# الكورس الثاني-حلقات

## المحاضرة ١

**Example:** (0) is a Maximal ideal of R.

**Example:** $((0), +_{12}, ._{12})$ is a maximal ideal of the ring $(Z_{17}, +_{17}, ._{17})$.

Q: find all Maximal ideal of $Z_{12}$

Solution: above its (**Pe.4**)

*Remark*: In general Z, we have (p) Maximal ideal in Z where p is a prime number.

المثاليات العظمى في حلقة $Z_n$ هي المثاليات التي تتولد بعوامل $n$ الأولي ة

Theorem: In the ring $(Z, +, .)$, $((p), +, .)$ is a maximal ideal $\Leftrightarrow$ p is a prime number.

Proof:

$\Rightarrow$ suppose that (P) is a maximal

ideal of $(Z, +, .)$ T.P: P is a

prime number??

Let P is not prime number

$\therefore P = k_1 . k_2$ ; $1 < k_1$

$< P$ , $1 < k_2 < P$ $\because$

$k_1 < P \Longrightarrow (P) \subset$

$(k_1) \rightarrow C!$

$\because k_2 < P \Longrightarrow (P) \subset (k_1) \rightarrow C!$

∴ P is a prime number

⟸ We have P

is a prime

number. T.P:

(P) is a

Maximal ideal?

Since P is a prime number, then all the ideal are $(2)$ , $(3)$ ,$(5)$,$(7)$ , $(11)$ , ...

$$0 \subset (4)$$

$$\subset (2) \rightarrow$$

$$Maxima$$

$$l\ 0 \subset$$

$$(6) \subset$$

$$(3) \rightarrow$$

$$Maxima$$

$$l$$

$$0 \subset (10) \subset (5) \rightarrow Maximal$$

Theorem: Let R be a comm. Ring with 1. Every proper ideal contains in maximal ideal.

Proof:

Let I ideal of $R \ni I \neq R$

$A = \{J : I \subseteq J \; and \; J \; ideale \; of \; R\}$

$A \neq \emptyset$ Choose $I \subset J_1 \subset J_2 \subset \cdots$

∴ By (Zorn’s Lemma); J is maximal

# المحاضرة ٢

Theorem: Let $(I, +, .)$ be only Maximal ideal of R. then there only 1 , 0 are idempotent elements

Proof:

Let $x \in R \ni x \neq 1, \neq 0$

If $x^2 = x \Longrightarrow x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \;\; or \;\; x = 1 \rightarrow C!$

OR

Let x any ideal

And x-1 another ideal

$x \notin$ Maximal

$(x) \subset I$ and $(x-1) \subset I$

From definition first conditional of ideal

$x - x + 1 = 1 \in I \rightarrow I = R \rightarrow C!$

***LUC (5)***

Theorem: If $(I, +, .)$ Is an ideal of comm. Ring with 1 $(R, +, .)$, then $I$ is a Maximal ideal $\Leftrightarrow$ $(\frac{R}{I}, +, .)$ Is a field.

Proof:

$\Rightarrow$ Let I be a

Maximal ideal

of R T.P: $(\frac{R}{I},$

$+ , .)$ Is a

field??

T.P: $\frac{R}{I}$ comm. rang with 1 and every element non zero has inverse

Since: R is comm. $\Longrightarrow \frac{R}{I}$ is comm.

R has 1 $\Longrightarrow \frac{R}{I}$ has $1 + I$

Let $I = 0 + I \neq a + I \in \frac{R}{I}$

$a + I \neq \emptyset \Rightarrow a \notin I$

$\because$ I Maximal $\Rightarrow (a) + I \in R$

$1 \in R \Longrightarrow 1 \in (a) + I$

$\Longrightarrow 1 = r(a) + i \, ; i \in I \, , r \in R$

$\Longrightarrow 1 = 1 - ra$

$\because i \in I \Longrightarrow 1 - ra \in I \Longrightarrow 1 + I \in ra + I =$

$(r + I).(a + I) \Longrightarrow (a + I)^{-1} = r + I \Longrightarrow \frac{R}{I}$

field.

$\Longleftarrow$ Let $(\frac{R}{I}, + , .)$ Be a field

T.P: I is a Maximal ideal of R??

Suppose that J is an ideal

of R $\ni I \subset J \subseteq R$ T.P: $J =$

$R$??

$\because I \subset J \Longrightarrow \exists\, a \in J\, ,\, a \notin I$

**المحاضرة ٣**

$\Longrightarrow a + I \neq I \Longrightarrow a + I \neq 0 + I$

$\because \frac{R}{I}$ field

$\exists\, b + I \in \frac{R}{I} \ni (a + I)(b + I) = 1 + I$

$\Longrightarrow 1 - a.b \in J \subset I \subset J$

$\because a \in J\, ,\, b \in R \Longrightarrow a.b \in J$

$\Longrightarrow 1 \in J \Longrightarrow J = R$ why??

$\therefore$ I is a maximal of R

اذا كان محايد الحلقة موجود داخل مثالي في الحلقة فهذا يعني ان المثالي يساوي الحلقة.

# Prime ideal

**المثاليات الاولية**

**Def:** Let R be a ring (commutative with 1). An ideal I is called prime if:-

$$a.b \in I \to either\ a \in I\ or\ b \in I\ ;\ \forall a, b \in R$$

متى يقال للمثالي انه مثالي اولي ؟

١. كل عنصران ينميان الى الحلقة مثل $a, b$
٢. حاصل ضرب العددين يوجد في المثالي
٣. اما احدهما ينتمي الى المثالي او كلاهم ا

Ex: Let $(Z, +, .)$ be a ring. Then (P) is prime ideal such that P is a prime number.

$(5) = \{ \dots , -10 , -5 , 0 , 5 , 10 , \dots \}$

Is a prime ideal

$(3) = \{ \dots , -9 , -6 , -3 , 0 , 3 , 6 , 9 , \dots \}$

Is a prime ideal

Remark: $(Z, +, .)$ is a prime ideal

$(\{0\}, + , .)$ is a prime ideal **In general**

Ex: $(Z_{14} , +_{14} , ._{14})$ is a ring but $(\{0\}, +_{14} , ._{14})$ is not prime ideal because

$2,7 \in Z_{14}$

$2 ._{14} 7 = 0$ .But $2 \notin$

$\{0\}$ and $7 \notin \{0\}$

Now:

First find all ideals

$2(7) = 14$

$I_2 = (2) = \{0 , 2 , 4 , 6 , 8 , 10 , 12 \}$

$I_2 = (7) = \{0 , 7 \}$

2 | 14
7 | 7
  | 1

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 2 | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| 3 | 0 | 3 | 6 | 9 | 12 | 1 | 4 | 7 | 10 | 13 | 2 | 5 | 8 | 11 |
| 4 | 0 | 4 | 8 | 12 | 2 | 6 | 10 | 0 | 4 | 8 | 12 | 2 | 6 | 10 |
| 5 | 0 | 5 | 10 | 1 | 6 | 11 | 2 | 7 | 12 | 3 | 8 | 13 | 4 | 9 |
| 6 | 0 | 6 | 12 | 4 | 10 | 2 | 8 | 0 | 6 | 12 | 4 | 10 | 2 | 8 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0 | 7 | 0 | 7 | 0 | 7 | 0 | 7 | 0 | 7 | 0 | 7 | 0 | 7 |
| 8 | 0 | 8 | 2 | 10 | 4 | 12 | 6 | 0 | 8 | 2 | 10 | 4 | 12 | 6 |
| 9 | 0 | 9 | 4 | 13 | 8 | 3 | 12 | 7 | 2 | 11 | 6 | 1 | 10 | 5 |
| 10 | 0 | 10 | 6 | 2 | 12 | 8 | 4 | 0 | 10 | 6 | 2 | 12 | 8 | 4 |
| 11 | 0 | 11 | 8 | 5 | 2 | 13 | 10 | 7 | 4 | 1 | 12 | 9 | 6 | 3 |
| 12 | 0 | 12 | 10 | 8 | 6 | 4 | 2 | 0 | 12 | 10 | 8 | 6 | 4 | 2 |
| 13 | 0 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

$1 ._{14} 3 = 3 \notin (2)$

Then there is not prime ideal in $Z_{14}$

Q: Let R be a ring (comm. with1). Then R is an integral domain ⇔ {0} is prime Solution:

⇒ Let R be an integral domain.

$\therefore a. b =$

$0 \to a =$

$0 \; or \; b =$

$0 \;\; \therefore \{0\}$

prime.

Why?

⇐ Let {0} is prime

$\therefore a. b = 0 \to a. b \in \{0\} \to a \in \{0\} \; or \; b \in \{0\} \quad \forall a, b \in R$

$a=0$ $b=0$

∴ R integral domain.

# المحاضرة ٤

Theorem: Let $(I, +, .)$ Be a prime ideal of a comm. ring R with 1. Then the quotient ring (R/I) is integral domain if and only if $(I, +, .)$ Is a prime ideal.

Proof: ⇒ Suppose

that $R$ is

integral

domain

$I$

T.P: I is a prime ideal ??

Let $a, b \in R \ni a.b \in I$.

We must prove that $a \in I \; or \; b \in I$ ??

$\because a.b \in I \rightarrow a.b + I = I \; (a + I)(b + I) = I = 0 + I$.

But $R$ integral domain → $R$_ has no zero divisors

$I$ $I$

$\rightarrow a + I = I \;\; or \;\; b + I = I$. Why ??

$\rightarrow a \in I \quad or \quad b \in I$

$\rightarrow I$

prime

idea.

Why

?? ⇐

suppo

se

that I prime ideal T.P: $\frac{R}{I}$ integral domain ??

Since R comm. $\rightarrow \frac{R}{I}$ comm.

Since R with 1 $\rightarrow \frac{R}{I}$ with $1+I$

We must prove $\frac{R}{I}$ has no zero divisors.

Let $a+I\,,\, b+I \in \frac{R}{I} \ni a, b \in R$

$(a+I)(b+I) = 0 + I = I$

$(a.b) + I = I \rightarrow a.b \in I$

But $I$ prime ideal $\rightarrow$ either $a \in I$ *or* $b \in I$

If: $a \in I \rightarrow a + I = I \rightarrow a + I = 0 + I$

Or: $b \in I \rightarrow b + I = I \rightarrow b + I = 0 + I$

$\rightarrow \frac{R}{I}$ has no zero divisor $\rightarrow \frac{R}{I}$ integral domain

Corollary: Every maximal ideal is prime ideal.

Proof:

Suppose that A is a maximal ideal of R

T.P: A is a prime

ideal ??

Suppose $x.y \in A \ \ \forall x, y \in A$ ? ?

If $x \notin A$ :-

Since A maximal , $x \notin A \rightarrow A + (x) = R$

But $1 \in R \rightarrow 1 \in A + (x)$

$\rightarrow 1 = a + rx , r \in R , a \in A$

$\therefore y = ay + y(rx)$ نضرب بـ

$= ay + rxy$

But $y \in R; a \in A \rightarrow ay \in A$

But $r \in R; x.y \in A \rightarrow r(xy) \in A$

$\rightarrow ay + r(xy) \in A \rightarrow y \in A \rightarrow A$ prime ideal

## المحاضرة ٥

**Def:** We say R principal ideal ring (P.I.R) if every if every ideal of R is principal (**it is generated by one element**)

Ex: (2) is principle in Z

المثالي الرئيسي: هو المثالي الذي يتولد بعنصر )مشابه تماما الى المثالي الفعلي( بفرق انه يشمل مثالي ) 0 و 1 )

1: هو الحلقة نفسها و 0: هو مجموعة الصفر فقط القانون العام له هو $(a) = \{r.a : r \in R\}$

ويكتب على

شكل حلقة

Ex: In Z

every

ideal is

principal.

**Remark: Z is a P.I.R**

**Def:** if R **integral domain** and every ideal of R is **principal** then R is called principal integral domain (**P.I.D**)

Theorem: Let R be a P.I.D. every non trivial ideal A is prime ⟺ A is maximal

## المحاضرة ٦

Remarks:

1. If R is P.I.D, then every non trivial ideal I is prime ⟺ is maximal
2. In $(Z, +, .)$ Every non trivial ideal I is maximal ⟺ I prime
3. If R is P.I.D, then the ideal $((a), +, .)$ is prime (maximal) in R ⟺ a prime number 4. $(Z, +, .)$ Is P.I.D (By (3) )

**Def:** Let R be a ring. Then the radical of R (Rad(R)) defined by:

$$Rad(R) = \cap \{M: M \text{ is Maximal ideal of } R\}$$

- $Rad(R) \neq \emptyset$
- $Rad(R) \subseteq R$
- 

$Rad$

$(R)$

ideal

in $R$

Q:

find

$Rad$

$(Z)$? ?

Ans: $Rad(Z) = \cap\{all\ Maximal\ ideals\}\ \ni (P)$ prime

$\therefore (2) \cap (3) \cap (5) \cap (7) \cap (11) \cap (13) \cap$

...

Q: find Rad $(Z_{12})$

Solution:

We must find all maximal ideals

Where $12 = (2)(6) = (3)(4) = (1)(12)$

$I_1 = (2) = \{0,2,4,6,8,10\}$

$I_2 = (3) = \{0,3,6,9\}$

$I_3 = (4) = \{0,4,8\}$

$I_4 = (6) = \{0,6\}$

$I_4$ and $I_3$ are not maximal ideals because $\subset I_1$. But $I_1$ and $I_2$ are both maximal ideal because there not ideal contains them

Then $Rad(Z_{12}) = I_1 \cap I_2 = \{0,2,4,6,8,10\} \cap \{0,3,6,9\} = \{0,6\}$

**Def:** We say the ring R is semi simple if $Rad(R) = 0$

**Def:** Let I be an ideal of R. Then $\sqrt{I} = \{r \in R\ ; r^n \in I\ , n \in Z^+\}$ **Remark:**

1. $\sqrt{I} \subseteq R$
2. $\sqrt{I} \neq \emptyset$ why?

المحاضرة ٧

Thereon: Let I be an ideal of R. Then if J an ideal of $R \Longrightarrow \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Proof:

Suppose that $x \in \sqrt{I \cap J}$

$\therefore x^n \in I \cap J ; n \in Z^+$

$\therefore x^n \in I \wedge x^n \in J$

$\therefore x \in \sqrt{I} \cap x \in \sqrt{J}$

$\therefore \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$

Suppose $y \in \sqrt{I} \cap \sqrt{J}$

$y \in \sqrt{y}, y \in \sqrt{J}$

$y^n \in I , y^m J$, $n, m \in Z^+$

$y^n . y^m \in I$, because $y^n \in I , y^m \in R$

$\therefore y^{n+m} \in I$

$\therefore y^k \in I$

Also, $y^n . y^m \in J$ because $y^m \in J , y^n \in R$

$\therefore y^{n+m} \in J \rightarrow y^k \in J$

$\therefore y^k \in I \wedge y^k \in J$

$\therefore y^k \in I \cap J$

$\therefore y \in \sqrt{I \cap J}$

$\therefore \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$

$\therefore \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Def: Let $f : R_1 \rightarrow R_2$ be a function we say f is ring homomorphism if

1. $f(a+b) = f(a) +_2 f(b)$
2. $f(a.b) = f(a) \cdot_2 f(b)$

Example: $f: R_1 \to R_2 \ni f(a) = \bar{0}, a \in R$

Solution:

$$f(a+b) = \bar{0} = \bar{0} +_2 \bar{0} = f(a) +_2 f(b)$$

$$f(a.b) = \bar{0} = \bar{0} ._2 \bar{0} = f(a) ._2 f(b)$$

$\therefore$ f is homeomorphism.

Example: $f: Z \to Z_e \ni f(a) = 2a$

Solution:

$$f(a+b) = 2(a+b) = 2a + 2b$$

$$= f(a) + f(b)$$

المحاضرة ٨

$$\neq f(a.b) = 2(ab)$$

$\therefore$ f isn't homo.

Theorem: Let $f: R_1 \to R_2$ be a ring homo. then (1) $f(0) = 0_2 \ni 0_2$ identity of $R_2$

Proof:

$$a + 0 = 0 + a = a \ \forall a \in R$$

$\therefore f(a+0) = f(0+a) = f(a)$

$\therefore f(a) +_2 f(0) = f(0) +_2 f(a) = f(a)$ (f homo.)

$\therefore f(0)$ is identity in $R_2$.

But $0_2$ is identity in $R_2$ (unique)

$\therefore f(0) = 0_2$

Def: Let $f: F_1 \to F_2$ be a function. Then f is filed homo. if

1. $f(a+b) = f(a) +_2 f(b)$
2. $f(a.b) = f(a) ._2 f(b)$

Def: Let $f: R_1 \to R_2$ be a ring homo. then Kernel of f define by.

$$Ker(f) = \{a \in R : f(a) = 0_2\}$$

Remark:

1.
$Ker(f) \neq \emptyset$

w
hy
?
2.
$Ker(f) \subseteq$
$R$
w
hy
?

Example: $f: R_1 \to R_2 \ni f(a) = 0_2 , a \in R.$

$$Ker(f) = \{a \in R : f(a) = 0_2\}$$
$$= \{a : a \in R\} = R$$

Example: $f: R_1 \to R_2 \ni f(a) = a$

$$Ker(f) = \{a \in R : f(a) = 0\}$$
$$= \{ \qquad\qquad\qquad \}$$

Def: A field F is called prime filed if it has no

proper subfield. Example: $Q , Z_3 , Z_5$

ملاحظة: أي حقل قواسم هو حقل اولي.

Remarks:

1. Any Quotient field of integral domain is a prime field
2. Any prime field is a quotient ideal.

Example: Let F be a field a field. Then 0 is maximal ideal and F is a minimal ideal

Example: $(Z , + , .)$

$Z_e$ maximal

$(1) \supset (2)$

$(1) \supset I \supset (2)$

Theorem: Let $f: R_1 \to R_2$ be a ring homo. Then Ker(f) is an ideal in R.

Proof:

$Ker \neq \emptyset$

Let $x, y \in Ker(f)$

$\Rightarrow f(x) = 0_2 , f(y) = 0_2$

T.P: $x - y \in Ker(f)$

T.P: $f(x - y) = 0_2$

$f(x - y) = f(x) - f(y) = 0_2 - 0_2 = 0$

Let $x \in Ker(f) , r \in R$

T.P: $rx , xr \in Ker(f)$

$f(rx) = f(r) . f(x) = f(r) . 0_2 = 0_2$

$\Rightarrow rx \in Ker(f)$

$xr \in Ker(f)$

$\therefore Ker(f)$ is ideal in R.

Theorem: if $f: R_1 \to R_2$ is a ring home. Then f is one to one $\Leftrightarrow Ker(f) = 0$

Proof:

Suppose f is one to one

T.P $Kre(f) = \{0\}$

Let $a \in Ker(f)$

$\Longrightarrow f(a) = 0_2$

$f$

$(a) = f(0_2)$

$\because$

f

$\therefore a = 0 \to Ker(f) = \{0\}$

المحاضرة ٩

$\Longleftarrow$ Suppose $Ker(f) = 0$

T.P: f is 1 -1

Let $a, b \in R_1 \ni f(a) = f(b)$

T.P: $a = b$
$f(a) = f(b)$

$f(a) - f(b) = 0$

$f(a - b) = 0 \to a - b \in Ker(f)$

$\Rightarrow a = b$

Corollary: The unique homo. otherwise, zero homo. is from $Z \to Z$ and it is identity ho
$(f(n) = n, n \in Z)$

Proof:

Let f be a non-zero

homo. from $Z \to Z$

T.P: f is identity

homo.

$(f(n) = n, n \in Z)$

Let $n \in {}^{+}Z$

$n = 1 + 1 + 1 + 1 + \cdots + 1$

$\therefore f(1) = f(1 + 1 + \ldots + 1)$

$= f(1) + f(1) + \cdots + f(1) = n.f(1)$

If n is negative $(n < 0)$

$\Rightarrow -n \in Z^{+} \Rightarrow f(n) = f\big(-(-n)\big) = -f(-n)$

$= -(-n)f(1) = n.f(1)$

If $n = 0 \to f(n) = f(0) = 0 = 0\,f(1)$

$\therefore f(n) = nf(1), n \in Z$

$\because f(1) = 1 \to f(n) = n \Rightarrow f$ unique homo.

**LUC (9)**

Def: Let $(I, +, .)$ Be an ideal of $(R, +, .)$. We define the set ***ann*(*I*)** by:

$$ann(I) = \{r \in R\,;\, r.a = 0, \forall a \in I\}$$

Q: prove that $(ann(I), +, .)$ Is an ideal of $(R, +, .)$

Proof:

Let:

$a = r_1 x \Rightarrow a = 0$

$b = r_2 y \Longrightarrow b = 0$

1- $a - b \Longrightarrow r_1 x - r_2 y = 0 - 0 = 0$

2- $ra \Longrightarrow r0 = 0 \ . \forall r \in R$

Def: Let R be a ring. Then R is called Boolean ring if R has identity and $a^2 = a$ , $\forall a \in R$. Ex: $(Z_2 ,+_2 , \cdot_2 )$ is a Boolean ring because $Z_2 = \{0 ,1\}$

$\Longrightarrow 0^2 = 0 , 1^2 = 1$

Ex: $(P(X), \Delta , \cap)$ is a Boolean ring because $\forall A \in P(X) \Rightarrow A^2 = A \cap A = A$

Ex: $R = \{ f : f : X \rightarrow Z_2 \}$ and we define

$(f + g)(x) = f(x) +_2 g(x)$
$(fg) = f(x) \cdot_2 g(x) \qquad \forall x \in X$

$\therefore$ R is a commutive ring with 1 and satisfy:

$$f(x) = 0 \ \ or \ f(1) = 1 \ \ , f \in R$$

$\therefore f^2 = f$

If $f(x) = 0 \Longrightarrow f^2(x) = f(x) \cdot_2 f(x) = 0 \cdot_2 0 = f(x)$
If $f(x) = 1 \Longrightarrow f^2(x) = f(x) \cdot_2 f(x) = 1 \cdot_2 1 = f(x)$

$$\therefore f^2 = f$$

$\therefore$ R is a Boolean ring

## المحاضرة ١٠

Theorem: Every Boolean ring is commutative and has Char = 2

Proof:

Suppose that R is a ring

and $a^2 = a \quad \forall\, a \in R$ Let $a$,

$b \in R \Longrightarrow a + b \in R$ . why?

$\Rightarrow a^2 = a \;, b^2 = b \;, (a+b)^2 = a + b$

$\rightarrow (a+b)^2 = a^2 + b^2 + ab + ba$

$\rightarrow a + b = a^2 + b^2 + ab + ba$

$= a + b + ab +$
$ba$

$\rightarrow ab + ba = 0$

$\rightarrow a.a + a.a = 0$

$a$

${}^2 + a^2 = 0$

$a$

$+ a = 0$

$2a = 0 \;\; \forall a \in R$

$\therefore$ Char(R) = 2

Also, $ab + ba = 0 \Longrightarrow$

$ab + ab + ba = ab$

$2ab + ba = ab$

$0 + ba = ab$

$\Longrightarrow$

$ab$

$=$

$ba$

$\forall a$,

$b \in$

$R$ ∴

R

com

m.

Theorem: If R is a Boolean ring and I an ideal (proper) of R, then I is a prime ideal ⟺ I is maximal ideal.

Proof:

Suppose

that I is

prime ideal

T.P: I is

maximal??

Assume that I ideal

of R ∋ $\subset$ $\subseteq I\ J\ R$

T.P: $J = R$ ??

∵ $\subset I\ \ J \rightarrow \exists$ $\in$ $a\ J\ , a \notin I$

∵ $\in$

$a$

$J$

$a$

$\in$

$R$

R Boolean

$\therefore a^2 = a$

$a(1-a) = 0 \in I$

$\because$ I prime $a \notin I$

$\therefore 1 - a \in J$

$\because \subset I \; J \Longrightarrow 1 - a \in J$

$\rightarrow (1-a) + a \in J \rightarrow 1 \in J \rightarrow J = R$

$\therefore$ I maximal

$\Leftarrow$

Suppose that I is maximal ideal

T.P: I is prime ideal

By "Every maximal ideal is prime ideal"

$\therefore$ I is prime ideal

## المحاضرة ١١

Theorem: Let be a proper ideal of Boolean ring R. Then I is a maximal $\Leftrightarrow \frac{R}{I} \cong Z_2$

Proof:

Since R is a Boolean ring $\rightarrow \frac{R}{I}$ is a Boolean ring

Since R is a Boolean ring

$\rightarrow$ R is a commutative with 1

$\rightarrow \frac{R}{I}$ is a commutative with 1

T.P: $\left(\frac{R}{I}\right)$ $(0+I)^2 = a+I$ , $\forall\, a+I \in \frac{R}{I}$

$(a+I)^2 \in \frac{R}{I}$

$(a+I)^2 = (a+I)(a+I) = a.a + I = a^2 + I$

$= a + I \quad (a^2 = a)$

$\rightarrow \frac{R}{I}$ Boolean ring

## The Boolean ring R is a filled $\Leftrightarrow R \cong Z_2$

Proof:

Let R be a Boolean ring.

$\therefore\ \forall\ a \in R \rightarrow a = a.1$

$= a.(a.a^{-1})$

$= (a.a)a^{-1}$

$= a^2 . a^{-1}$

$= a.a^{-1} = 1$

$\because R = \{0,1\}$

$\Longrightarrow R \cong Z_2$

But I maximal $\Leftrightarrow \frac{R}{I}$ field

I maximal $\Leftrightarrow \frac{R}{I}$ Boolean ring

I maximal $\Leftrightarrow \frac{R}{I} \cong Z_2$

## Theorem: Every Boolean ring R is semi simple

Proof:

Let R be a Boolean ring

∴ R has identity

element.

Why?? and $a^2$

$= a \ \forall a \in R$.

T. P. R is a semi simple ring

T. P.

Rad

(R)=

0 (

$\sqrt{R} = 0$)

supp

ose

that

Rad

$\neq 0$

$\therefore \exists \ a \in$

Rad

(R).

$\ni a \neq 0$

$\therefore \exists$ homomorphism

from R to $Z_2$

$\ni f(a) = 1$

∴ Ker(f) ideal

(proper) in R

$\therefore\ \exists$ maximal

ideal M in R $\ni$

$Ker(f) \subseteq M$

But $1 - a \in Ker(f)$

$\therefore 1 - a \in M\ (Ker(f) \subseteq M)$

Since $a \in M$ (because $a \in Rad(R)$ )

$Rad\ (R) = \bigcap\{$ Maximal ideal$\}$

$\rightarrow 1 - a + a \in M$

$\rightarrow 1 \in M \rightarrow M = R$ C!

Because M Maximal in R

$\therefore Rad(R) = \{0\}$

R semi simple ring

Definition: We define Boolean algebra is a Mathematical system $(B, \vee, \wedge)$ such that $\vee, \wedge$ two binary operation on B and $B \neq \emptyset$ and satisfy the following

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- $\vee, \wedge$ commutative on B.
  i.e.: $a \wedge b = b \wedge a$ , $a \vee b = b \vee a\ \forall a, b \in B$.
- $\vee, \wedge$ distributive with them ; others ;
  $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$
  $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$
- $\exists\ 1, 0$ identity elements with $\vee, \wedge$
  $\ni a\vee 0 = a\ \&\ a \wedge 1 = a\ \forall a \in B$ For each element $a \in B\ \exists\ a' \in B$
  $\ni a \vee a' = 1\ \&\ a \wedge a' = 0$

(a’ is called complement of a)

Example: $(P(X),\cup,\cap)$, $x \neq 0$ is a Boolean algebra.
$0 = \emptyset$, $1 = x$

Example: Let B be a set of Positive integer numbers which represent divisors of 10 B = {1,2,5,10}

We define $\vee$ , $\wedge$ on B by:

$\forall$ a,b$\in$ B $\rightarrow$

g.c.d (a, b) =a $\wedge$ b

L.c.m (a, b) = a $\vee$ b . Then

(B , $\vee$ , $\wedge$ ) is a Boolean algebra

| $\vee$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 10 |
| 2 | 2 | 2 | 10 | 10 |
| 5 | 5 | 10 | 5 | 10 |
| 10 | 10 | 10 | 10 | 10 |

| $\wedge$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 2 |
| 5 | 1 | 1 | 5 | 5 |
| 10 | 1 | 2 | 5 | 10 |

identity element of $\vee$ is 1

identity element of $\wedge$ is 10

1’ = 10 , 2’ = 5 , 5’ = 2 , 10’ = 1