University Of Anbar

College of Education For Pure Sciences

Department of Mathematics

LECTURES IN ADVANCED COMPLEX ANALYSIS

Preparation by

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LECTURE 1

Conformal Maps

We continue to study analytic functions as mappings. Let us understand the geometric meaning of the derivative. Since derivative are now complex-valued , the calculus interpretation of the derivative no longer makes sense. But we are going to find a new one .

1. Geometric Interpretation of the Derivative

Let $f : E \to \mathbb{C}$ be analytic and $f'(z_0) \neq 0$ where $z_0 \in E$ is fixed point and $w_0 = f(z_0)$. Write $f'(z_0)$ in exponential form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w(z_0)}{\Delta z} = Ae^{i\alpha} \tag{1.1}$$

where as before $\Delta w(z_0) = f(z_0 + \Delta z) - f(z_0)$. The limit in (1.1) doesn't depend on how $\Delta z \to 0$. So we take two different paths γ_1, γ_2 non-tangential at z_0 .

Let $\Gamma_1 = f(\gamma_1), \Gamma_2 = f(\gamma_2)$ be the images of γ_1, γ_2 under f .

From (1.1),

$$\alpha = \arg f'(z_0) = \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z \tag{1.2}$$

Since α is independent of the way $\Delta z \to 0$, (1.2) yields

$$\underbrace{\lim_{\Delta z \to z, z \in \gamma_1} \arg \Delta w}_{\emptyset_1} - \underbrace{\lim_{\Delta z \to z, z \in \gamma_1} arg \Delta z}_{\varphi_1} = \underbrace{\lim_{\Delta z \to z, z \in \gamma_2} arg \Delta w}_{\emptyset_2} - \underbrace{\lim_{\Delta z \to z, z \in \gamma_2} arg \Delta z}_{\varphi_2}$$

$$\Rightarrow \quad \emptyset_1 - \varphi_1 = \emptyset_2 - \varphi_2$$

$$\Rightarrow \quad \underbrace{\emptyset_2 - \emptyset_1}_{\Delta\emptyset} = \underbrace{\varphi_2 - \varphi_1}_{\Delta}$$

$$\Rightarrow \qquad \Delta\emptyset = \Delta\varphi \tag{1.3}$$

Equation (1.3) reads that an analytic function f preserves angles at each point z_0 such that $f'(z_0) \neq 0$.

Read (1.1) now differently

$$)z\Delta\,,_0z(\epsilon + z\Delta^{\alpha i}eA = w\Delta \qquad \Rightarrow \qquad \lim_{\Delta z \to 0} \frac{\Delta w(z_0)}{\Delta z} = Ae^{i\alpha}$$

where

$$\lim_{\Delta z \to 0} \frac{\varepsilon(z_0,\Delta z)}{\Delta z} = 0\ .$$

Recalling the definition of the differential

$$dw = Ae^{i\alpha}\Delta z \qquad \Rightarrow \quad |dw| = A|\Delta z|.$$

2. Conformal Map

Definition 1.1. A map , $f : E \to f(E)$ is called <u>conformal</u> if it preserves angles and has a constant dilation at any $z_0 \in E$.

Proposition 1.2 . An analytic function f is conformal at z_0 if and only if $f'(z_0) \neq 0$.

Proof.

Let z_0 be a point in E and C_1 and C_2 be two smooth curves passing through z_0 with tangent T_1 and T_2, respectively . Let θ_1 and θ_2 denotes the angles of inclination at T_1 and T_2, respectively .

K_1 and K_2 (images curves) that pass through $w_0 = f(z_0)$ will have tangents T_1^* and T_2^*, respectively .

$\gamma = \arg[f'(z_0)] + \arg[z'(0)] = \alpha + \theta$ when $\alpha = \arg[f'(z_0)]$,

$\gamma_1 = \alpha + \theta_1$ and $\gamma_2 = \alpha + \theta_2$.

We conclude $\gamma_2 - \gamma_1 = \theta_2 - \theta_1$.

That the angle $\gamma_2 - \gamma_1$ from K_1 to K_2 is the same magnitude and orientation as the angle $\theta_2 - \theta_1$ from C_1 to C_2 . Therefore , $w = f(z)$ is conformal mapping .

LECTURE 2

Möbius Transforms (Linear Fractional Transforms

Definition 2.1. The function

$$\varphi(z) := \frac{az+b}{cz+d} \text{ ,}where\ (ad \neq bc\ ) \tag{2.1}$$

is called the <u>*Möbius transform*</u> .

Proposition 2.2. Let the Möbius transform , φ, be defined by (2.1). Then

(1) φ is analytic on $\mathbb{C} \setminus \{-\frac{d}{c}\}$
(2) φ is univalent and conformal on $\mathbb{C} \setminus \{-\frac{d}{c}\}$
(3) the inverse, φ^{-1} of φ, is given by

$$\varphi^{-1}(z) := \frac{dz-b}{-cz+a} .$$

Proof . Since (1) is clear, we will begin by considering (2). Simply differentiate (2.1) by the quotient rule to conclude that

$$\varphi'(z) = \frac{dz-b}{(cz+d)^2} \neq 0.$$

So, by Proposition 1.2, φ is conformal on $\mathbb{C} \setminus \{-\frac{d}{c}\}$. To prove that φ is univalent , we write

$$\begin{aligned} \frac{az+b}{cz+d} = w \quad &\Leftrightarrow \quad az+b = czw+dw \\ &\Leftrightarrow \quad (a-cw)z = dw-b \\ &\Leftrightarrow z = \frac{dw-b}{a-cw} = \frac{dw-b}{-cw+a} \ \text{ ,}\forall\, w \neq \frac{a}{c} \end{aligned} \tag{2.2}$$

which implies that φ is univalent .Thus ,(2) is done . Moreover, switching $w \leftrightarrow z$ in (2.2) yields (3) . □

Example . There are four fundamental example at Möbius transform $f : \hat{\mathbb{C}} \to E$

a) Translations : $z \to a + b = \frac{1.z+b}{0.z+1}$, $(b \in \mathbb{C})$. (Parabolic transformation).

b) Rotations : $z \to az = \frac{\sqrt{a}\,.z+0}{0.z+(\sqrt{a})^{-1}}$, $(|a| = 1)$. (Elliptic transformation).

c) Expansions : $z \to \lambda z = \frac{\sqrt{\lambda}\,.z+0}{0.z+(\sqrt{\lambda})^{-1}}$, $(\lambda > 1 \; or \; 0 < \lambda < 1)$.

(Hyperbolic transformation).

d) Inversion : $z \to \frac{1}{z}$.

Proposition 2.3. Let φ be a Möbius transform . Then

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$$

where $\varphi_1 \; and \; \varphi_3$ are linear mappings , and φ_2 is an inversion , i.e $\varphi_2(z) = \frac{1}{z}$. Moreover if $c \neq 0$

$$\varphi_1(z) = cz + d$$

$$\varphi_2(z) = \frac{1}{z}$$

$$\varphi_3(z) = \frac{a}{c} + \left(b - \frac{ad}{c}\right) z \, .$$

If $c \neq 0$ then φ is linear .

The proof is left as an exercise .

The importance of the Möbius transform is due to the following proposition .

Proposition 2.4. A Möbius transform φ transforms linear and circles to lines and circles (i.e. φ preserves lines and circles).

Proof . We show first that the statement holds for a linear $\varphi_1(z)$. Recall that any line L in $\mathbb{R}^2$ can be written in the parametric form $(x(t), y(t))$ where $x(t)$ and $y(t)$ are linear (real) functions of t . Consider now, $\varphi_1\big(z(t)\big) = cz(t) + d,$ where $z(t) + iy(t)$. Denoting $c = c_1 + ic_2 \; and \; d = d_1 + id_2$

$$\varphi_1(z(t)) = \underbrace{c_1x(t) - c_2y(t) + d_1}_{\text{linear function of } t} + i\underbrace{(c_2x(t) + c_1y(t) + d_2)}_{\text{linear function of } t}.$$

It follows that $\varphi_1(z(t)) := u(t) + iv(t)$ where $u(t)$ and $v(t)$ are linear functions . Hence, $\varphi_1(z(t))$ represents a line in the w −plane .

It follows that $|w - w_0| = \rho|c|$. That is, $w \in C_{\rho|C|}(w_0), i.e.$ $\varphi_1 C_\rho(z_0) = C_{\rho|C|}(\varphi_1(z_0))$.

It remains to show that the mapping $w = \frac{1}{z}$ transforms circles and lines into circles and lines . Note that when a point $w = u + iv$ is the image of a nonzero $z = x + iy$ under the transformation $w = \frac{1}{z}$, writing $w = \frac{\bar{z}}{|z|^2}$ yields that

$$u = \frac{x}{x^2+y^2} \quad , v = \frac{-y}{x^2+y^2} \quad . \tag{2.3}$$

Moreover, since $z = \frac{1}{w} = \frac{\bar{w}}{|w|^2}$,

$$x = \frac{u}{u^2+v^2} \quad , y = \frac{-v}{u^2+v^2} \quad . \tag{2.4}$$

Now note that when A,B,C, and D are all real numbers satisfying the condition $B^2 + C^2 > 4AD$, the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{2.5}$$

represents an arbitrary circle or line, where $A \neq 0$ for a circle and $A = 0$ for a line . If $A \neq 0$ it is indeed necessary for $B^2 + C^2 > 4AD$ since by completing the square we may rewrite (2.5) as

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

Note that if $A = 0$ then $B^2 + C^2 > 0$.Thus , either B or C is greater than 0 . Now observe that if x and y satisfy (2.5), by (2.4) we may substitute . After simplification we conclude that u and v satisfy the following equation

$$D(u^2 + v^2) + Bu - Cv + A = 0\,, \tag{2.6}$$

which also represents a circle or line . Conversely, if u and v satisfy (2.6), it follows from (2.3) that x and y satisfy (2.5).

Now by (2.5) and (2.6) we may conclude that

(1) a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z–plane is transformed into a circle not passing through the origin in the w–plane ;
(2) a circle ($A \neq 0$) through the origin ($D = 0$) in the z–plane is transformed into a line that does not pass through the origin in the w–plane ;
(3) a line ($A = 0$) not passing through the origin ($D \neq 0$) in the z–plane is transformed into a circle not passing through the origin in the w–plane ;
(4) a line ($A = 0$) through the origin ($D = 0$) in the z–plane is transformed into a line that does not pass through the origin in the w–plane .

This concludes our proof . □

It is left as an exercise to the reader to come up with a more elegant proof of Proposition 2.4.

Proposition 2.5. Given that φ_1 and φ_2 are Möbius transform ,it is follows that $\varphi_2 \circ \varphi_1$ is Möbius transform .

The proof is left as an exercise.

Proposition 2.6. Assume $\{z_1\ ,\ z_2\ ,\ z_3\}$ and $\{w_1\ ,\ w_2\ ,\ w_3\}$ are sets of distinct numbers . Then there exist a Möbius transform , φ , where

$$\varphi(z_i) = w_i, \qquad i = 1,2,3.$$

Moreover, φ can be explicitly constructed by

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} \ .$$

Proof . Consider $z_1\ ,\ z_2\ ,\ z_3\ \neq \infty$. Let

$$S(z) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

$\Rightarrow \qquad S(z_1) = 1 \ \ , \ \ S(z_2) = 0 \quad , \quad S(z_3) = \infty$.

Let $T(z) = \frac{(z-w_2)(z_1-w_3)}{(z-w_3)(z_1-w_2)}$, then $T(w_1) = 1 \ \ , \ \ T(w_2) = 0 \quad , \quad T(w_3) = \infty$.

If we define $\varphi(z) = S^{-1} \circ T(z)$, then $\varphi(zi) = T^{-1} \circ S(z_i) = w_i$, $(i = 1,2,3)$.

In case $z_1 = \infty$ then we let $S(z) = \frac{z-z_2}{z-z_3}$ …. Similarly for T.

In case $z_2 = \infty$ then we let $S(z) = \frac{z_1-z_3}{z-z_3}$ …. Similarly for T.

In case $z_3 = \infty$ then we let $S(z) = \frac{z-z_2}{z_1-z_2}$ …. Similarly for T.

<u>Uniqueness exercise.</u>

Example .Find the Möbius transform φ which maps $-1, 0\ , 1$ to the point $-i, 1\ , i$.

Assume $\varphi(z) = \frac{az+b}{cz+d}$. Since $\varphi(0) = \frac{b}{d} = 1 \quad \Rightarrow \quad b = d$ and $\varphi(z) = \frac{az+b}{cz+b}$

Similarly……..complete .

Example .Show that $\varphi(z) = \frac{z-1}{z+1}$ maps $y > 0$ onto $v > 0$ and $x \ \ axis$ onto $u \ \ axis$.

We first note that when the number z is real , so is the number w . Consequently, since the image of the real axis $y = 0$ is either a circle or line it must be the real axis $v = 0$, for at point w in the point w −plaine ,

$$v = Im\, w = Im\ \frac{(z-1)(z+1)}{(z+1)\overline{(z+1)}} = \frac{2y}{|z+1|^2}\ , \quad (z \neq -1)$$

v, y have the same sign ,

$x\ axis \ \rightarrow u\ axis$ (since $\varphi(z)\ is\ 1-1$) .

Exercise 2.7 Let U be the open unit disk in $\mathbb{C}$, and let $U^+ = \{z \in \mathbb{C} : Im\ z > 0, |z| < 1\}$.

Exhibit a one –one conformal mapping from U^+ onto U .

(Hint: consider $\varphi_0(z) = \frac{1-z}{1+z}$) .

Exercise 2.8. Show that

$\varphi(z) = \frac{i-z}{i+z}$ maps $\mathbb{C}^+$ onto $\mathbb{D}$.

LECTURE 3

Complex Integration

Definition 3.1. A piecewise smooth curve γ with parametric interval $[a,b]$ is said to be reparametrization of $\tilde{\gamma}(t)$, $\tilde{a} \le t \le \tilde{b}$ if and only if there is c'-map $\alpha:[\tilde{a},\tilde{b}] \to [a,b]$ such that $\alpha'(t)=0, \alpha(\tilde{a}) = a$, $\alpha(\tilde{b}) = b$ and $\tilde{\gamma}(t) = \gamma(\alpha(t))$, some time γ and $\tilde{\gamma}$ are said to be equivalent . Suppose f is continuous in D(open) containing all the points of $\gamma(t)$. Then we have

$$\int_{\tilde{\gamma}} f(z)\,dz = \int_{\tilde{\gamma}} f(\tilde{\gamma}(t))\,\tilde{\gamma}'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\gamma(\alpha(t))\gamma'(\alpha t)\alpha'(t)dt$$

$$. \int_{\alpha(\tilde{a})}^{\alpha(\tilde{b})} f(\gamma(\alpha(t))\alpha'(t)dt = \int_{a}^{b} f(\gamma(\alpha(t))\alpha'(t)dt = \int_{\gamma} f(z)\,dz =$$

Therefore, if in immaterial which parametrization is useful .

Example. Evaluate $I = \int_{\gamma} f(z)dz \quad , f(z) = z^n \quad , \gamma(t) = e^{it} \quad , 0 \le t \le 2\pi$

γ : is an arc of any circle centered of the origin,

$$I = r^{n+1} \int_{0}^{2\pi} ie^{(n+1)it}\,dt \quad [\gamma\,(t) = re^{it}]$$

given
$$I = \begin{cases} r^{n+1}\left[\dfrac{e^{it(n+1)}}{n+1}\right]_0^{2\pi} & if \quad n \ne -1 \\ 2\pi i & if \quad n = -1 \end{cases}$$

i.e.
$$\int_{\gamma} f(z)dz = \begin{cases} 0 & if \quad n \ne -1 \\ 2\pi i & if \quad n = -1 \end{cases}$$

continuous hold id γ and f are replaced by any circle centered at z_0 and $(z-z_0)^n$ respectively . This mean that

$$\int_{|z-z_0|=1} (z-z_0)^n \, dz = \begin{cases} 0 & if \quad n \neq -1 \\ 2\pi i & if \quad n = -1 \end{cases}$$

Theorem 3.2. Let γ be smooth curve and let f , g be continuous functions on D (open) continuous β ($[a, b]$) and let $\alpha \in \mathbb{C}$ .Then

1) $\int_{\gamma} f(z)dz = -\int_{-\gamma} f(z)dz$
2) $\int_{\gamma} [\alpha \ f(z) + g(z)]dz = \alpha \int f(z)dz - \int g(z)dz$

   If γ_1, γ_2 are two paths such that $\gamma_1(b) = \gamma_2(a)$
3) $\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$
4) If $L = L(\gamma)$ is the length of the curve and $M = .\max_{t\in[a,b]} |f(\gamma(t))|$, then $\left|\int_{\gamma} f(z)dz\right| \leq ML$.

Proof.

$$\int_{-\gamma} f(z)dz = \int_a^b f(\gamma(b+a.t))\, d(\gamma(b+a.t))$$

$$= \int_a^b f(\gamma(s))\, d\gamma(s)) \quad \text{by chang variable form } t \text{ to } s = b + a.t$$

$$= -\int_a^b f(\gamma(s))\, d\gamma(s)) = -\int_{\gamma} f(z)dz$$

1) Follows 2) follows from definition and linearity property of Riemann integral
3) As γ_1, γ_2 are curve with $\gamma_1(b) = \gamma_2(a)$, $\gamma = \gamma_1 + \gamma_2$, is then

$$\gamma(t) = \begin{cases} \gamma_1(2t-a) \ , \ a \leq t \leq (b+a)/2 \\ \gamma_2(2t-b) \ , \ (b+a)/2 \leq t \leq a \end{cases}$$

and the assertion now follows the definition of the noting that for γ_1, γ_2 a reparametrization has been made

4) $L = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt$ and for a real-valued Riemann integral function φ on $[a, b]$, we know that $\left|\int_a^b \varphi(t)dt\right| \le \int_a^b |\varphi(t)|dt$ …………………(*) .

If $\int_\gamma f(z)dz = 0$, then is nothing to prove .Therefore we assume $\int_\gamma f(z)dz \ne 0$ and write

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(\mathrm{t})\,\mathrm{dt} = \mathrm{Re}^{\mathrm{i}\theta}, \quad \text{where} \quad R > 0 \quad \text{and} \quad \theta = \arg(\int_\gamma f(z)dz).$$

Therefore, we have $\int_a^b \mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(\mathrm{t})\,\mathrm{dt} = \mathrm{e}^{-\mathrm{i}\theta}\int_a^b f(\gamma(t))\gamma'(\mathrm{t})\,\mathrm{dt} = \mathrm{R}$

$$= \mathrm{Re}\int_a^b \left|\mathrm{e}^{\mathrm{i}\theta} f(\gamma(t))\gamma'(\mathrm{t})\right| dt = \int_a^b \mathrm{Re}[\mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t)\,]d\mathrm{t}$$

Apply (*), with $\varphi(t) = \mathrm{Re}[\mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(\mathrm{t})]$, to get

$$\mathrm{R} = \left|\int_\gamma f(z)dz\right| \le \int_a^b \mathrm{Re}\left|\mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t)\right| d\mathrm{t}$$

$$= \int_a^b |f(\gamma(t))|\,|\gamma'(\mathrm{t})|\,dt\,.$$

Since $|f(\gamma(t))| \le M$ for all $t \in [a, b]$, and since fore positive integrands the Riemann integral is larger integrand , we have 4 .

Theorem 3.3. Let $\{f_n\}$ be a sequence of continuous functions, suppose that $f_n \to f$ uniformly on a smooth curve γ . Then

$$\lim_{n\to\infty}\int_\gamma f_n(z)dz = \int_\gamma f(z)dz$$

Proof . Let $\varepsilon > 0$ and f_n converge uniformly on γ with parametric integral $[a, b]$. Then there is an N such that

$$|f_n(\gamma(\mathrm{t})) - f(\gamma(\mathrm{t}))| < \varepsilon \text{ for } t \in [a,, b] \text{ and } n \ge N$$

by above theorem (3.2) , we have

$$\left|\int_{\gamma} f_n(z)dz - \int_{\gamma} f(z)dz\right| = \left|\int_{\gamma}[f_n(z) - f(z)]dz\right|$$

$$\left|\int_a^b [f_n(\gamma(t) - f(\gamma(t))]\gamma'(t)\,\mathrm{dt}\right| \le \int_a^b \left|f_n(\gamma(t)\right| - \left|f(\gamma(t))\right|\left|\gamma'(t)\right|\mathrm{dt}\ \varepsilon\int_a^b \left|\gamma'(t)\right|\mathrm{dt},\quad n \ge N =$$

$\rightarrow 0 \quad as\ n \rightarrow \infty$ for arbitrary small $\varepsilon > 0$.

Theorem 3.4. If $f = u + iv$ is analytic in open set D containing the smooth path γ with parametric interval $[a, b]$. i.e. $\gamma(\ [a, b]) \subset D$, then
$\int_{\gamma} f'(z)dz = f(\gamma(b)) - f(\gamma(a))$.

Proof . H.W

The Cauchy Theorem

Note : a region is analytic is an open set .

The simplest region of Cauchy's Theorem utilizes a theorem from calculus known as "Green's Theorem ".

"Given two real valued function $M(x, y)$ and $N(x, y)$, which continuous with their partial derivatives "

$$\int_{\gamma} M(x,y)dx + N(x,y)dy = \iint_{\substack{\Omega \\ \text{interionof } \gamma}} \left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{dy}\right]dxdy$$

Theorem 3.5 . If f is analytic with f' is continuous inside and on simple closed curve γ , then $\int_{\gamma} f(z)dz = 0$.

Proof . Let $f(z) = u(x, y) + iv(x, y) and\ \Omega = Int\Upsilon$ .Then

$$\int_{\gamma} f(z)dz = \int_{\gamma}(udx - vdy) + i\int_{\gamma}(udy + vdx).$$

By assumption , and by Green's Theorem

$$\int_{\gamma} f(z)dz = \iint_{\Omega}(-v_x - u_y)dxdy + i\iint_{\Omega}(u_x - v_y)dxdy$$

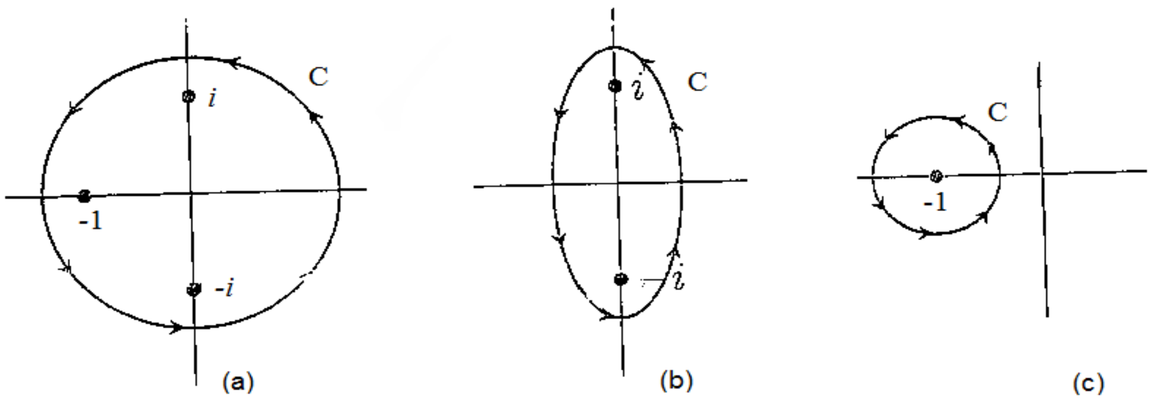
by C. R. equation we have result and assents that the integral of a function, analytic in a simple connected domain D along any closed curve $\gamma \subset$ D , is always zero .

Cauchy Goursat Theorem

Theorem 3.6. If f is analytic in a simple connected region D $\subset \mathbb{C}$. Then $\int_{\gamma} f(z)dz = 0$, (γ: closed contour , $\gamma \subset$ D).

Theorem 3.7. Let f be analytic on a (multiply) connected domain and let $C_1, C_2, \dots, C_n$ be simply connected region with positive oriented in D such that for all k , $Int\ C_k \subset Int\ C$ and let C_k are disjoint . If f is analytic on $Int\ C\ /\ \{\bigcup_{k=1}^{n} Int\ C_k\}$, then $\int_{C} f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$.

Example .Evaluate $\int_{C} \frac{dz}{(z+1)(z^2+1)}$ for the following contour



Solution .

Note that $f(z) = \frac{1}{(z+1)(z^2+1)}$ analytic on $C\ /\ \{-1, i, -i\}$.

$$f(z) = \frac{1/2}{z+1} + \frac{1/(1+i)2i}{z-i} + \frac{1/(1-i)(-2i)}{z+i}$$

$$=\frac{1}{2}\left(\frac{1}{z+1}+\frac{1}{i-1}\cdot\frac{1}{z-i}-\frac{1}{1+i}\cdot\frac{1}{z+i}\right)$$

we establish positively oriented circles of radius r , and respectively of center $z_1 = -1, z_2 = i, z_3 = -i$ (take $r = \frac{1}{4}$), C_r are disjoint , therefore a_1, a_2, a_3 being coefficient , such that

$$\int_{C_r(z_k)} f(z)dz = \int_{C_r(z_k)} \frac{a_1 dz}{z - z_1} + \int_{C_r(z_k)} \frac{a_2 dz}{z - z_2} + \int_{C_r(z_k)} \frac{a_3 dz}{z - z_3}$$

$$= \int_{C_r(z_k)} \frac{a_k dz}{z - z_k} \text{ (by linearity)}$$

By Cauchy theorem

since $\quad z_j \notin Int\ C_r(z_k)$, $\quad j \neq k \displaystyle\int_{C_r(z_k)} \frac{a_j dz}{z - z_j} = 0$

Now , $\displaystyle\int_{C_r(z_k)} \frac{a_k dz}{z - z_k} = a_k \int_{C_r(z_k)} \frac{dz}{z - z_k}$.

Take $\ z = z_k + re^{it}\ ,0 \ \leq t\ \leq 2\pi$

$$\int = \int_0^{2\pi} \frac{re^{it} i dt}{re^{it}} = \int_0^{2\pi} i dt = 2\pi i$$

$$\text{then , } \int_{C_r(z_k)} \frac{a_k dz}{z - z_k} = 2\pi i a_k$$

a) All $C_r(z_k)$ defined above an in Int C .Hence by above theorem , we have

$$\int_C \frac{dz}{(z+1)(z^2+1)} = \frac{1}{2}\left(\int_{C_r(-1)} \frac{dz}{z+1} + \frac{1}{i-1}\int_{C_r(i)} \frac{dz}{z-i} - \frac{1}{i+1}\int_{C_r(-i)} \frac{dz}{z+i}\right)$$

$$= \frac{1}{2}(2\pi i + \frac{1}{i-1}2\pi i - \frac{1}{i+1}2\pi i) = \pi i(1-1) = 0$$

b) Only $C_r(i)$ and $C_r(-i)$ are Int C , hence

$$\int_C \frac{dz}{(z+1)(z^2+1)} = \frac{1}{2}\left(\frac{1}{i-1}\int_{C_r(i)} \frac{dz}{z-i} - \frac{1}{i+1}\int_{C_r(-i)} \frac{dz}{z+i}\right)$$

$$= \frac{1}{2}(2\pi i\,(\frac{1}{i-1} - \frac{1}{i+1}) = -\pi i$$

c) $C_r(-1) \in Int\ C$, then

$$\int_C \frac{dz}{(z+1)(z^2+1)} = \frac{1}{2}\left(\int_{C_r(-1)} \frac{dz}{z+i} \right) = \frac{1}{2}(2\pi i) = \pi i$$

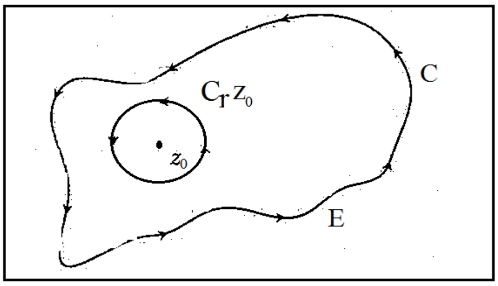
The Cauchy formula

Theorem 3.8 . Let f be analytic on E and $C \subset E$ be positively oriented simple contour . Then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz = f(z_0), \qquad \forall\, z_0 \in \text{Int C}.$$

Proof. Since $\frac{f(z)}{z - z_0}$ analytic on $E/\{z_0\}$, $C \sim C_r(z_0)$ for some r .

By Theorem $(*)$



$$\int_C \frac{f(z)dz}{z - z_0} = \int_{C_r(z_0)} \frac{f(z)dz}{z - z_0} = \int_{C_r(z_0)} \frac{f(z) - f(z_0)}{z - z_0} dz + f(z_0) \int_{C_r(z_0)} \frac{dz}{z - z_0},$$

then we have

$$\left| \int_{C_r(z_0)} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \le \int_{C_r(z_0)} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |dz| \le \max_{z \in C_r(z_0)} |f(z) - f(z_0)| \int_{C_r(z_0)} \frac{|dz|}{|z - z_0|}$$

$$\to o \quad \text{as } r \to 0$$

take $r \to 0$, we have

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) + \underbrace{\lim_{r \to 0} \int_{C_r(z_0)} \frac{f(z) - f(z_0)}{z - z_0} dz}_{\to 0}$$

LECTURE 4

Corollary 4.1 . If f is analytic on $D_r(z_0)$ and continuous on $\overline{D}_r(z_0)$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)d\theta \ .$$

Example. Evaluate $\int_c \frac{dz}{(z+1)(z^2+1)}$ of as in above example by using Cauchy formula

solution .Let $f_1(z) = \frac{1}{z^2+1}$, $f_2(z) = \frac{1}{(z+1)(z+i)}$, $f_3(z) = \frac{1}{(z+1)(z-i)}$

$$C_1 = C_r(-1)\ ,\quad C_2 = C_r(i)\ \ , C_3 = C_r(-i) \quad (r \text{ small enough })$$

therefore, C_1 , C_2 and C_3 are disjoint , also note that f_k is analytic on $C_k \cup \text{Int}\, C_k$, k = 1,2,3,...

$$\int_C f(z)dz = \int_{C_1} \frac{f_1(z)}{z+1}dz + \int_{C_2} \frac{f_2(z)}{z-i}dz + \int_{C_3} \frac{f_3(z)}{z+i}dz = 2\pi i\left(f_1(-1) + f_2(i) + f_3(-i)\right)$$

$$= 2\pi i \int_C \frac{1}{2} + \frac{1}{1-i} - \frac{1}{i+1} = \pi i(1-1) = 0$$

b) , c) left exercise .

The Maximum Modulus Principle

Definition 4.2. Let D be any subset of $\mathbb{C}$. A complex function defined on D is said to have (local) maximum modulus of $a \in D$, if there exists a $\delta > 0$ such that $D_\delta(a) \subset D$ and $|f(z)| \leq |f(a)|$ for all $z \in \epsilon_\delta(a)$, a minimum of $|f|$ is similar defined .

Theorem 4.3. (M.M.P.) suppose that f is analytic in D and $a \in D$ such that $|f(z)| \leq |f(a)|$ holds for all $z \in D$. Then f is constant .

Example. Let $f(z) = \sin z$ and $A = \{z : |z| \leq 1\}$. Then

$$|f(x+iy)| \leq |\sin x||\csc ecy|$$

and so $|f|$ attains its maximum value

$$1 + \frac{1}{3i} + \frac{1}{5i} + \cdots \text{, on A at } a = -1$$

Similarly we see that $|g|$, where $g(z) = e^z$, attains its maximum value e on A at $a = 1$.

Note that the minimum value of $|f|$ is attained at on interior point of E with out f being constant.

If we take $f(z) = z$ for $z \in E_r$. Then

$$|f(z)| = |z| \geq 0 = |f(0)|$$

then the minimum value attains at the origin .

The maximum value of $f(x + iy) = \sqrt{x^2 + y^2}$ is attained at the boundary point $|z| = r$.

Theorem 4.4.(Maximum M. T.).Let f be analytic function in boundary domain D and continuous on $\overline{D}$. Then $|f|$ attains its maximum on boundary ∂D of D .

Proof. We know that a continuous function on a compact set is boundary, therefore by hypothesis f is boundary on $\overline{D}$ and the maximum value of $|f|$ is attained at some point of $\overline{D}$ .By M.M.P. if cannot be in D so it must be on boundary ∂D .

Note : In M.M.T. that D is boundary cannot be dropped for instance if we consider

$$f(z) = e^{-iz}\ with\ D = \{z : Imz > 0\},$$

then $|f(\zeta) = 1$ on $\partial D = \{\zeta : Im\zeta = 0\}$.

But for $z = x + iy \in D$

$$|f(x + iy)| = e^y \to \infty\ as\ y \in \mathbb{R}, y \to +\infty$$

i.e. f itself note boundary .

Another example , see

$$f(z) = e^{e^z}, \quad z \in D = \{z : |Im\, z| < \tfrac{\pi}{2}\}. \text{ Then}$$

for $\quad a + ib \in \partial D = \{\zeta : |Im\, \zeta| = \frac{\pi}{2}\}$

$$\left| f(a+ib) = \left| e^{e^{a \mp i\frac{\pi}{2}}} \right| \right| = \left| e^{\mp ie^{a}} \right| = 1$$

Again M.M.T. fails

if $z = x \in \mathbb{R} \subset D, \quad f(x) = e^{e^x} \to \infty \quad as\ x \to +\infty$.

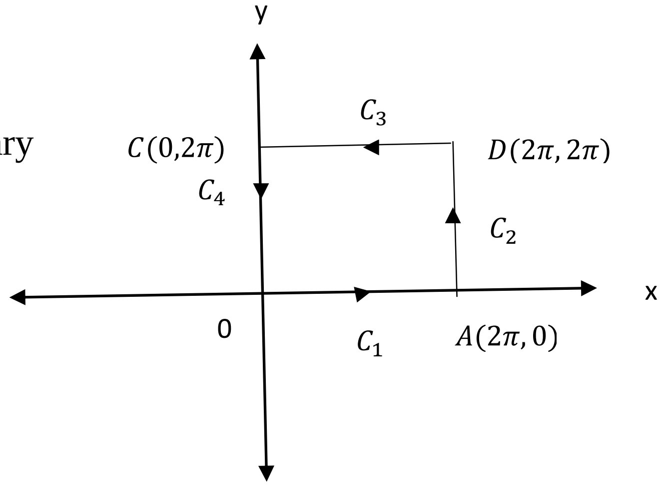
Example: find $\max_{z\in \bar{D}} |f(z)|$ for

a) $f(z) = cosz$ with $D = \{z = x + iy : 0 < x, y < 2\pi\}$.
b) $f(z) = cosz$ for $z \in \mathbb{C}$.

Solution

a) $|f(z)| = \sqrt{sinh^2y + cos^2x}$

by M.M.T , maximum attain on boundary



$$\partial D = [0,2\pi] \cup [2\pi, 2\pi + 2\pi i] \cup [2\pi + 2\pi i, 2\pi] \cup [i2\pi, 0].$$

For $z \in [0,2\pi]$, we have $z = x + i(0)$ with $0 \le x \le 2\pi$,

then $|cosz|$ has maximum 1 at $z = 0,2\pi$.

For $z \in [2\pi, 2\pi + 2\pi i]$, we have $z = 2\pi + iy$ with $0 \le y \le i\pi$,

then $|cosz|$ has maximum $\sqrt{1 + sinh^2(2\pi)}$ at $z = 2\pi + 2\pi i$, since $sinhy$ increasing of y .

For $z \in [0 + 2\pi]$, we have $z = x + 2\pi i$ with $0 \le x \le 2\pi$,

then $|cosz|$ has maximum $\sqrt{1 + sinh^2(2\pi)}$ at $z = 0 + 2\pi i\ , \pi + 2\pi i$.

For $z \in [2\pi i, 0]$, we have $z = 2\pi + iy$ with $0 \le y \le i\pi$,

then $|cosz|$ has maximum $\sqrt{1 + sinh^2(2\pi)}$.

Hence $\max_{z\in \bar{D}} |\cos z| = \sqrt{1 + sinh^2(2\pi)} = cosh2\pi$.

b) **H.W**

Theorem 4.5 (Minimum M. T.).

If is a non-constant analytic function in a bounded D and $f(z) \neq 0$ for any $z \in D$, that $|f|$ cannot attain its minimum in D .

Theorem 4.6 (open mapping Theorem)

A non-constant analytic function maps open sets onto open sets .

Schwarz Lemma .

If f is analytic and satisfies $|f(z)| < 1$ in E and $f(0) = 0$**,** then $|f(z)| \leq |z|$ for each $z \in E$ and $|f'(0)| \leq 1$.

Theorem 4.7 . Let $f: E \to E$ be analytic having n zeros of the origin .Then

i) $|f(z)| \leq |z|^n$ for all $z \in E$
ii) $|f^n(0)| \leq n!$

and equality in (i) and (ii) for some point $z_0 \neq 0$ occurs iff

iii) $f(z) = \epsilon z^n$ with $|\epsilon| = 1$.

Note that if $n = 1$, then we obtain Schwarz Lemma .

Proof .

Since f has $n - th$ order zero at the origin , the function g defined by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & , \quad \text{for } z \in E/\{0\} \\ f^n(0)\big/ n! & \quad \text{for } z = 0 \end{cases}$$

is analytic in E . Let ζ such that $0 < |\zeta| < 1$ and choose r , such that $|\zeta| < r < 1$. Since $|f(z)| < 1$ for every $z \in E$, the M.M.P. applied to $g(z)$ yields

$$|g(\zeta)| \leq \max_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)| = \max_{|z|=r} \left| \frac{f(z)}{z^n} \right| \leq \frac{1}{r^n} .$$

Since $g(\zeta) \leq \frac{1}{r^n}$ for each r , then $g(\zeta) \leq 1$

therefore, $g(\zeta) \leq 1$ for each $z \in E$ and this is same.

a) (i). Since $\left|\frac{f^n(0)}{n!}\right| = |g(z)|$ and $g(0) \leq 1$.

ii) follows.

In case $|f(z_0)| = |z_0|^n$ for some z_0 with $0 < |z| < 1$, then $|g(z_0)| = 1$. Therefore $|g(z)|$ achieves its maximum modulus at z_0 . So g must reduce to a constant ; that is f takes the form $f(z) = \epsilon z^n$, when $|\epsilon| = 1$. Then same argument at $z_0 = 0$, shows (iii) holds when $|f'(0)| \leq n!$.

LECTURE 5

Consequences of the Cauchy formula

Lemma 5.1. For all $z \in D$, we have

$$\frac{1}{(1-z)^n} = \sum_{k=1}^{\infty} \frac{(n+k-1)!}{k!\,(n-1)!} z^k$$

Proof. The geometric series

$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ is uniformly convergent on $D_r(0)$ for $r < 1$, then by the following (*)

and $\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{1-z}\right) = \frac{(n-1)!}{(1-z)^n}$, therefore, the assertion is follows:

$\Big\{$ (*) If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $D_R(z_0)$ then $f(z)$ is infinitely differentiable on $D_R(z_0)$ and

$f^k(z) = \sum_{n=0}^{\infty} n(n-1).......(n-k+1)a_n(z-z_0)^{n-k}$, k = 1,2,.... prove that $\Big\}$

Theorem 5.2. Let f be analytic on a simply connected region E and $C \subset E$ be a positively oriented simple contour . Then f is differentiable on E infinitely many times and

$$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta .$$

Corollary 5.3. (Morera's Theorem). Let f be continuous on a simply connected domain E and $\int_C f(z)dz = 0$ for all $C \subset E$. Then f is analytic on E.

Liouville's Theorem :

If f is bounded and entire then f is constant an immediate consequences of Lioville's Theorem we conclude that if f is analytic in the extended complex plane , then f is constant . This is due to the fact that if f is analytic at $z = \infty$, then $\lim_{|z|\to\infty} f(z)$ is finite . Let this limit be L

i.e. given $\epsilon > 0$ $\exists R > 0$ such that $|f(z)| - |L| \leq |f(z) - L| < \epsilon$ where $|z| > R$, and so, in particular f is bonded for $|z| > R$ and thus by continuity of f on compact set $\{ z: |z| \leq R \}$, f is bounded on the whole od $\mathbb{C}$. Hence f is constant .

Another interesting application of Lioville's Theorem is that f is entire and $|f(z)| > M > 0$, M is fixed for all $z \in \mathbb{C}$ then f is constant . This because the given condition imply that $f'(z)$ exists , $f(z) \neq 0$ in $\mathbb{C}$, and so

$$\frac{d}{dz}\left(\frac{1}{f(z)}\right) = -\frac{f'(z)}{f^2(z)}$$

exists for all $z \in \mathbb{C}$. Thus $\frac{1}{f(z)}$ is analytic on $\mathbb{C}$ and $\left|\frac{1}{f(z)}\right| < \frac{1}{M}$ for all $z \in \mathbb{C}$. Now applying Lioville's Theorem to $\frac{1}{f(z)}$, therefore f is constant .

Taylor series

Definition 5.4. We say a sequence of function $\{f_n\}$ defined on C converges uniformly to f if

$$\sup|f_n(z) - f(z)| \to 0 \text{ as } n \to \infty$$

Theorem 5.5. Let $\{f_n(z)\}$ be a sequence of continuous function on a contour C of finite length . Then f_n converges uniformly implies that

$$\int_C f_n(z)dz \to \int_C f(z)dz \qquad , n \to \infty$$

Proof. Note that

$$\left| \int_C f_n(z)dz - \int_C f(z)dz \right| = \left| \int_C (f_n(z)dz - f(z))dz \right|$$

$$\leq \int_C |f_n(z) - f(z)||dz|$$

$$\leq \int_C \sup_{z\in C} |f_n(z) - f(z)||dz| \leq \int_C \sup_{z\in C} |f_n(z) - f(z)||C|$$

$$\to 0 \text{ as } n \to \infty$$

Corollary 5.6. Let C be a contour of finite length and $\{f_n\}$ be a sequence of continuous functions on C . If $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on C , then

$$\int_C \sum_{n=0}^{\infty} f_n(z)dz = \int_C \sum_{n\geq 0}^{\infty} f_n(z)dz$$

Proof . Exercise

Theorem 5.7. (Taylor Theorem): Every analytic function on a disk $D_r(z_0)$ can be uniquely expanded into the Taylor series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $a_n = \frac{1}{n!} f^n(z_0)$ and the above series in absolutely convergent $D_r(z_0)$

LECTURE 6

Theorem 6.1. (Taylor Theorem): Every analytic function on a disk $D_r(z_0)$ can be uniquely expanded into the Taylor series $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n = \frac{1}{n!} f^n(z_0)$ and the above series in absolutely convergent $D_r(z_0)$

Proof.

Let $g(z) = f(z+z_0)$, its clear that g will be analytic on $D_r(0)$ and by Cauchy's Theorem

$$g(z) = \frac{1}{2\pi i} \int_{C_r(0)} \frac{g(\zeta)}{\zeta - z} d\zeta \qquad \forall z \in D_r(0) \ , \ r < R$$

Now

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1-\frac{z}{\zeta}} = \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n \ , \left|\frac{z}{\zeta}\right| < 1$$

since $\sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n$ is uniformly converge to $\frac{1}{1-\frac{z}{\zeta}}$ then

$$\frac{1}{2\pi i} \int_{C_r(0)} \frac{g(\zeta)}{\zeta - z} d\zeta \ = \frac{1}{2\pi i} \int_{C_r(0)} \sum_{n=0}^{\infty} \frac{g(\zeta)}{\zeta} \left(\frac{z}{\zeta}\right)^n d\zeta$$

by above Corollary $= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(0)} \frac{g(\zeta)}{\zeta} d\zeta \right) z^n = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \frac{g^n(0)}{n!}$.

Thus , $g(z) = \sum_{n=0}^{\infty} a_n z^n$.

Recall, $f(z)=g(z+z_0)$ and note that $g^n(0)=f^n(z_0)$.

Uniqueness Exercise .

Laurent series

Suppose that f not defined , or not analytic at a point z_0. Then we cannot express if in neighborhood at z_0 as a convergent power series expansion at the form

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for, if we could do so then f would be analytic at z_0 . A series at the form

$\sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$ can be thought of as a power series in the variable $\dfrac{1}{z-z_0}$ letting $\zeta=\dfrac{1}{z-z_0}$, then the above series we will be of the form $\sum_{n=0}^{\infty} b_n \zeta^n$.

Definition 6.2. A Laurent series about z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} A_n (z-z_0)^n=\sum_{n=0}^{\infty} A_n (z-z_0)^n+\sum_{n=1}^{\infty} A_{-n} (z-z_0)^{-n}$$

which analytic function in annulus $R_1<|z-z_0|<R_2$. As a motivation for Laurent series , we consider

$$f(z)=\frac{1}{(z-a)(z-b)} \quad , a\neq b .$$

Then f is analytic every where except $z=a,b$, then we cannot express if in the neighborhood of a as a convergent series of positive power of $z-a$, so

$$f(z)=\frac{1}{a-b}\left[\frac{1}{(z-a)(z-b)}\right] \quad \text{.If } 0<|a|<|z|<|b| \text{, then } \left|\frac{z}{b}\right|<1, \left|\frac{a}{z}\right|<1.$$

$$f(z)=\frac{1}{a-b}\left[\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n\right]$$

$$=\frac{1}{a-b}\left[\sum_{n=1}^{\infty}\frac{a^{n-1}}{z^n}-\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}\right]=\frac{1}{a-b}\left[\sum_{n=-\infty}^{-1}a^{-n-1}z^n+\sum_{n=0}^{\infty}\frac{-z^n}{b^{n+1}}\right]$$

therefore, $f(z)=\sum_{n=-\infty}^{\infty}A_n z^n$, where

$$A_n=\begin{cases}-\dfrac{1}{(a-b)b^{n-1}} & \text{if } n\geq 0\\ \dfrac{1}{(a-b)a^{n+1}} & \text{if } n\leq -1\end{cases}$$

is Laurent series of f .

If $|z|>|b|>|a|$ (then $\left|\frac{b}{z}\right|<1,\ \left|\frac{a}{z}\right|<1$), then we have

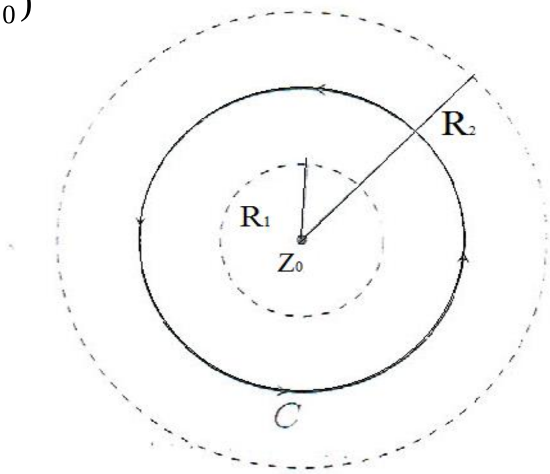
$$f(z)=\frac{1}{a-b}\left[\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{b^n}{z^{n+1}}\right]=\frac{1}{a-b}\left[\sum_{n=-\infty}^{-1}\frac{a^n-b^n}{z^{n+1}}\right]\ .$$

Theorem 6.3. (Laurent Theorem).

Any function $f(z)$ that is analytic on an annulus $R_1<|z-z_0|<R_2$ can be expressed into the Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n \text{ where } a_n=\frac{1}{2\pi i}\int_C\frac{g(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\quad ,n\in Z$$

where , C is any contour like .



Proof.

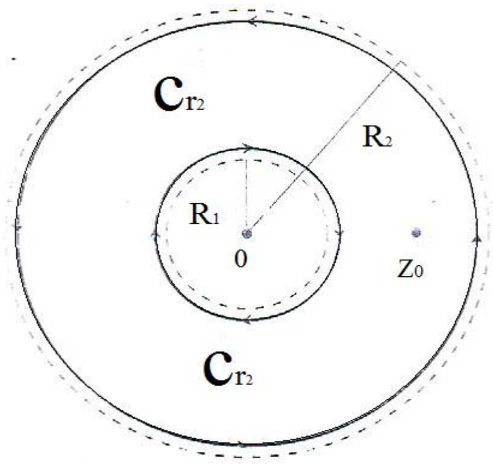
Let $g(z)=f(z+z_0)$, $R_1<|z-z_0|<R_2$, By Cauchy formula

$$g(z)=\frac{1}{2\pi i}\int_{C_{r_2}\cup(-C_{r_1})}\frac{g(\zeta)}{\zeta-z}d\zeta \;=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{g(\zeta)}{\zeta-z}d\zeta \;+\frac{1}{2\pi i}\int_{-C_{r_1}}\frac{g(\zeta)}{\zeta-z}d\zeta \;=g_1(\mathrm{z})+\mathrm{g}_2(\mathrm{z})$$

where C_{r_1} and C_{r_2} an a like in .

But we know that

$$\frac{1}{\zeta-z}=\frac{1}{\zeta}\frac{1}{1-\frac{z}{\zeta}}=\frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^{n}.$$



$C_{r_2}\cup(-C_{r_1})$

If $\zeta\in C_{r_2}$, then $\left|\frac{z}{\zeta}\right|<1$ and $\frac{1}{\zeta-z}$ uniformly convergent on $IntC_{r_2}$ and hence

$$g_2(z)=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{g(\zeta)}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^{n}d\zeta \quad =\sum_{n=0}^{\infty}a_n z^n$$

where $$a_n=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{g(\zeta)}{\zeta^{n+1}}d\zeta \quad =\frac{\mathrm{g}^{\mathrm{n}}(0)}{n!},\, n\geq 0$$

If $\zeta\in C_{r_1}$, then $\left|\frac{\zeta}{z}\right|<1$ and $\frac{1}{\zeta-z}=-\frac{1}{z}\cdot\frac{1}{1-\frac{\zeta}{z}}=-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{\zeta}{z}\right)^{n}$ converges uniformly on $ExtC_{r_1}$ ($ExtC=\mathbb{D}\backslash\overline{Int\ C}$) and hence

$$g_1(z)=\frac{1}{2\pi i}\int_{-C_{r_1}}g(\zeta)\left(-\frac{1}{\zeta}\right)\sum_{n=0}^{\infty}\left(\frac{\zeta}{z}\right)^{n}d\zeta \quad =-\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{-C_{r_1}}g(\zeta)\,\zeta^{n}\quad d\zeta\right)z^{-(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_{r_1}}\frac{g(\zeta)}{\zeta^{-n+1}}\quad d\zeta\right)z^{-n}$$

then substitute in $g(z)$, we have

$$g(z) = g_1(z) + g_2(z) = \sum_{n=-\infty}^{\infty} a_n z^n \text{ , where}$$

$$a_n = \frac{1}{2\pi i} \begin{cases} \displaystyle\int_{C_{r_1}} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta \ , \ n \leq -1 \\ \displaystyle\int_{C_{r_2}} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta \ , \quad n \geq 0 \end{cases}$$

Note: two contours are said to be equivalent with respect to connected region ϵ if one can be continuous deformed in E in to the other

But $C_{r_1} \sim C_{r_2} \sim \mathrm{C}$, where C is contour in $R_1 < |z| < R_2$ and hence

$$a_n = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta \ , \ n \leq -1$$

therefore, the Laurent replantation of an analytic function is not unique and depends on the of choice of annulus for example

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \qquad 0 < |\mathrm{z}| < 1$$

$$= \sum_{n=1}^{\infty} z^n \quad , \qquad 1 < |\mathrm{z}| < \infty$$

LECTURE 7

Classification of singularities

Definition 7.1: A point z_0 is called an isolated singularity of an analytic function $f(z)$ if there exists some $R > 0$ such that f is analytic on $D_R(z_0)[z_0]$ {puncture disk}.

If a function $f(z)$ ha an isolated singularity at z_0 , then by the Laurent Theorem if can be expressed as

$$f(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = f_1(z) + f_2(z)$$

on some annulus $D_R(z_0)[z_0]$.

Now, if

1) If $f_1(z) = 0$, then z_0 is called removable singularity

2)If $f_1(z) = \sum_{n=1}^{N} a_{-n} z^{-n}, N < \infty$ and $a_{-N} \neq 0$, then z_0 is called a pole of order N

3)If $f_1(z)$ has infinitely many non-zeros terms , then z_0 is called an essential singularity .

Example. Discuss the singularities of $g(z) = \dfrac{1}{f(z)}$, where

$$f(z) = u(x,y) + iv(x,y) = \sin\left(\frac{x}{|z|^2}\right)\cosh\left(\frac{y}{|z|^2}\right) - i\cos\left(\frac{x}{|z|^2}\right)\sinh\left(\frac{y}{|z|^2}\right)$$

Solution.

f is analytic every where except $z = 0$, therefore, the singularity of g at $z = 0$ and the points where $f(z) = 0$.

i.e. when $u(x,y) = 0$ and $iv(x,y) = 0$

since $\cosh\left(\dfrac{y}{|z|^2}\right) > 1$, $u(x,y) = 0$ implies $\sin\left(\dfrac{x}{|z|^2}\right) = 0$.........(1)

in this case $\cos\left(\dfrac{x}{|z|^2}\right) \neq 0$ .Thus $v(x,y) = 0$, implies that $\sinh\left(\dfrac{y}{|z|^2}\right) = 0$.........(2) .

Thus (1) $\frac{x}{|z|^2} = n\pi, \quad n = 0, \mp 1, \mp 2, \ldots\ldots$

And (2) yields $= 0$.

Further, $y = 0$ gives $x = \frac{1}{n\pi}$ for n=0, $\mp$1,.... . This means that $u(x,y) = 0$ and $iv(x,y) = 0$ holds if and only if $x = \frac{1}{n\pi}$ for n=0, $\mp$1,.... . Thus the singularities of g are at points $x = \frac{1}{n\pi}$ for n=0, $\mp$1,.... , and their limit point $z = 0$ .Thus g has an isolated singularities at $z = \frac{1}{n\pi}$ and non-isolated singularities at the limit point $z = 0$.

Analytic Continuation

In an important concept because if provides a method for making the domain of definition of an analytic functions as large as possible . Usually analytic functions are defined by means of some mathematical expressions such as polynomial infinite series , integrals etc. The domain of definition of such an analytic is often representation as such of analytic function dose not provide any direct information as to whether we could have a function analytic in a domain larger that the circular domain of convergence which coincides with the given function . The make this point more precise, let us start by examining the analytic continuation of the function

$f(z) = \sum_{n=0}^{\infty} z^n$, it is convergent for $|z| < 1$ and diverges for $|z| > 1$. On the other hand the sum of series for $|z| < 1$ is $\frac{1}{1-z}$. Now $F(z) = \frac{1}{1-z}$ defined for all values of $z \neq 1$ and analytic for $\check{\mathbb{C}}\{1\}$ $(f\left(\frac{1}{z}\right) = (1 - z^{-1})^{-1} = \frac{z}{z-1}$ is analytic at 0 , $F(z)$ analytic at ∞).

$f(z) = F(z)$ for $z \in \check{\mathbb{C}}\{1\} \cap D = \{z: |z| < 1\}$, and we call F analytic continuation of f from D into $\check{\mathbb{C}}\{1\}$.

Next, we consider $G(z) = \int_0^{\infty} \exp[(z-1)t]dt$.

If $Re\, z < 1$, then $G(z) = \left.\dfrac{e^{(z-1)t}}{z-1}\right|_0^\infty = \dfrac{1}{1-z}$.

Thus the integral is convergent in the half-plane $H = \{z\colon Re\, z < 1\}$ and represent the sum function $\dfrac{1}{1-z}$ for $z \in H$, we have $F(z) = G(z)$ for $z \in H \cap D$, F the continuation of point at $z = 1$.

Definition 7.2. Let f and F are two functions such

1) f is defined and analytic on E .
2) F is defined and analytic in E , such that $E_1 \cap E \neq \varphi$ and $E_1 \supset E$
3) $f(z) = F(z)$ for $z \in E \cap E_1$.

Then we call F the analytic continuation of f from E into E_1 .

Example. If $E_1 = \mathbb{C} \setminus \{z\colon Re\, z \geq 1, Im\, z = 1\}$, $Log(1-z)$ is the analytic continuation of the power series $\sum_{n=1}^{\infty} \dfrac{z}{n}$ D into E_1.

<u>Calculus of Residues</u>

Definition 7.3. Let z_0 be isolated singularity of C simply connected contour enclosing z_0 and laying in the domain of analyticity of f . Then

$$\frac{1}{2\pi i}\int_C f(z)dz = \mathrm{Res}\left[f(z), z_0\right].$$

Theorem 7.4 . If f has a removable singularity of z_0 , then $\mathrm{Res}\left[f(z), z_0\right] = 0$

Proof .

Since f has removable singularity at z_0 , then there is

$$g(z) = \begin{cases} f(z) \;, & 0 < |z - z_0| < \delta \\ \lim\limits_{z \to z_0} f(z), & z = z_0 \end{cases}$$

Hence by Cauchy Theorem

$$\int_C g(\zeta)d\zeta = 0, \quad C = \{\zeta : |\zeta - z_0| = r < \delta\}$$

Clearly $g(z) = f(z)$ for $z \in C$. Then $\int_C f(z)dz = 0.$, then $\operatorname{Res}[f(z), z_0] = 0$.

Theorem 7.5. If f has a pole of order n at z_0, then

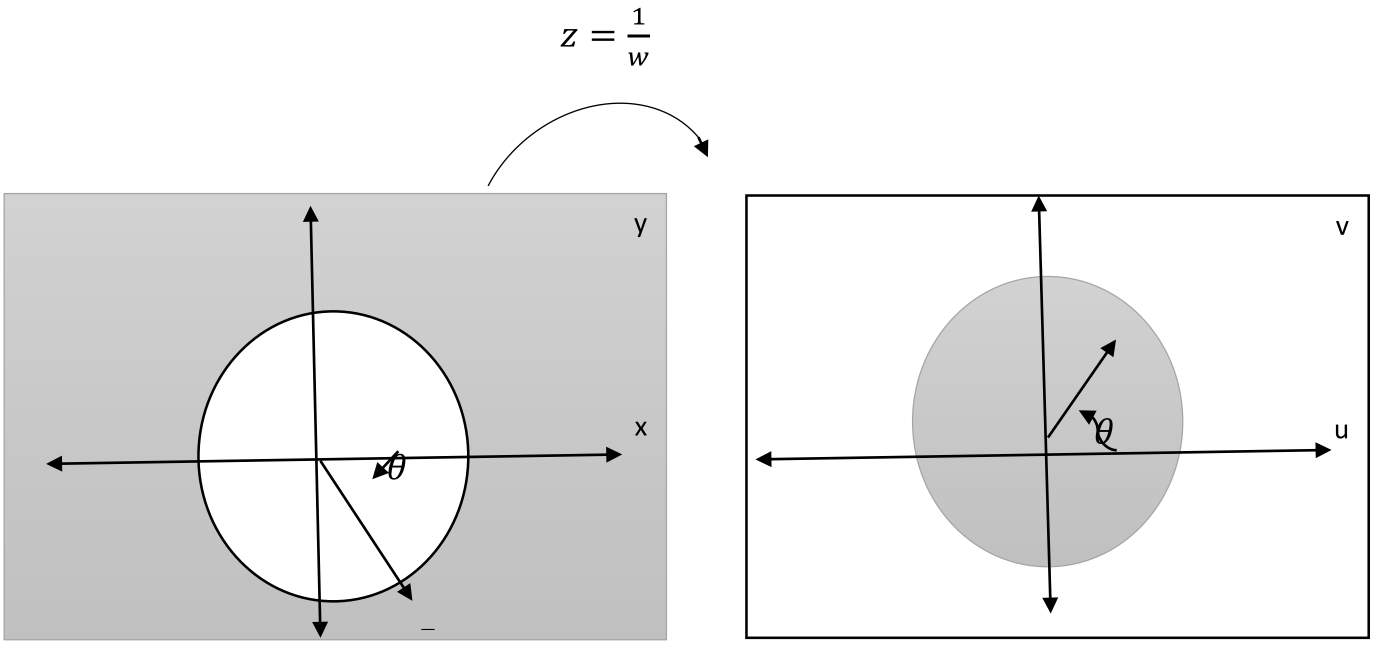
$$\operatorname{Res}[f(z), z_0] = \frac{1}{n-1} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \Big]_{z - z_0}$$

Proof. Exercise

Corollary 7.6 . If z_0 is a simple pole , then $\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

Residue of the point at infinity

Consider $z = \frac{1}{w}$, if we set $z = Me^{-i\theta}$, then $w = M^{-1}e^{i\theta}$. Thus the point $z_0 = \rho e^{-i\theta}$, $\rho > M$ outside $|z| = M$ corresponds to a point $w_0 = \rho^{-1}e^{i\theta}$ inside $|w| = \frac{1}{M}$ (as in figure)



z-plane w-plane

Let f be analytic in a deleted neighborhood of the point at infinity . Then by Laurent

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \ , \ |z| > R, \ 0 \le \mathrm{R} < \infty$$

Define $C' = \{z : |z| = M > R, M$ is sufficiently Laurent$\}$

Put $z = Me^{-i\theta}$, . Then $\int_C f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_C z^k dz$

$$= \sum_{k=-\infty}^{\infty} ia_k M^{k+1} \int_0^{2\pi} e^{-i(k+1)\theta} d\theta = -ia_{-1} \int_0^{2\pi} d\theta = -2\pi i a_{-1} .$$

That is $\operatorname{Res}[f(z),\infty] = \frac{1}{2\pi i} \int_C f(z)dz = -a_{-1}$

Further we observe that $z = Me^{-i\theta}$ with $M = \frac{1}{R'}$

$$\operatorname{Res}[f(z),\infty] = \frac{1}{2\pi i} \int_C f(z)dz = -\frac{1}{2\pi i} \int_0^{2\pi} f(Me^{-i\theta}) iMe^{-i\theta} d\theta$$

$$= -\frac{1}{2\pi i} \int_0^{2\pi} f\left(\frac{1}{R'e^{i\theta}}\right) \frac{d(R'e^{i\theta})}{\left(R'e^{i\theta}\right)^2}$$

$$= \frac{1}{2\pi i} \int_C f\left(\frac{1}{w}\right) \frac{dw}{w^2} = -\operatorname{Res}\left[\frac{f(\frac{1}{w})}{w^2}, 0\right]$$

where $C = \{w : |w| = \frac{1}{R'}$ is described in the anti-clock wise direction .

Hence , $\operatorname{Res}[f(z),\infty] = -\operatorname{Res}\left(\frac{f(\frac{1}{z})}{z^2}, 0\right)$

Example. Consider the function $f(z) = 1 + z^{-1}$. Then $F(w) = f\left(\frac{1}{w}\right) = 1 + w$ and $\lim_{w \to 0} F(w) = 1$. Thus F has removable singularity at $w = 0$ and therefore the point at infinity is a removable singularity of f . Further we have $\operatorname{Res}[f(z),\infty] = -1$

Exercise . (1) prove Liouville's Theorem by using the above information .

(2) Determine the residue of all singularity of f ,

$$f(z)=\frac{z^n e^{\frac{1}{z}}}{1+z},\ n\in\mathbb{N}.$$

Residue Theorem

If f is analytic in a domain E except for isolated singularities $a_1,a_2....a_n$, then any closed curve γ in $E_\gamma$on which none of the points a_k lie we have $\int_\gamma f(z)dz=2\pi i\sum_{k=1}^{n} n(\gamma;a_k)\operatorname{Res}[f(z);a_k]$

$$n(\gamma;a_k)\begin{cases} o & \text{if } a_k \text{ is in the unbounded component of } C\backslash\{\gamma\} \\ 1 & \text{if } a_k \text{ is inside } \gamma \end{cases}$$

Now, if γ is a simple closed curve then under the hypothesis of above Theorem , we have .

Theorem 7.7. $\int_\gamma f(z)dz=2\pi i\sum_{k=1}^{n} n(\gamma;a_k)\operatorname{Res}[f(z);a_k]$item the sum is taken over all a_k inside γ .

Theorem 7.8. Let f be analytic with the exception of finitely many isolated singularities at a_k in the extended complex plane .Then the sum of all residues (residue at infinity included of f equal zero.)

Example . Evaluate $1=\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{2n+3m-1}}{(z^2+a)^n(z^3+b)^m}dz$, $a,b\in\mathbb{C}\{0\}$,

where $R>\max\{\sqrt{|a|},|b|^{\frac{1}{3}}\}$, $n\in\mathbb{Z}$.

solution. Let $f(z)=\frac{z^{2n+3m-1}}{(z^2+a)^n(z^3+b)^m}$

$z^2+a=0$ z_1,z_2 are poles of order n .

$z^3+b=0$ z_3,z_4,z_5 are poles of order m , therefore

$$I=\sum_{j=1}^{5}\operatorname{Res}(f(z);z_j)$$

by above theorem , we have $\sum_{j=1}^{5} \operatorname{Res}(f(z);z_j) + \sum_{j=1}^{5} \operatorname{Res}(f(z);\infty) = 0$.

As calculation of the residue is quite difficult , thus

$$I = \operatorname{Res}(f(z);\infty) = \operatorname{Res}(\frac{1}{z^2} f(\frac{1}{z});0)$$

$$\operatorname{Res}\left[\frac{1}{a(1+az^2)^n(1+bz^3)^m},0\right] = 1.$$

Example. $I = \dfrac{1}{2\pi i} \displaystyle\int_{|z|=2} \frac{1}{(z-3)(z^n-1)} dz$, n=1,2,...

$I = \operatorname{Res}(f(z);z_j)$, where z_j are nothing but the n-th roots of unity .

As previous example , we must have

$$\sum_{j=1}^{n} \operatorname{Res}(f(z);z_j) = -\operatorname{Res}(f(z);3) + \operatorname{Res}(f(z);\infty)$$

We note that $\operatorname{Res}(f(z);3) = \lim(z-3)f(z) = \dfrac{1}{3^n-1}$ and

$$\operatorname{Res}(f(z);\infty) = -\operatorname{Res}(\frac{1}{z^2} f(\frac{1}{z});0) = -\operatorname{Res}\left(\frac{z^{n-1}}{(1-3z)(1-z^n)};0\right) = 0.$$

Hence $I = \dfrac{1}{3^n-1}$.

Integral at type $\displaystyle\int_{\alpha}^{2\pi+\alpha} R(\cos\theta,\sin\theta)d\theta$.

$R(\cos\theta,\sin\theta)$ is a rational function of $cos\theta$ and $sin\theta$,

if $z = re^{i\theta}$, $|r| = 1$, then $\cos\theta = \dfrac{z^2+1}{2z}$, $\quad \sin\theta = \dfrac{z^2-1}{2iz}$, $d\theta = \dfrac{dz}{iz}$

$$\int_C f(z)dz = \int_C R\left(\frac{z^2+1}{2z},\frac{z^2-1}{2iz}\right)\frac{dz}{iz} = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z),a_k)$$

a_k inside C (C positive direction).

Example. Evaluates $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, a,b real , $|b| < |a|$

Solution.

$I = \frac{2}{b}\int_C f(z)dz$, $f(z) = \frac{1}{z^2 + (2iaz/b) - 1}$, C={z:|z|=1} the only singularities of f are the poles

$$-i\left(\frac{a}{b} \mp \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right) = \begin{cases} \alpha & \text{for + sign} \\ \beta & \text{for - sign} \end{cases}$$

Since $|b| < |a|$, then

i) $0 < b < a$
ii) $a < b < 0$
iii) $b < 0 < a$
iv) $a < 0 < b$

Therefore if is enough to consider either (i) and (ii) or (ii) and (iv),

since $\alpha\beta = -1$, then one root inside C

i) Suppose $0 < b < a \Longrightarrow \frac{a}{b} > 1$ β inside $|z| < 1$, then by Residue Theorem

$$I = \frac{2}{b}\int_C \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2}{b}\left[2\pi i \operatorname{Res}(f(z);\beta)\right]$$

$$\frac{2}{b}\left[2\pi i \lim_{z\to\beta}(z-\beta)f(z)\right] = \frac{4\pi i}{b}\left(\frac{1}{\beta-\alpha}\right),$$

that is $I = \frac{2\pi}{\sqrt{a^2 - b^2}}$

(iii) if $b < 0 < a \Longrightarrow \frac{b}{a} < 0$ and so the pole at $a = \alpha$ inside $|z| < 1$, using the residue theorem we have

$$I = \frac{4\pi i}{b(\alpha - \beta)}$$

LECTURE 8

Example: Evalute $I = \int_0^{2\pi} \frac{d\theta}{1+\alpha^2 - 2\alpha\cos\theta}$, if $\alpha > 0$

Solution .

$$I = \frac{i}{\alpha}\int_C f(z)dz, \qquad f(z) = \frac{1}{(2-\alpha)(1-\frac{1}{\alpha})}$$

where $C = \{z \in \mathbb{C} : |z| = 1\}$, singular points are the singular poles at $z_1 = \alpha,\ z_2 = \frac{1}{\alpha}$.

If $0 < \alpha < 1$ then $z = \alpha$ is inside |z|<1 other outside $I = \frac{i}{\alpha}\{2\pi i \operatorname{Res}(f(z), \alpha)\}$

$$= -\frac{2}{\alpha}\lim_{z\to\alpha}(z-\alpha) = \frac{2\pi}{1-\alpha^2} \quad , 0 < \alpha < 1.$$

Similarly for $\alpha > 1$, we deduce that $I = \frac{-2\pi}{1-\alpha^2}$, $\alpha > 1$. Integral of type $\int_{-\infty}^{\infty} f(x)dx$.

The improper integral of continuous functions $f(x)$ defined on $[0, \infty)$ is defined by $\int_0^{\infty} f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$.

If $f(x)$ is continuous for all , its improper integral over $-\infty < x < \infty$ is defined by

$I = \int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx = \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$, and both of limits exists .

We can now write $\lim_{R,S\to\infty}\int_{-R}^{S} f(x)dx$.

Now, Cauchy principle value (P.V.) of above integral $P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$, provided this single limit exists .

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}\left[\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx\right]$$

$$= \lim_{R\to\infty}\int_{-R}^{0} f(x)dx + \lim_{R\to\infty}\int_{0}^{R} f(x)dx$$

We start by the following example .

Example: Let $f(x)=\dfrac{1}{1+x^2}$ and consider $\int_{-\infty}^{\infty} f(z)dz$ with the path of integration is the line $z = y = 0$. By Cauchy integral formula $J=\int_C \dfrac{(z+i)^{-1}}{z-i}dz=\pi$

Write $J = J_1 + J_2$ with

$$J_1=\int_{-R}^{R}\frac{1}{1+x^2} \quad , J_2=\int_{\mathrm{Re}^{i\theta}}^{\mathrm{Re}^{i\pi}}\frac{dz}{1+z^2}=\int_0^{\pi}\frac{d(\mathrm{Re}^{i\theta})}{1+(\mathrm{Re}^{i\theta})^2}d\theta$$

Since $|J_2|\le\int_0^{\pi}\dfrac{R}{R^2-1}\,\mathrm{d}\theta\to 0$ as $\mathrm{R}\to\infty$, we deduce that $J=\int_{-\infty}^{\infty}\dfrac{dx}{1+x^2}=\pi$,

then $J=\int_0^{\infty}\dfrac{dx}{1+x^2}=\dfrac{\pi}{2}$.

Now, we discuss the following example

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^{2n}}, \quad n=1,2,........$$

Solution .

Observe that $\int_{-\infty}^{\infty} f(x)dx = 2\int_{-\infty}^{\infty} f(x)dx$, $\quad f(z)=\dfrac{dx}{1+z^{2n}}$

Poles of f are located at $2n-th$ roots at -1 , $a_k=e^{i\frac{(2k+1)\pi}{2n}}$, $\quad k=0,1,....,2n-1$

Let $C=[0,R]\cup\{z:z=\mathrm{Re}^{i\theta},0\le\theta\le\dfrac{\pi}{n}\}\cup\{z:z=re^{\frac{i\pi}{n}},\ 0<r<R\}$

$= [0,R]\cup\Gamma R\cup\gamma_R$.

$$\operatorname{Res}[f(z),a_0]=\lim_{z\to a_0}\frac{1}{2n\ z^{2n-1}}=\lim_{z\to a_0}\left[\frac{-2}{2n}\right]=\frac{-a_0}{2n}$$

where a_0 is a simple pole lie inside C . Then by Residue Theorem

$$\int_C f(z)dz=\left(\int_{[R,R]}+\int_{\Gamma R}+\int_{\Gamma R}\right)f(z)dz$$

$$=2\pi i\sum \operatorname{Res}(f(x),C]$$

$$=2\pi i\left[\frac{-a_0}{2n}\right]=\frac{-i\pi e^{\frac{i\pi}{2n}}}{n}$$

since $\quad |f(z)|\le\frac{1}{|z|^{2n}-1}\le\frac{2}{|z|^{2n}}$, as R $\to\infty$

we have $\quad \left|\int_{\Gamma R}f(z)dz\right|\le\frac{2}{R^{2n}}\frac{\pi R}{n}\to 0$ as R $\to\infty$

and so $\quad \int_{\Gamma R}f(x)dx\to 0$ as R $\to\infty$

Next, $\quad \int_{\gamma R}f(z)dz=-\int_0^R f\left(re^{\frac{i\pi}{n}}\right)\mathrm{d}\left(re^{\frac{i\pi}{n}}\right)=-e^{\frac{i\pi}{n}}\int_0^R\frac{dr}{1+r^{2n}}$

as above with $R\to\infty$

$$(1-e^{\frac{i\pi}{n}})\int_0^\infty\frac{dx}{1+x^{2n}}=\frac{i\pi e^{\frac{i\pi}{2n}}}{n}$$

that is $\displaystyle\int_0^\infty\frac{dx}{1+x^{2n}}=\frac{\pi}{n}\left[\frac{i}{e^{\frac{\pi i}{2n}}-e^{-\frac{\pi i}{2n}}}\right]=\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)}$

Example . $\displaystyle\int_0^\infty\frac{\sin^2 x}{x^2+1}dx$

Solution.

Consider $f(z)=\dfrac{1-e^{i2z}}{1+z^2}$ (since $\sin^2 x=(\dfrac{1-\cos 2x}{2})$ singular point $a_0=i$, $a_1=-i$

$$\operatorname{Res}[f(z),i]=\lim_{z\to i}\frac{1-e^{i2x}}{z+i}=-\frac{i}{2}\left(1-e^{-2}\right)$$

Since i is the only pole inside C , $C=[-R,R]\cup\Gamma R$

$\Gamma_R=\{z:|z|=R,\ \ 0\le anyz\le\pi\}$

then $\displaystyle\int_{-R}^{R}f(x)dx+\int_{\Gamma_R}f(z)dz=2\pi i\left[-\frac{i}{2}(1-e^{-2})\right]=\pi(1-e^{-2})$

since $\displaystyle\left|\int_{\Gamma_R}f(z)dz\right|\le\int_{\Gamma_R}|f(z)||dz|=\frac{2}{R^2-1}\int_{\Gamma_R}|dz|=\frac{2\pi R}{R^2-1}\to 0$ as $R\to\infty$

we have $\displaystyle\int_{-R}^{R}f(x)dx=\pi(1-e^{-2})$, by equation real parts

then, then, $\displaystyle\int_0^{\infty}\frac{\sin^2 x}{x^2+1}dx=\frac{\pi(1-e^{-2})}{2}$

Integration Involving Branch Cuts

Definition 8.1: A function $f(z)$ is called meromorphic if f is analytic on the whole of $\mathbb{C}$ except for poles .

Since z^{α} is multivalued , we must first specify the branch of z^{α} defined as follows $z^{\alpha}=e^{\alpha(\ln r+i\theta)}=r^{\alpha}(\cos\alpha\theta+i\sin\alpha\theta),\quad 0\le\theta\le 2\pi$.

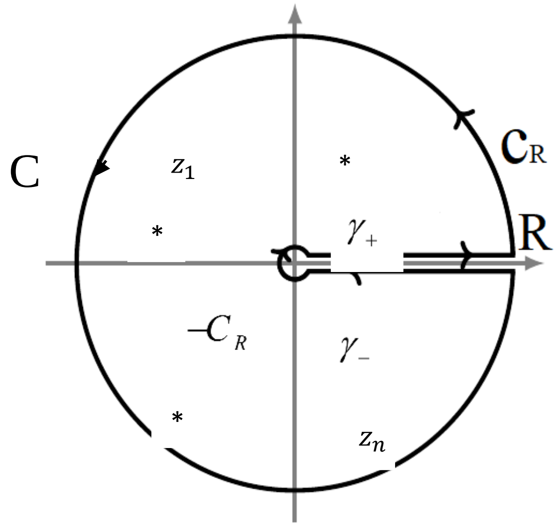
If $\alpha\in\mathbf{Z},\gamma_+$ the necessity of the branch cut disappears as $e^{i\alpha.2\pi}$, $\forall\alpha\in\mathbf{Z}$.

Proposition 8.2 . Let (x) , $x\in\mathbb{R}^+$, have meromorphic continuation $f(z)$, assume $f(z)$ has a finite number of poles $\left[z_k\right]_{k=1}^{n}$, $z_k\in\mathbb{R}^+$ and $zf(z)$ has a removable singularity at ∞ .

$$\forall\alpha\in(0,1)\qquad\int_0^{\infty}x^{-\alpha}f(x)dx=\frac{2\pi i}{1-e^{-2\pi i\alpha}}\sum_{k=1}^{n}\operatorname{Res}\left[z^{\alpha}f(z),z_k\right],\ z^{\alpha}\text{ defined on } C\setminus\square +$$

Proof .

Let $\varphi(z)=z^{-\alpha}f(z)$. Then its analytic on $C\setminus\mathbb{R}^{+}$ except of finite number of poles $[z_k]_{k=1}^{n}$. Take C as in figure



$$2\pi i\sum_{k=1}^{n}\operatorname{Res}(\varphi(z),z_k)=\left[\underbrace{\int_{C_R}}_{I_1}+\underbrace{\int_{-C_R}}_{I_2}+\underbrace{\int_{\gamma+}}_{I_3}+\underbrace{\int_{\gamma-}}_{I_4}\right]\varphi(z)dz$$

$\lim_{z\to\infty} zf(z)$ excite and $\exists M>0$ such that $|zf(z)|\le M$ for any $|z|=R$. Hence for all $z\in C_R$, $|\varphi(z)|=|z^{\alpha}||f(z)|\le|z|^{-\alpha}\dfrac{M}{|z|}=\dfrac{M}{R^{1+\alpha}}$

Therefore, $|I_1|\le\int_{C_R}|\varphi(z)||dz|\le\dfrac{M}{R^{1+\alpha}}2\pi R=\dfrac{2\pi M}{R^{\alpha}}\to 0$ as $R\to\infty$

and $\quad |I_2|\le\int_{C_\varepsilon}|\varphi(z)||dz|=\int_{C_\varepsilon}|z|^{-\alpha}|f(z)||dz|$

$$\le\sup_{|z|=\varepsilon}|f(z)|\ .\ \underbrace{2\pi\varepsilon\varepsilon^{-\alpha}}_{=2\pi\varepsilon\varepsilon^{1-\alpha}}\to 0 \text{ as } \varepsilon\to\infty$$,bounded as 0 not pole

$$|I_3|=\int_{\varepsilon}^{R}x^{-\alpha}f(x)dx$$

I_4: if $z \in \gamma_-$ then $z = xe^{2\pi}, x > 0$, then $z^{-\alpha} = x^{-\alpha}e^{-2\pi i\alpha}$, therefore

$$I_4 = \int_{\gamma_-} z^{-\alpha} f(z)dz = \int_R^{\varepsilon} x^{-\alpha} e^{-2\pi i\alpha} f(x)dx = -e^{-2\pi i\alpha} I_3$$

then $I_4 + I_3 = (1-e^{-2\pi i\alpha})I_3 = (1-e^{-2\pi i\alpha})\int_{\varepsilon}^{R} x^{-\alpha} f(x)dx$

$$= 2\pi i\sum_{k=1}^{n}\text{Res}(\varphi, z_k) - I_1 - I_2$$

then $\displaystyle\int_R^{\varepsilon} x^{-\alpha} f(x)dx = \frac{1}{1-e^{-2\pi i\alpha}}\left(2\pi i\sum_{k=1}^{n}\text{Res}(\varphi, z_k) - I_1 - I_2\right)$

take limit as $R \to \infty$, $\varepsilon \to 0$, we have the required results .

Example . $\displaystyle I = \int_0^{\infty}\frac{dx}{x^{\alpha}(x+1)}$, $0<\alpha<1$

Note that $\dfrac{1}{1+x}$, ≥ 0 , has a meromorphic continuous into $\mathbb{C}$ with one simple pole at $z = -1$

$\displaystyle\lim_{z\to\infty}\frac{z}{z+1} = 1$, by applying the above proposition

$$I = \frac{2\pi i}{1-e^{-2\pi i\alpha}}\frac{1}{(-1)^{\alpha}} = \frac{2\pi i}{e^{-\pi i\alpha} - e^{-\pi i\alpha}} = \frac{\pi}{\sin\pi\alpha}$$

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