COURSE 2- GROUP 2

Quotient Group

LECTURE 1

Here is a fundamental construction of a new group from a given group.

Theorem (1): Let G/K denotes the family of all the cosets of a subgroup K of G. If K is a normal subgroup, then:

a K bK = abK

for all a, b□G, and G/K is a group under this operation

Definition (2): The group G/K is called the **quotient group**; when G is finite, its order G/K is the index [G:K] (presumably, this is the reason quotient groups are so called).

We can now prove the converse of Proposition 2.91(ii).

Proposition (3): Every normal subgroup K of a group G is the kernel of some homomorphism.

Proof:

Define the natural map π: G G/K by π(a) = a K. With this notation, the formula a K bK =abK can be rewritten as π(a)π(b) = π(ab); thus, π is a (surjective) homomorphism. Since K is the identity element in G/K,

ker π = {a G : π(a) = K } = {a G : a K = K } = K

LECTURE 2

First Isomorphism Theorem

The following theorem shows that every homomorphism gives rise to an isomorphism, and that quotient groups are merely constructions of homomorphic images.

Theorem (1): (First Isomorphism Theorem)

If f: G→H is a homomorphism, then:

G/ ker f = im f

Where im f = f(H). In more detail, if we put ker f = K , then the function ϕ : G/K → f(H) is given by:

ϕ: a K r→ f (a) for each aG, is an isomorphism.

Proof:

It is clear that ker f is a normal subgroup of G, and we can easily show that ϕ is well-defined. Let us now see that ϕ is a homomorphism. Since f is a homomorphism and $\phi(a K) = f(a)$,

$\phi(a K bK) = \phi(abK) = f(ab) = f(a) f(b) = \phi(a K)\phi(bK)$.

Also ϕ is surjective and injective Therefore, $\phi: G/K \to \text{im } f$ is an isomorphism.

Remark (2):

1. Here is a minor application of the first isomorphism theorem. For any group G, the identity function $f: G \to G$ is a surjective homomorphism with ker f = 127

LECTURE 3

Proposition (3):

1. If H and K are subgroups of a group G, and if one of them is a normal subgroup, then HK is a subgroup of G. Moreover, HK = KH.

2. If both H and K are normal subgroups, then HK is a normal subgroup.

Proof:

1. Assume first that K is normal in G. We claim that HK = KH. If hk = HK, then:

$hk = hkh^{-1}h = k1\ h = KH$

where $k_1 = hkh^{-1}$, then $k_1$in K, because K is normal subgroup .

Hence, HK = KH. For the reverse inclusion, write $kh = hh^{-1}kh = hk_2$ = HK, where k_2 = h-1kh.
(Note that the same argument shows that HK = KH if H is normal subgroup of G.)

LECTURE 4

Third Isomorphism Theorem

In the following lecture we study the third important theorem of fundamental isomorphism theorem.

Theorem (1): (Third Isomorphism Theorem)
If H and K are normal subgroups of a group G with $K \leq H$, then H/K is normal in G/K and
(G/K)/(H/K) = G/H. 2

Proof:

Define f: G/K□G/H by f(a K) = a H. Note that f is a (well- defined function, for if a G = b K, then a-1b□K But K□H, thus a-1b□ H, and so a H= b H , and we are done. It is easy to see that f is an epimorphism.

Now ker f = H/K. Also clearly H/K is a normal subgroup of G/K. Since f is monomorphism, so by the first isomorphism theorem we have: (G/K)/(H/K) □ G/H

The third isomorphism theorem is easy to remember: the K’s in the fraction (G/K)/(H/K) can be canceled. One can better appreciate the first isomorphism theorem after having proved the third one. The quotient group (G/K)/(H/K) consists of cosets (of H/K) whose representatives are themselves cosets (of G/K). 3

Here is another construction of a new group from two given groups.
Definition (2): If H and K are groups, then their **direct product**, denoted by HxK, is the set of all ordered pairs (h, k) equipped with the following operation:
$(h, k)(h1, k1) = (hh1, kk1)$
It is routine to check that HXK is a group [the identity element is (e, e1) and $(h, k)^{-1} = (h^{-1}, k^{-1})$.
Remark (3): let G and h be groups.Then HxK is abelian if and only if both H and K are abelian.
We end the tenth lecture by the following example.

Example: Zx2Z is the direct product between (Z, +) and (2Z, +) groups.

The identity element is (0, 0), and the inverse element of (a, b) is (-a, -b).

We now show that HK is a subgroup. Since $e \square H$ and $e \square K$, we have $e = e \cdot e \square HK$. If $hk \square HK$, then $(hk)^{-1} = k^{-1} h^{-1} \square KH = HK$. If $hk, h_1k_1 \square HK$, then $h_1^{-1} kh_1 = ke \square K$ and
$Hkh_1 k_1 = hh_1(h_1^{-1} kh_1)k_1 = (hh_1)(kek_1) \square HK$.
Therefore, HK is a subgroup of G.
2. If $g \square G$, then:
$ghkg^{-1} = (ghg^{-1})(gkg^{-1}) \square HK$
Therefore, HK is normal in G. $ghkg^{-1} = (ghg^{-1})(gkg^{-1}) \square HK$. Therefore, HK is normal in G.
Definition (1): If H and K are subgroups of a finite group G, then then the **Product Formula** is:

$$|HK||H \cap K| = |H||K|$$

LECTURE 5

One can shorten the list of items needed to verify that a subset is, in fact, a subgroup.
Proposition (4): A subset H of a group G is a subgroup if and only if H is nonempty and, whenever $x, y \in H$, then $x y^{-1} \in H$.
Proof: If H is a subgroup, then it is nonempty, for $1 \in H$. If $x , y \in H$, then $y^{-1} \in H$, by part (iii) of the definition, and so $x y^{-1} \in H$, by part (ii).
Conversely, assume that H is a subset satisfying the new condition. Since
H is nonempty, it contains some element, say, h. Taking $x = h = y$, we see that $e = hh^{-1} \in H$, and so

part (i) holds. If $y \in H$, then set $x = e$ (which we can now do because e

$\in H$), giving $y^{-1} = ey^{-1} \in H$, and so part (iii) holds. Finally, we know that $(y^{-1})^{-1} = y$, by. Hence, if $x , y \in H$, then $y^{-1} \in H$ and so $x y = x (y^{-1})^{-1} \in H$. Therefore, H is a subgroup of G. Since every subgroup contains e, one may replace the hypothesis "H is nonempty" in Proposition by "$e \in H$".

Note that if the operation in G is addition, then the condition in the proposition is that H is a nonempty subset of G such that $x, y \in H$ implies $x- y \in H$.

Proposition (5): Let G be a finite group, and a G. Then the order of a, is the number of elements in (a).

Definition (6): If G is a finite group, then the number of elements in G, denoted by |G|, is called the **order of G**.

Definition (7): If X is a subset of a group G, such that X generates G, then G is called **finitely generated**, and G generated by **X**.

In particular; If G = ({a}), then G is generated by the subset X = {a}.

Definition (8):

A group G is called **cyclic** if G = (a); that is G can be generated by only one element say a, and this element is called a generator of G.

LECTURE 6

If H and K are subgroups of a group G with H is normal in G, then HK is a subgroup of G and $H \cap K$ is normal in K.

Moreover:

$K/(H \cap K)$ □ HK/H

Proof:

We begin by showing first that HK/H makes sense, and then describing its elements. Since H is normal subgroup of G, then HK is a subgroup of G. Normality of H in HK follows :

from a more general fact: if H □ S□ G and if H is normal in G, then H is normal in S.
We can easily show that each coset x H □ HK/H has the form k H for some k□K. It follows that the function f: K□ HK/H, given by f(k) = k H, is surjective. Moreover, f is a homomorphism, for it is the restriction of the natural map π: G → G/H. Since ker π = H, it follows that ker f = H ∩ K and so H ∩ K is a normal subgroup of K . The first isomorphisim theorem gives:
K /(H ∩ K) □HK/H

Remark (3):

The second isomorphism theorem gives the product formula in the special case when one of the subgroups is normal: if K /(H ∩ K) □ H K /H , then: |K /(H ∩ K)| = |H K /H |, and so |H K ||H ∩ K | = |H ||K |.

LECTURE 7

Definition (6): If f: G → H is a homomorphism, define
kernel f = {x □ G : f (x) = e}
and **image f** = {h □ H : h = f (x) for some x □G}.
We usually abbreviate kernel f to ker f and image f to im f
So that if f: G □ H is a homomorphism and B is a subgroup of H then f−1(B) is a subgroup of G containing ker f .
Note: Kernel comes from the German word meaning “grain” or “seed” (corn comes from the same word).
Its usage here indicates an important ingredient of a homomorphism, we give it without proof.

Proposition: Let f: G $\rightarrow$ H be a homomorphism.
(i) ker f is a subgroup of G and im f is a subgroup of H .

(ii) If x □ ker f and if a □ G, then ax a−1□ ker f.
(iii) f is an injection if and only if ker f = {e}.

Normal Subgroups

Definition (1): A subgroup K of a group G is called **normal,** if for each k □ K and g □ G imply gkg−1□ K. that is gKg-1 □ G for every g□G.

Definition (2):

Define the **center of a group G,** denoted by Z (G), to be

Z (G) = {z □ G: zg = gz for all g □ G};

that is, Z (G) consists of all elements commuting with every element in G. (Note that the equation zg
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= gz can be rewritten as z = gzg−1, so that no other elements in G are conjugate to z.

Remark (3):

Let us show that Z (G) is a subgroup of G. We can easily show that Z(G is subgroup of G. It is clear that Z(G)≠ since 1 ∈ Z (G), for 1 commutes with everything. Now, If y, z □ Z (G), then yg = gy and zg = gz for all g □ G. Therefore, (yz)g = y(zg) = y(gz) = (yg)z = g(yz), so that yz commutes with everything, hence yz □ Z (G). Finally, if z □ Z (G), then zg = gz for all g □ G; in particular, zg−1 = g−1 z. Therefore,

gz−1 = (zg−1)−1 = (g−1z)−1 = z−1g

(we are using (ab)−1 = b−1a−1 and (a−1)−1 = a).

So that Z(G) is subgroup pf G.

Clearly che center Z (G) is a normal subgroup; since if z Z (G) and g G, then
$gzg^{-1} = zgg^{-1} = z$ Z (G)

A group G is abelian if and only if Z (G) = G. At the other extreme are groups G for which Z (G) = {1}; such groups are called centerless. For example, it is easy to see that Z (S_3) = {1}; indeed, all large symmetric groups are centerless.

{1}. By the first isomorphism theorem, we have
G/{1} □ G

2. Given any homomorphism f:G□H , one should immediately ask for its kernel and its image; the first isomorphism theorem will then provide an isomorphism
G/ ker f □ im f . Since there is no significant difference between isomorphic groups, the first isomorphism theorem also says that there is no significant difference between quotient groups and homomorphic images.