In that case, we write $f : A \xrightarrow{1-1} B$.

Remark 4.3.2

A function $f : A \xrightarrow{1-1} B$ is one-to-one if and only if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad \text{or equivalently} \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2).$$

Example 4.3.3

Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x - 1$. Show that f is one-to-one.

Solution:

Assume that $f(a) = f(b)$, then $5a - 1 = 5b - 1 \Rightarrow 5a = 5b \Rightarrow a = b$. Thus, f is 1-1.

Example 4.3.4

Determine whether $f : \mathbb{R} \to \mathbb{R}$ is one-to-one, where $f(x) = \dfrac{1}{x^2 + 1}$.

Solution:

Assume that $f(a) = f(b)$, then

$$\frac{1}{a^2 + 1} = \frac{1}{b^2 + 1} \Rightarrow a^2 + 1 = b^2 + 1 \Rightarrow a^2 = b^2 \Rightarrow a = \pm b.$$

Therefore, f is not 1-1. For instance, $f(1) = f(-1)$ while $1 \neq -1$.

Example 4.3.5

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^{m-1}(2n - 1)$. Show that f is one-to-one.

Solution:

Assume that $f(a, b) = f(x, y)$ for $(a, b), (x, y) \in \mathbb{N} \times \mathbb{N}$. Then, $2^{a-1}(2b - 1) = 2^{x-1}(2y - 1)$. Consider the following cases:

1. if $a > x$: $2^{a-1}(2b - 1) = 2^{x-1}(2y - 1) \Rightarrow \underbrace{2^{a-x}(2b - 1)}_{\text{even}} = \underbrace{(2y - 1)}_{\text{odd}}$ which is impossible.
2. if $a < x$: $2^{a-1}(2b - 1) = 2^{x-1}(2y - 1) \Rightarrow \underbrace{(2b - 1)}_{\text{odd}} = \underbrace{2^{x-a}(2y - 1)}_{\text{even}}$ which is impossible.
3. if $a = x$: $2^{a-1}(2b - 1) = 2^{x-1}(2y - 1) \Rightarrow (2b - 1) = (2y - 1) \Rightarrow b = y$.

Thus, the only possible case is the third case which suggests that $(a, b) = (x, y)$. Therefore, f is 1-1.

Theorem 4.3.2

Let A, B, and C be three sets. Then,

1. If $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} C$, then $g \circ f : A \xrightarrow{1-1} C$.
2. If $f : A \to B$ and $g : B \to C$, and $g \circ f : A \xrightarrow{1-1} C$, then $f : A \xrightarrow{1-1} B$.

Proof:

1. Assume that $(g \circ f)(x) = (g \circ f)(y)$ for some $x, y \in A$. Then, $g(f(x)) = g(f(y))$. Since, g is 1-1, $f(x) = f(y)$, and since f is 1-1 as well, $x = y$. Therefore, $g \circ f$ is 1-1.
2. Assume that $f(x) = f(y)$ for $x, y \in A$. Then $g(f(x)) = g(f(y))$ implies that $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is 1-1, $x = y$. Thus, f is 1-1.

Remark 4.3.3

HORIZONTAL LINE TEST: Let $f : A \to B$ be a given function. Then,

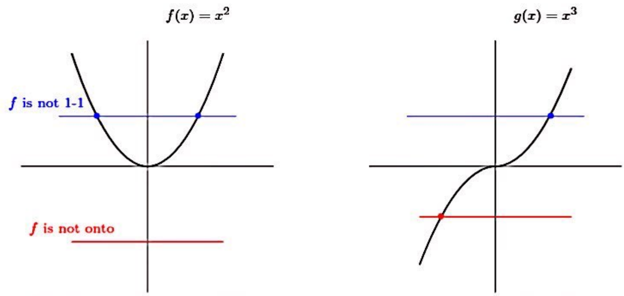
1. f is onto B iff for all $b \in B$, the horizontal line $y = b$ intersects the graph of f at least once.
2. f is one-to-one iff for all $b \in B$, the horizontal line $y = b$ intersects the graph of f at most once.

Example 4.3.6

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two function. Use the Horizontal line test to decide whether $f(x) = x^2$ and $g(x) = x^3$ are onto, one-to-one, or neither.

Solution:

We apply the horizontal line test on both f and g. In f, we see that on one place the line crosses the curve in two points, so f is not one-to-one, and it does not cross the curve in another place so it is not onto. However, in g, the line crosses the curve exactly once in any place, so it is one-to-one and onto.



Definition 4.3.3

Let $f : A \to B$ be a function. If the **inverse relation** f^{-1} of f is a function, then we say that f^{-1} is the **inverse function** of f. In particular, if f^{-1} is a function, then $f^{-1} : B \to A$ is defined by

$$f^{-1} = \{(y, x) : (x, y) \in f\}.$$

Example 4.3.7

Let $f = \{(1, 2), (4, 2)\}$ be a function. Decide whether f^{-1} is a function.

Solution:

No. Since $f^{-1} = \{(2, 1), (2, 4)\}$ where 2 is mapped to two distinct elements.

Theorem 4.3.3

Let $f : A \to B$ and $g : B \to A$. Then, $g = f^{-1}$ iff $f \circ g = I_B$ and $g \circ f = I_A$, where $I_A : A \to A$ is the **identity function** defined by $I_A(x) = x$ for all $x \in A$.

Example 4.3.8

Let $f(x) = 2x + 1$ and let $g(x) = \dfrac{x-1}{2}$. Show that $g = f^{-1}$.

Solution:

For all $x \in \mathbb{R}$, $(f \circ g)(x) = f(g(x)) = f(\frac{x-1}{2}) = 2\frac{x-1}{2} + 1 = x - 1 + 1 = x = I_{\mathbb{R}}$. Therefore, $g = f^{-1}$.

Theorem 4.3.4

Let $f : A \to B$ be a function. Then,

1. f^{-1} is a function from $\mathrm{Rng}(f)$ to A iff f is one-to-one.
2. If f^{-1} is a function, then f^{-1} is one-to-one.

Proof:

1. " $\Rightarrow$ ": Assume that f^{-1} is a function. Let $f(x) = f(y) = z$, then $(x, z), (y, z) \in f$. Thus, $(z, x), (z, y) \in f^{-1}$. Since f^{-1} is a function, $x = y$. Therefore, f is 1-1.
   " $\Leftarrow$ ": Assume that f is 1-1. Let $(x, y), (x, z) \in f^{-1}$ (we need to show that $y = z$). Then, $(y, x), (z, x) \in f$. Since f is 1-1, $y = z$. Thus, f^{-1} is a function. By Definition 3.1.6, $\mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f)$ and $\mathrm{Rng}(f^{-1}) = \mathrm{Dom}(f)$.
2. Assume that f^{-1} is a function. Let $f^{-1}(x) = f^{-1}(y) = z$, then $(x, z), (y, z) \in f^{-1}$. Thus, $(z, x), (z, y) \in f$ and since f is a function, $x = y$. Therefore, f^{-1} is 1-1.

Definition 4.3.4

A function $f : A \to B$ is called a **1-1 corresponding** or a **bijection** if it is both 1-1 and onto B. In that case, we write $f : A \xrightarrow[onto]{1-1} B$.

Theorem 4.3.5

Let $f : A \xrightarrow[onto]{1-1} B$ and $g : B \xrightarrow[onto]{1-1} C$. Then,

1. $g \circ f : A \xrightarrow[onto]{1-1} C$ is a bijection.
2. $f^{-1} : B \xrightarrow[onto]{1-1} A$ is a bijection.

Proof:

1. By Theorem 4.3.1 and Theorem 4.3.2, if f and g are one-to-one and onto, the composite function $g \circ f$ is also one-to-one and onto.
2. By Theorem 4.3.4, if f is one-to-one, then f^{-1} is a function and hence it is a one-to-one

function. To show that f^{-1} is onto A, let $a \in A$. Then, $f(a) = b \in B$. Thus, $(a, b) \in f$ and hence $(b, a) \in f^{-1}$ and therefore $f^{-1}(b) = a$.

Section 4.4: Images of Sets

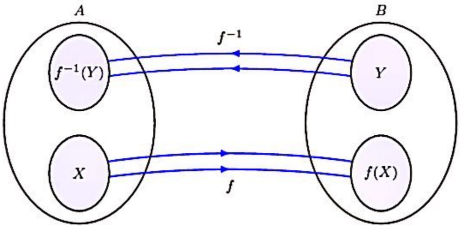
Definition 4.4.1

Let $f : A \to B$. If $X \subseteq A$, the **image of** X or image set of X is

$$f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\}.$$

If $Y \subseteq B$, then the **inverse image of** Y is

$$f^{-1}(Y) = \{x \in A : f(x) = y \text{ for some } y \in Y\}.$$

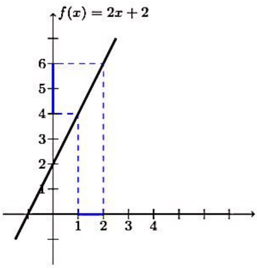


Example 4.4.1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 2$. Find $f(\{1,4\})$, $f([1,2])$, $f(\mathbb{N})$, $f^{-1}(\{2,3\})$, and $f^{-1}([2,4])$.

Solution:

- $f(\{1,4\}) = \{4, 10\}$.
- $f([1,2]) = [4,6]$.
- $f(\mathbb{N}) = \{4, 6, 8, 10, 12, \cdots\}$.
- $f^{-1}(\{2,3\}) = \{0, \frac{1}{2}\}$.
- $f^{-1}([2,4]) = [0,1]$.



$f([1,2])$

References

Abdullah Al-Azemi.Foundations of Mathematics .2019.Mathematics Department .Kuwait University