First year/ 1st Semester - Chemical and Petroleum Engineering Department

By

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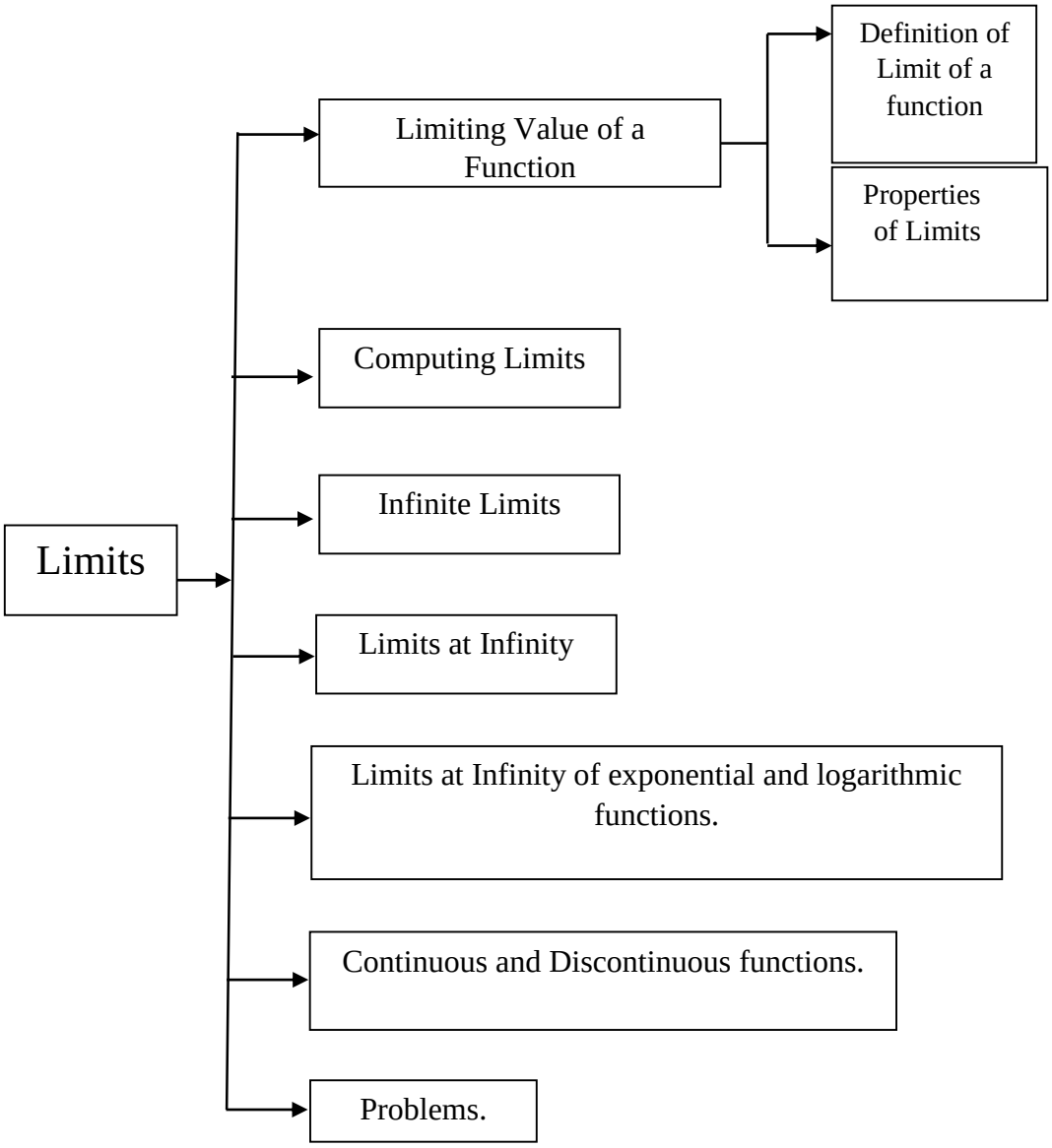
Lecture – Two ***Limits***



Limits

1- *Rates of Change*.

We want to determine just how fast *f(x)* is changing at some point, say x=a. This is called the **instantaneous rate of change** or sometimes just **rate of change** of *f(x)* at x=a.

To compute the average rate of change of *f(x)* at $x = a$ all we need to do is to choose another point, say x, and then the average rate of change will be,

$$A.R.C. = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(x) - f(a)}{x - a}$$

Example 1

Suppose that the amount of air in a balloon after *t* hours is given by

$$V(t) = t^3 - 6t^2 + 35$$

Estimate the instantaneous rate of change of the volume after 5 hours.
Solution:

$$A.R.C. = \frac{V(t) - V(5)}{t - 5} = \frac{t^3 - 6t^2 + 35 - 10}{t - 5} = \frac{t^3 - 6t^2 + 25}{t - 5}$$

To estimate the instantaneous rate of change of the volume at $t = 5$ we just need to pick values of t that are getting closer and closer to $t = 5$. Here is a table of values of t and the average rate of change for those values.

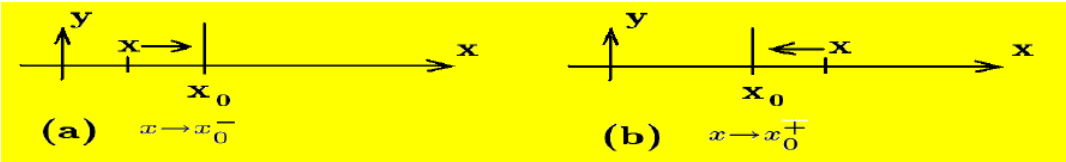
t	*A.R.C.*	*t*	*A.R.C.*
6	25.0	4	7.0
5.5	19.75	4.5	10.75
5.1	15.91	4.9	14.11
5.01	15.0901	4.99	14.9101
5.001	15.009001	4.999	14.991001
5.0001	15.00090001	4.9999	14.99910001

2- ***Limiting Value of a Function.***

The notation $\lim_{x\to x_0} f(x) = \ell$ is used to denote **the limiting value of a function** $f(x)$ **as** x **approaches the value** x_0**, but** $x \neq x_0$. Note that the limit statement $\lim_{x\to x_0} f(x)$ is dependent upon values of $f(x)$ for x near x_0, **but not for** $x = x_0$. One must examine the values of $f(x)$ both for x_0^+ values (values of x slightly greater than x_0) and for x_0^- values (values of x slightly less than x_0). These type of limiting statements are written

$$\lim_{x\to x_0^+} f(x) \qquad \text{and} \qquad \lim_{x\to x_0^-} f(x)$$

and are called **right-hand and left-hand limits** respectively.



There may be situations where

(a) $f(x_0)$ is not defined

(b) $f(x_0)$ is defined but does not equal the limiting value (ℓ).

(c) the limit $\lim_{x\to x_0} f(x)$ might become unbounded,

In which case one can write a statement stating "no limit exists as $x \longrightarrow x_0$

Some limits are easy to calculate, for example $\lim_{x\to 2}(3x+1) = 7$, is a limit of the form $\lim_{x\to x_0} f(x) = f(x_0)$, where $f(x_0)$ is the value of $f(x)$ at $x = x_0$, if the value $f(x_0)$

Some limiting values produce the **indeterminate forms** $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and these resulting forms **must be analyzed further to determine their limiting value.** For example, the limiting value $\frac{0}{0}$ may reduce to a zero value, a finite quantity or it may become an infinite quantity as the following examples illustrate.

$$\lim_{x\to 0} \frac{3x^2}{2x} = \lim_{x\to 0} \frac{3}{2}x = 0, \quad \text{a zero limit}$$

$$\lim_{x\to 0} \frac{3x}{2x} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}, \quad \text{a finite limit}$$

$$\lim_{x\to 0} \frac{3x}{2x^2} = \lim_{x\to 0} \frac{3}{2x} = \infty, \quad \text{an infinite limit}$$

$$\lim_{x\to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x\to x_0} \frac{(x-x_0)(x+x_0)}{(x-x_0)} = 2x_0, \quad \text{a finite limit}$$

$$\lim_{x\to 3} x^2 = 9$$

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

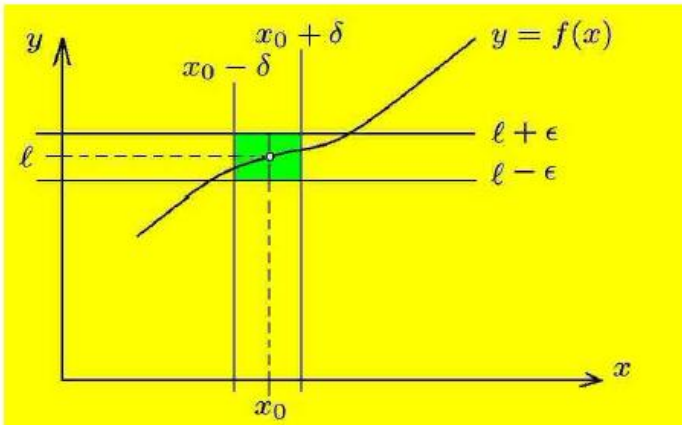
$$\lim_{x\to \infty} \left(3 + \frac{1}{x}\right) = 3$$

a- Definition of Limit of a function.

Let $f(x)$ be defined and single-valued for all values of x in some deleted neighborhood of the point x_0. A number ℓ is called a limit of $f(x)$ as x approaches x_0, written $\lim_{x \to x_0} f(x) = \ell$, if for every small positive number $\epsilon > 0$ there exists a number δ such that[6]

$$|f(x) - \ell| < \epsilon \qquad \textit{whenever} \qquad 0 < |x - x_0| < \delta$$

Then one can write $f(x) \to \ell$ ($f(x)$ approaches ℓ) as $x \to x_0$ (x approaches x_0). Note that $f(x)$ need not be defined at the point x_0 in order for a limit to exist.



The above definition must be modified if restrictions are placed upon how x approaches x_0. For example, the limits $\lim_{x \to x_0^+} f(x) = \ell_1$ and $\lim_{x \to x_0^-} f(x) = \ell_2$ are called the **right-hand and left-hand limits** associated with the function $f(x)$ as x approaches the point x_0. Sometimes the right-hand limit is expressed $\lim_{x \to x_0^+} f(x) = f(x_0^+)$ and the left-hand limit is expressed $\lim_{x \to x_0^-} f(x) = f(x_0^-)$. The $\epsilon - \delta$ definitions associated with these left and right-hand limits is exactly the same as given above with the understanding that for right-hand limits x is restricted to the set of values $x > x_0$ and for left-hand limits x is restricted to the set of values $x < x_0$.

b- Properties of Limits

Properties

First we will assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and that c is any constant. Then,

1. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
   In other words we can "factor" a multiplicative constant out of a limit.

2. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
   So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by "+" or "-".

3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
   We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4. $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
   As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is any real number
   In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, *etc.*). In the case that n is an integer this rule can be thought of as an extended case of **3**.

   For example consider the case of $n = 2$.

$$\begin{aligned}\lim_{x\to a}\left[f(x)\right]^2 &= \lim_{x\to a}\left[f(x)f(x)\right] \\ &= \lim_{x\to a} f(x)\lim_{x\to a} f(x) && \text{using property 3} \\ &= \left[\lim_{x\to a} f(x)\right]^2\end{aligned}$$

The same can be done for any integer n.

6. $\lim\limits_{x\to a}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim\limits_{x\to a} f(x)}$

This is just a special case of the previous example.

$$\begin{aligned}\lim_{x\to a}\left[\sqrt[n]{f(x)}\right]&=\lim_{x\to a}\left[f(x)\right]^{\frac{1}{n}}\\&=\left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}}\\&=\sqrt[n]{\lim_{x\to a} f(x)}\end{aligned}$$

7. $\lim\limits_{x\to a} c = c,\qquad c$ is any real number

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x)=c$.

8. $\lim\limits_{x\to a} x = a$

As with the last one you should be able to convince yourself of this by drawing the graph of $f(x)=x$.

9. $\lim\limits_{x\to a} x^n = a^n$

This is really just a special case of property **5** using $f(x)=x$.

Example 1 Estimate the value of the following limit

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$$

Solution:

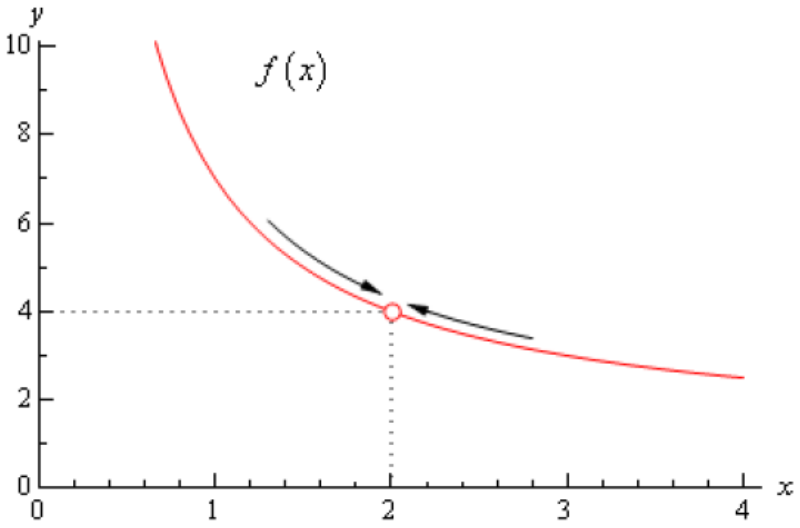
x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

Note that we made sure and picked values of x that were on both sides of $x = 2$ and that we moved in very close to $x = 2$ to make sure that any trends that we might be seeing are in fact correct.

Also notice that we can't actually plug in $x = 2$ into the function as this would give us a division by zero error. This is not a problem since the limit doesn't care what is happening at the point in question.

From this table it appears that the function is going to 4 as x approaches 2, so

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4 .$$



First, notice that there is a rather large open dot at $x = 2$. This is there to remind us that the function (and hence the graph) doesn't exist at $x = 2$.

As we were plugging in values of x into the function we are in effect moving along the graph in towards the point as $x = 2$. This is shown in the graph by the two arrows on the graph that are moving in towards the point.

3- ***Computing Limits.***

The purpose of this section is to develop techniques for dealing with

$$\lim_{x \to a} f(x) = f(a)$$

Some of these limits that will not allow us to just use this fact.

Example 1 Evaluate the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

First let's notice that if we try to plug in $x = 2$ we get,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{0}{0}$$

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$
$$=\lim_{x\to 2}\frac{x+6}{x}$$

Therefore, the limit is,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{x+6}{x}=\frac{8}{2}=4$$

Example 2 Evaluate the following limit.

$$\lim_{h\to 0}\frac{2(-3+h)^2-18}{h}$$

Solution

In this case we also get 0/0 and factoring is not really an option. However, there is still some simplification that we can do.

$$\lim_{h\to 0}\frac{2(-3+h)^2-18}{h}=\lim_{h\to 0}\frac{2(9-6h+h^2)-18}{h}$$
$$=\lim_{h\to 0}\frac{18-12h+2h^2-18}{h}$$
$$=\lim_{h\to 0}\frac{-12h+2h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(-12+2h)}{h}$$
$$=\lim_{h\to 0}-12+2h=-12$$

Example 3

Given the function,

$$g(y)=\begin{cases} y^2+5 & \text{if } y<-2 \\ 1-3y & \text{if } y\geq -2 \end{cases}$$

Compute the following limits.

(a) $\lim_{y\to 6} g(y)$

(b) $\lim_{y\to -2} g(y)$

Solution

(a) $\lim_{y\to 6} g(y)$

$$\lim_{y\to 6} g(y) = \lim_{y\to 6} 1-3y$$
$$= -17$$

(b) $\lim_{y\to -2} g(y)$

Notice that both of the one sided limits can be done here since we are only going to be looking at one side of the point in question. So let's do the two one-sided limits and see what we get.

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2+5 \qquad \text{since } y\to 2^- \text{ implies } y<-2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^+} 1-3y \qquad \text{since } y\to 2^+ \text{ implies } y>-2$$
$$= 7$$

So, in this case we can see that,

$$\lim_{y\to -2^-} g(y) = 9 \neq 7 = \lim_{y\to -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y\to -2} g(y)$$

doesn't exist.

Example 4

Evaluate the following limit.

$$\lim_{y\to -2} g(y) \qquad \text{where, } g(y) = \begin{cases} y^2+5 & \text{if } y<-2 \\ 3-3y & \text{if } y\geq -2 \end{cases}$$

Solution

The two one-sided limits this time are,

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2+5 \qquad \text{since } y\to 2^- \text{ implies } y<-2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^-} 3-3y \qquad \text{since } y\to 2^+ \text{ implies } y>-2$$
$$= 9$$

The one-sided limits are the same so we get,

$$\lim_{y\to -2} g(y) = 9$$

4- ***Infinite Limits.***

Definition

We say

$$\lim_{x\to a} f(x) = \infty$$

if we can make *f(x)* arbitrarily large for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

We say

$$\lim_{x\to a} f(x) = -\infty$$

if we can make *f(x)* arbitrarily large and negative for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

These definitions can be appropriately modified for the one-sided limits as well.

Example 1 Evaluate each of the following limits.

$$\lim_{x\to 0^+} \frac{1}{x} \qquad \lim_{x\to 0^-} \frac{1}{x} \qquad \lim_{x\to 0} \frac{1}{x}$$

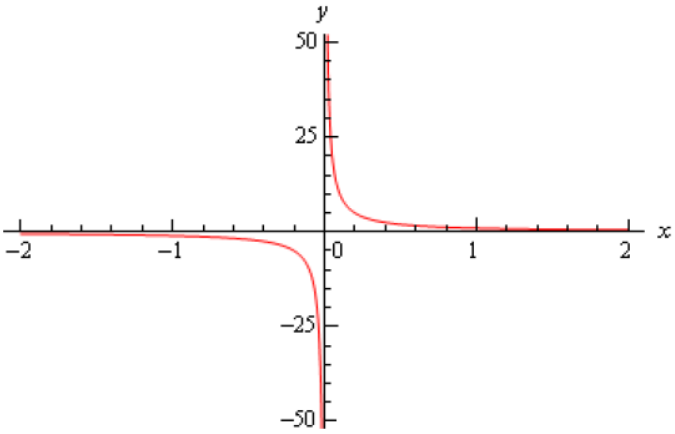
So, here is a table of values of *x*'s from both the left and the right. Using these values we'll be able to estimate the value of the two one-sided limits and once we have that done we can use the fact that the normal limit will exist only if the two one-sided limits exist and have the same value.

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000

So, from our definition above it looks like we should have the following values for the two one sided limits.

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Another way to see the values of the two one sided limits here is to graph the function.



Finally, the normal limit, in this case, will not exist since the two one-sided limits have different values.

So, in summary here are the values of the three limits for this example.

$$\lim_{x\to 0^+}\frac{1}{x}=\infty \qquad \lim_{x\to 0^-}\frac{1}{x}=-\infty \qquad \lim_{x\to 0}\frac{1}{x} \text{ doesn't exist}$$

Example 2 Evaluate each of the following limits.

$$\lim_{x\to 0^+}\frac{6}{x^2} \qquad \lim_{x\to 0^-}\frac{6}{x^2} \qquad \lim_{x\to 0}\frac{6}{x^2}$$

Solution:

The result should then be an increasingly large positive number. It looks like we should have the following value for the right-hand limit in this case,

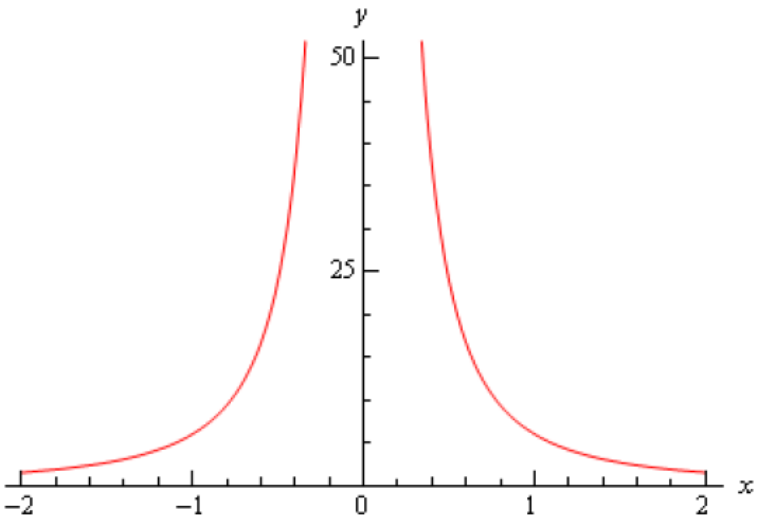
$$\lim_{x\to 0^+}\frac{6}{x^2}=\infty$$

The result, as with the left hand limit, will be an increasingly large positive number and so the left-hand limit will be,

$$\lim_{x\to 0^-}\frac{6}{x^2}=\infty$$

Now, in this example, unlike the first one, the normal limit will exist and be infinity since the two one-sided limits both exist and have the same value. So, in summary here are all the limits for this example as well as a quick graph verifying the limits.

$$\lim_{x\to 0^+}\frac{6}{x^2}=\infty \qquad \lim_{x\to 0^-}\frac{6}{x^2}=\infty \qquad \lim_{x\to 0}\frac{6}{x^2}=\infty$$



Example 3 Evaluate each of the following limits.

$$\lim_{x\to -2^+}\frac{-4}{x+2} \qquad \lim_{x\to -2^-}\frac{-4}{x+2} \qquad \lim_{x\to -2}\frac{-4}{x+2}$$

Solution

Let's again start with the right-hand limit. With the right hand limit we know that we have,

$$x > -2 \qquad \Rightarrow \qquad x + 2 > 0$$

Also, as x gets closer and closer to -2 then $x + 2$ will be getting closer and closer to zero, while staying positive as noted above. So, for the right-hand limit, we'll have a negative constant divided by an increasingly small positive number. The result will be an increasingly large and negative number. So, it looks like the right-hand limit will be negative infinity.

For the left hand limit we have,

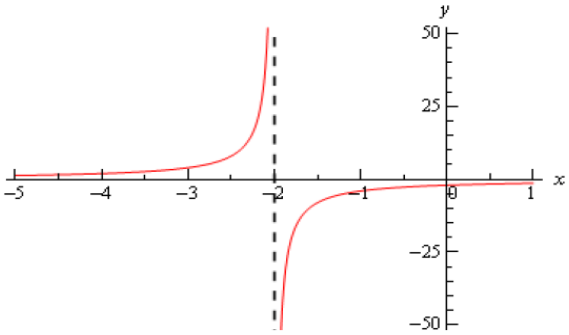
$$x < -2 \qquad \Rightarrow \qquad x + 2 < 0$$

and $x + 2$ will get closer and closer to zero (and be negative) as x gets closer and closer to -2. In this case then we'll have a negative constant divided by an increasingly small negative number. The result will then be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

Finally, since two one sided limits are not the same the normal limit won't exist.

Here are the official answers for this example as well as a quick graph of the function for verification purposes.

$$\lim_{x \to -2^+} \frac{-4}{x+2} = -\infty \qquad \lim_{x \to -2^-} \frac{-4}{x+2} = \infty \qquad \lim_{x \to -2} \frac{-4}{x+2} \text{ doesn't exist}$$



At this point we should briefly acknowledge the idea of vertical asymptotes. Each of the three previous graphs have had one. Recall from an Algebra class that a vertical asymptote is a vertical line (the dashed line at $x = -2$ in the previous example) in which the graph will go towards infinity and/or minus infinity on one or both sides of the line.

In an Algebra class they are a little difficult to define other than to say pretty much what we just said. Now that we have infinite limits under our belt we can easily define a vertical asymptote as follows,

Definition

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a} f(x) = \pm\infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at $x = a$.

Using this definition we can see that the first two examples had vertical asymptotes at $x = 0$ while the third example had a vertical asymptote at $x = -2$.

Definition

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty \qquad \lim_{x\to a} f(x) = \pm\infty$$

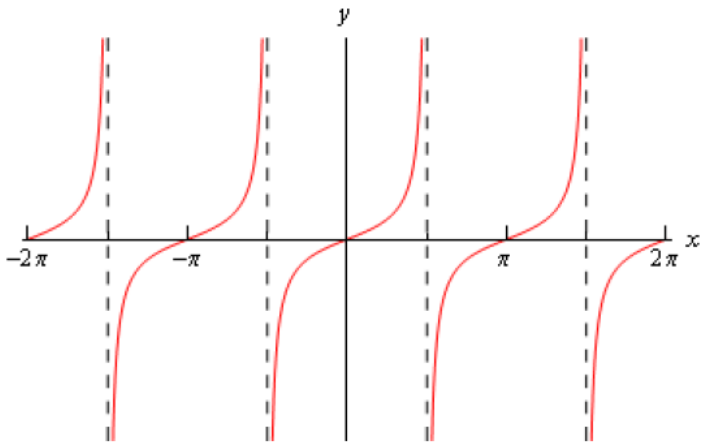
Example 4

Evaluate both of the following limits.

$$\lim_{x\to\frac{\pi}{2}^+} \tan(x) \qquad \lim_{x\to\frac{\pi}{2}^-} \tan(x)$$

Solution

Here's a quick sketch of the graph of the tangent function.



From this it's easy to see that we have the following values for each of these limits,

$$\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty \qquad \lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty$$

Note that the normal limit will not exist because the two one-sided limits are not the same.

Facts

Given the functions $f(x)$ and $g(x)$ suppose we have,

$$\lim_{x\to c} f(x) = \infty \qquad \lim_{x\to c} g(x) = L$$

for some real numbers c and L. Then,

1. $\lim_{x\to c}\left[f(x) \pm g(x)\right] = \infty$

2. If $L > 0$ then $\lim_{x\to c}\left[f(x)g(x)\right] = \infty$

3. If $L < 0$ then $\lim_{x\to c}\left[f(x)g(x)\right] = -\infty$

4. $\lim_{x\to c}\dfrac{g(x)}{f(x)} = 0$

5- ***Limits At Infinity.***

In the previous section we saw limits that were infinity and it's now time to take a look at limits at infinity. By limits at infinity we mean one of the following two limits.

$$\lim_{x\to\infty} f(x) \qquad\qquad \lim_{x\to-\infty} f(x)$$

In fact, many of the limits that we're going to be looking at we will need the following two facts.

Fact 1

1. If r is a positive rational number and c is any real number then,

$$\lim_{x\to\infty}\frac{c}{x^r}=0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x<0$ then,

$$\lim_{x\to-\infty}\frac{c}{x^r}=0$$

Example 1 Evaluate each of the following limits.

(a) $\lim_{x\to\infty}\left(2x^4-x^2-8x\right)$

(b) $\lim_{t\to-\infty}\left(\frac{1}{3}t^5+2t^3-t^2+8\right)$

Solution

(a) $\lim_{x\to\infty}\left(2x^4-x^2-8x\right)$

So, let's see what we get if we do that. As x approaches infinity, then x to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

$$\lim_{x\to\infty}\left(2x^4-x^2-8x\right)=\infty-\infty-\infty$$

This is one of those **indeterminate forms** that we first started seeing in a previous section.
So, we need a way to get around this problem. What we'll do here is factor the largest power of x out of the whole polynomial as follows,

$$\lim_{x\to\infty}\left(2x^4-x^2-8x\right)=\lim_{x\to\infty}\left[x^4\left(2-\frac{1}{x^2}-\frac{8}{x^3}\right)\right]$$

Now for each of the terms we have,

$$\lim_{x\to\infty}x^4=\infty \qquad\qquad \lim_{x\to\infty}\left(2-\frac{1}{x^2}-\frac{8}{x^3}\right)=2$$

The first limit is clearly infinity and for the second limit we'll use the fact above on the last two terms. Therefore using Fact 2 from the previous section we see value of the limit will be,

$$\lim_{x\to\infty}\left(2x^4-x^2-8x\right)=\infty$$

(b) $\lim_{t\to-\infty}\left(\frac{1}{3}t^5+2t^3-t^2+8\right)$

We'll work this part much quicker than the previous part. All we need to do is factor out the largest power of t to get the following,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5+2t^3-t^2+8\right)=\lim_{t\to-\infty}\left[t^5\left(\frac{1}{3}+\frac{2}{t^2}-\frac{1}{t^3}+\frac{8}{t^5}\right)\right]$$

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of t we're factoring out, t^5 in this case.

Now all we need to do is take the limit of the two terms. In the first don't forget that since we're going out towards $-\infty$ and we're raising t to the 5th power that the limit will be negative (negative number raised to an odd power is still negative). In the second term we'll again make heavy use of the fact above to see that is a finite number.

Therefore, using the a modification of the Facts from the previous section the value of the limit is,

$$\lim_{t\to-\infty}\left(\tfrac{1}{3}t^5+2t^3-t^2+8\right)=-\infty$$

Example 2 Evaluate both of the following limits.

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7} \qquad \lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7}$$

Since both the numerator and denominator are polynomials we can use the above fact to determine the behavior of each. Doing this gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\frac{\infty}{-\infty}$$

This is yet another ***indeterminate form,***

So, when we have a polynomial divided by a polynomial we're going to proceed much as we did with only polynomials. We first identify the largest power of x in the denominator (and yes, we only look at the denominator for this) and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{x^4\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)}{x^4\left(-5+\frac{7}{x^4}\right)}$$

Once we've done this we can cancel the x^4 from both the numerator and the denominator and then use the Fact 1 above to take the limit of all the remaining terms. This gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{8}{x^3}}{-5+\frac{7}{x^4}}$$
$$=\frac{2+0+0}{-5+0}$$
$$=-\frac{2}{5}$$

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn't matter which infinity we are going towards we will get the same value for the limit.

$$\lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=-\frac{2}{5}$$

Let's take a look at an example where we get different answers for each limit.

Example 3 Evaluate each of the following limits.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x} \qquad \lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}$$

Solution.
So we need to factor an x out of the numerator and the denominator. When we are done factoring the x out we will need an x in both of the numerator and the denominator. To get this in the numerator we will have to factor an x^2 out of the square root so that after we take the square root we will get an x.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{\sqrt{x^2\left(3+\frac{6}{x^2}\right)}}{x\left(\frac{5}{x}-2\right)}$$
$$=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$\sqrt{x^2}=|x|$$

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

Now, we can't just cancel the x's. We first will need to get rid of the absolute value bars. To do this let's recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we are going out to plus infinity so we can safely assume that the x will be positive and so we can just drop the absolute value bars. The limit is then,

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+6}}{5-2x} = \lim_{x\to\infty} \frac{x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2} = \frac{\sqrt{3+0}}{0-2} = -\frac{\sqrt{3}}{2}$$

For the second limit the initial work will be the same up until we reach the following step.

$$\lim_{x\to-\infty} \frac{\sqrt{3x^2+6}}{5-2x} = \lim_{x\to-\infty} \frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

In this limit we are going to minus infinity so in this case we can assume that x is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

$$\lim_{x\to-\infty} \frac{\sqrt{3x^2+6}}{5-2x} = \lim_{x\to-\infty} \frac{-x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}$$

$$= \frac{\sqrt{3}}{2}$$

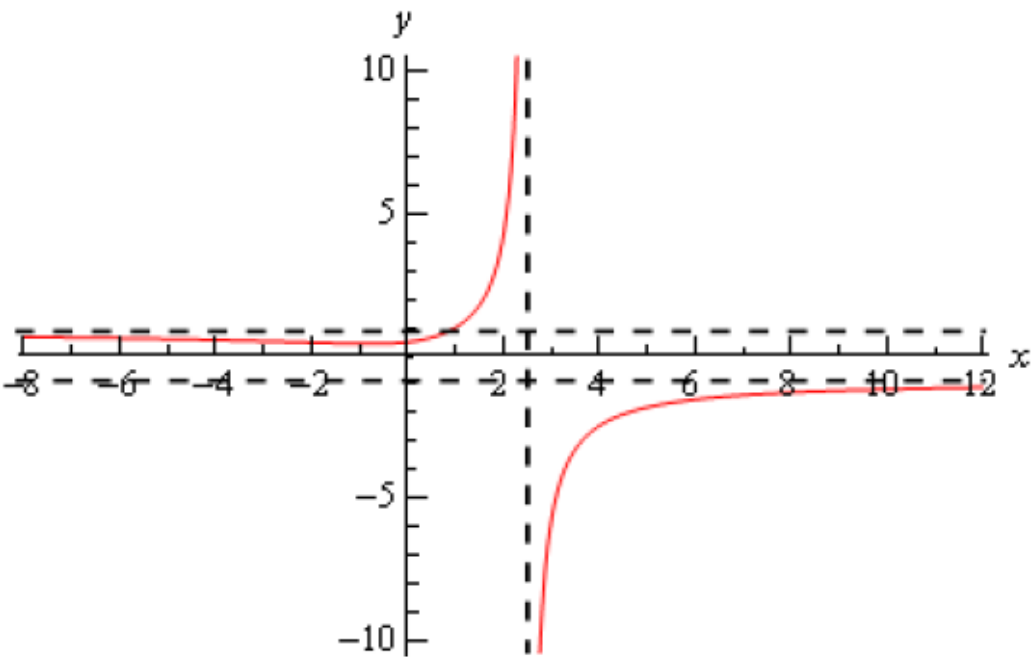
The **horizontal asymptotes** defined in terms of limits

Definition

The function $f(x)$ will have a horizontal asymptote at $y=L$ if either of the following are true.

$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$

The function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.



Example 4 Evaluate each of the following limits.

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3} \qquad\qquad \lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}$$

Solution.

There is a larger power of z in the numerator but we ignore it. We ONLY look at the denominator when doing this! So doing the factoring gives,

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to\infty}\frac{z^3\left(\frac{4}{z}+z^3\right)}{z^3\left(\frac{1}{z^3}-5\right)}$$

$$=\lim_{z\to\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}$$

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the z^3 in the numerator will be going to plus infinity in the limit and so the limit is,

$$\lim_{z\to\infty}\frac{4z^2+z^6}{1-5z^3}=\frac{\infty}{-5}=-\infty$$

Now, let's take a look at the second limit. Note that the only different in the work is at the final "evaluation" step and so we'll pick up the work there.

$$\lim_{z\to-\infty}\frac{4z^2+z^6}{1-5z^3}=\lim_{z\to-\infty}\frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}=\frac{-\infty}{-5}=\infty$$

6- Limits at Infinity of exponential and logarithmic functions.

a- exponential functions.

Let's start by taking a look at a some of very basic examples involving exponential functions

Example 1 Evaluate each of the following limits.

$$\lim_{x\to\infty}\mathbf{e}^{x} \qquad \lim_{x\to-\infty}\mathbf{e}^{x} \qquad \lim_{x\to\infty}\mathbf{e}^{-x} \qquad \lim_{x\to-\infty}\mathbf{e}^{-x}$$

Solution

There are really just restatements of facts given in the basic exponential section of the review so we'll leave it to you to go back and verify these.

$$\lim_{x\to\infty}\mathbf{e}^{x}=\infty \qquad \lim_{x\to-\infty}\mathbf{e}^{x}=0 \qquad \lim_{x\to\infty}\mathbf{e}^{-x}=0 \qquad \lim_{x\to-\infty}\mathbf{e}^{-x}=\infty$$

Example 2 Evaluate each of the following limits.

(a) $\lim_{x\to\infty}\mathbf{e}^{2-4x-8x^2}$

(b) $\lim_{t\to-\infty}\mathbf{e}^{t^4-5t^2+1}$

(c) $\lim_{z\to0^+}\mathbf{e}^{\frac{1}{z}}$

Solution

(a) $\lim_{x\to\infty}\mathbf{e}^{2-4x-8x^2}$

In this part what we need to note (using Fact 2 above) is that in the limit the exponent of the exponential does the following,

$$\lim_{x\to\infty}\left(2-4x-8x^2\right)=-\infty$$

So, the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous set of examples. So, the answer here is,

$$\lim_{x\to\infty}\mathbf{e}^{2-4x-8x^2}=0$$

(b) $\lim_{t\to-\infty}\mathbf{e}^{t^4-5t^2+1}$

Here let's first note that,

$$\lim_{t\to-\infty}\left(t^4-5t^2+1\right)=\infty$$

The exponent goes to infinity in the limit and so the exponential will also need to go to infinity in the limit. Or,

$$\lim_{t\to-\infty}\mathbf{e}^{t^4-5t^2+1}=\infty$$

(c) $\lim\limits_{z\to 0^+} \mathbf{e}^{\frac{1}{z}}$

So, let's first note that using the idea from the previous section we have,

$$\lim_{z\to 0^+} \frac{1}{z} = \infty$$

Here's the answer to this part.

$$\lim_{z\to 0^+} \mathbf{e}^{\frac{1}{z}} = \infty$$

Example 3 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right)$

(b) $\lim\limits_{x\to-\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right)$

Solution.

(a) $\lim\limits_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right)$

Let's start by just taking the limit of each of the pieces and see what we get.

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right) = \infty - \infty + \infty + 0 - 0$$

The first three are a problem however as they present us with another indeterminate form.
When dealing with polynomials we factored out the term with the largest exponent in it.
What do we mean by the "largest" exponent? So, since $10x$ is the largest of the three exponents there we'll "factor" an $\mathbf{e}^{10x}$ out of the whole thing.

$$\frac{-9\mathbf{e}^{-15x}}{\mathbf{e}^{10x}} = -9\mathbf{e}^{-15x-10x} = -9\mathbf{e}^{-25x}$$

Doing factoring on all terms then gives,

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right) = \lim_{x\to\infty}\left[\mathbf{e}^{10x}\left(1 - 4\mathbf{e}^{-4x} + 3\mathbf{e}^{-9x} + 2\mathbf{e}^{-12x} - 9\mathbf{e}^{-25x}\right)\right]$$

$$\lim_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x} + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right) = \infty$$

To simplify the work here a little all we really needed to do was factor the $\mathbf{e}^{10x}$ out of the "problem" terms (the first three in this case) as follows,

$$\begin{aligned}\lim_{x\to\infty}\left(\mathbf{e}^{10x} - 4\mathbf{e}^{6x} + 3\mathbf{e}^{x}\right) &= \lim_{x\to\infty}\left[\mathbf{e}^{10x}\left(1 - 4\mathbf{e}^{-4x} + 3\mathbf{e}^{-9x}\right) + 2\mathbf{e}^{-2x} - 9\mathbf{e}^{-15x}\right] \\ &= (\infty)(1) + 0 - 0 \\ &= \infty\end{aligned}$$

(b) $\lim\limits_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)$

Taking the limits gives,

$$\lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=0-0+0+\infty-\infty$$

So, the last two terms are the problem here as they once again leave us with an indeterminate form. Instead we're going to use "largest" to refer to the exponent that is farther away from zero. Using this definition of "largest" means that we're going to factor an $\mathbf{e}^{-15x}$ out.

Again, remember that to factor this out all we really are doing is dividing each term by $\mathbf{e}^{-15x}$ and then subtracting exponents. Here's the work for the first term as an example,

$$\frac{\mathbf{e}^{10x}}{\mathbf{e}^{-15x}}=\mathbf{e}^{10x-(-15x)}=\mathbf{e}^{25x}$$

As with the first part we can either factor it out of only the "problem" terms (*i.e.* the last two terms), or all the terms. For the practice we'll factor it out of all the terms. Here is the factoring work for this limit,

$$\lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=\lim_{x\to-\infty}\left[\mathbf{e}^{-15x}\left(\mathbf{e}^{25x}-4\mathbf{e}^{21x}+3\mathbf{e}^{16x}+2\mathbf{e}^{13x}-9\right)\right]$$

Finally, after taking the limit of the two terms (the first is infinity and the second is a negative, finite number) and recalling the Facts from the Infinite Limit section we see that the limit is,

$$\lim_{x\to-\infty}\left(\mathbf{e}^{10x}-4\mathbf{e}^{6x}+3\mathbf{e}^{x}+2\mathbf{e}^{-2x}-9\mathbf{e}^{-15x}\right)=-\infty$$

Example 4 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty}\dfrac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$

(b) $\lim\limits_{x\to-\infty}\dfrac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$

Solution.

(a) $\lim\limits_{x\to\infty}\dfrac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$

We look at the denominator and determine the exponential function with the "largest" exponent which we will then factor out from both numerator and denominator.
So, we'll factor an $\mathbf{e}^{4x}$ out of both then numerator and denominator. Once that is done we can cancel the $\mathbf{e}^{4x}$ and then take the limit of the remaining terms. Here is the work for this limit,

$$\lim_{x\to\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}=\lim_{x\to\infty}\frac{\mathbf{e}^{4x}\left(6-\mathbf{e}^{-6x}\right)}{\mathbf{e}^{4x}\left(8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}\right)}$$
$$=\lim_{x\to\infty}\frac{6-\mathbf{e}^{-6x}}{8-\mathbf{e}^{-2x}+3\mathbf{e}^{-5x}}$$
$$=\frac{6-0}{8-0+0}$$
$$=\frac{3}{4}$$

(b) $\displaystyle\lim_{x\to-\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}$

In this case we're going to minus infinity in the limit and so we'll look at exponentials in the denominator with negative exponents in determining the "largest" exponent. There's only one however in this problem so that is what we'll use. Do NOT use the exponential from the numerator, even though that one is "larger" than the exponential in the denominator.

Here is the work for this part.

$$\lim_{x\to-\infty}\frac{6\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{8\mathbf{e}^{4x}-\mathbf{e}^{2x}+3\mathbf{e}^{-x}}=\lim_{x\to-\infty}\frac{\mathbf{e}^{-x}\left(6\mathbf{e}^{5x}-\mathbf{e}^{-x}\right)}{\mathbf{e}^{-x}\left(8\mathbf{e}^{5x}-\mathbf{e}^{3x}+3\right)}$$
$$=\lim_{x\to-\infty}\frac{6\mathbf{e}^{5x}-\mathbf{e}^{-x}}{8\mathbf{e}^{5x}-\mathbf{e}^{3x}+3}$$
$$=\frac{0-\infty}{0-0+3}$$
$$=-\infty$$

b- Logarithms Functions.

Let's take a quick look at some basic limits involving logarithms.

Example 5 Evaluate each of the following limits.

$$\lim_{x\to0^+}\ln x \qquad\qquad \lim_{x\to\infty}\ln x$$

Solution

As with the last example I'll leave it to you to verify these restatements from the basic logarithm section.

$$\lim_{x\to0^+}\ln x=-\infty \qquad\qquad \lim_{x\to\infty}\ln x=\infty$$

Example 6 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to\infty}\ln\left(7x^3-x^2+1\right)$

(b) $\displaystyle\lim_{t\to-\infty}\ln\left(\frac{1}{t^2-5t}\right)$ [S

Solution

(a) $\lim\limits_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right)$

So, let's first look to see what the argument of the log is doing,

$$\lim_{x\to\infty}\left(7x^3 - x^2 + 1\right) = \infty$$

The argument of the log is going to infinity and so the log must also be going to infinity in the limit. The answer to this part is then,

$$\lim_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right) = \infty$$

(b) $\lim\limits_{t\to-\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$

Using the techniques from earlier in this section we can see that,

$$\lim_{t\to-\infty} \frac{1}{t^2 - 5t} = 0$$

Therefore, not only does the argument go to zero, it goes to zero from the right. This is exactly what we need to do this limit.

So, the answer here is,

$$\lim_{t\to-\infty} \ln\left(\frac{1}{t^2 - 5t}\right) = -\infty$$

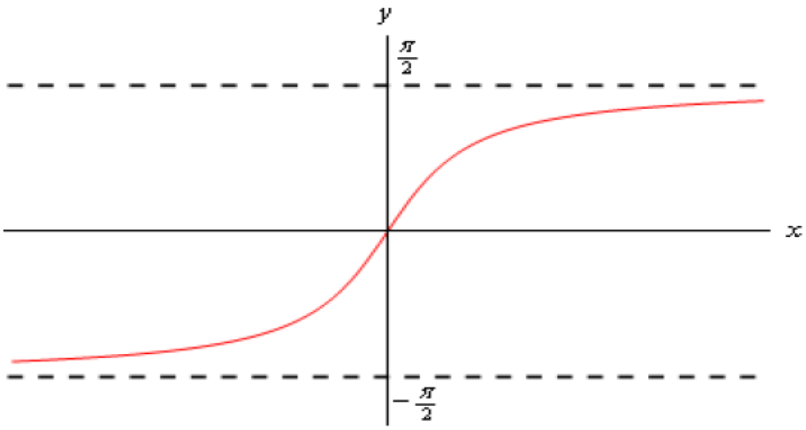
Example 7 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty} \tan^{-1} x$

(b) $\lim\limits_{x\to-\infty} \tan^{-1} x$

Solution.

The basic limits involving inverse tangents and can easily be found by examining the following sketch of inverse tangents.



(a) $\lim\limits_{x\to\infty} \tan^{-1} x$

As noted above all we really need to do here is look at the graph of the inverse tangent. Doing this shows us that we have the following value of the limit.

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

(b) $\lim\limits_{x\to-\infty} \tan^{-1} x$

Again, not much to do here other than examine the graph of the inverse tangent.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

7- *Continuous and Discontinuous Functions.*

Definition

A function $f(x)$ is said to be **continuous** at $x = a$ if

$$\lim_{x\to a} f(x) = f(a)$$

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Note that this definition is also implicitly assuming that both $f(a)$ and $\lim\limits_{x\to a} f(x)$ exist. If either of these do not exist then the function will not be continuous at $x = a$.

This definition can be turned around into the following fact.

Fact 1

If $f(x)$ is continuous at $x = a$ then,

$$\lim_{x\to a} f(x) = f(a) \qquad \lim_{x\to a^-} f(x) = f(a) \qquad \lim_{x\to a^+} f(x) = f(a)$$

A function $f(x)$ is called a **continuous function** over the interval $a \le x_0 \le b$ if for all points x_0 within the interval

(i) $f(x_0)$ is well defined

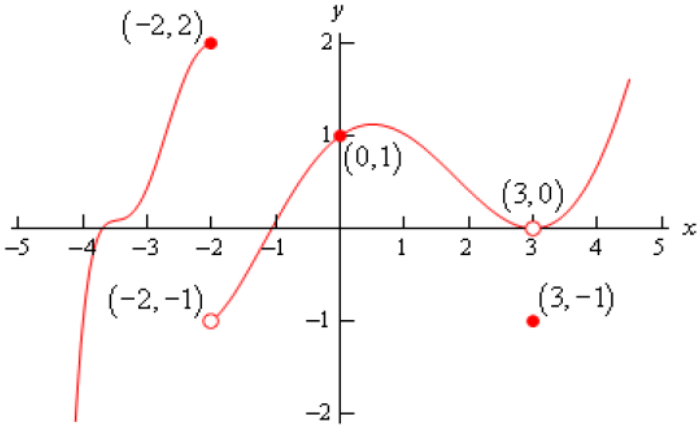
(ii) $\lim\limits_{x\to x_0} f(x)$ exists

(iii) $\lim\limits_{x\to x_0} f(x) = f(x_0)$

Polynomial functions are continuous functions. Rational algebraic functions, represented by the quotient of two polynomials, are continuous except for those points where the denominator becomes zero. The trigonometric functions,

exponential functions and logarithmic functions are continuous functions over appropriate intervals.

Example 1 Given the graph of *f(x)*, shown below, determine if *f(x)* is continuous at $x = -2$, $x = 0$, and $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First $x = -2$.

$$f(-2) = 2 \qquad \lim_{x \to -2} f(x) \text{ doesn't exist}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it as this graph does.

Now $x = 0$.

$$f(0) = 1 \qquad \lim_{x \to 0} f(x) = 1$$

The function is continuous at this point since the function and limit have the same value.

Finally $x = 3$.

$$f(3) = -1 \qquad \lim_{x \to 3} f(x) = 0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

Example 2 Determine where the function below is not continuous.

$$h(t)=\frac{4t+10}{t^2-2t-15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$$t^2-2t-15=(t-5)(t+3)=0$$

So, the function will not be continuous at $t=-3$ and $t=5$.

Fact 2

If $f(x)$ is continuous at $x=b$ and $\lim_{x\to a} g(x)=b$ then,

$$\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)$$

Example 3 Evaluate the following limit.

$$\lim_{x\to 0} \mathbf{e}^{\sin x}$$

Solution

Since we know that exponentials are continuous everywhere we can use the fact above.

$$\lim_{x\to 0} \mathbf{e}^{\sin x}=\mathbf{e}^{\lim_{x\to 0}\sin x}=\mathbf{e}^0=1$$

Example 4

Evaluate the following function

$$f(x)=\frac{x^2-1}{x-1}$$

not defined at the point $x=1$

The function is discontinues at x=1

The limit $\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=2$ exists

The function can be made continuous at x=1

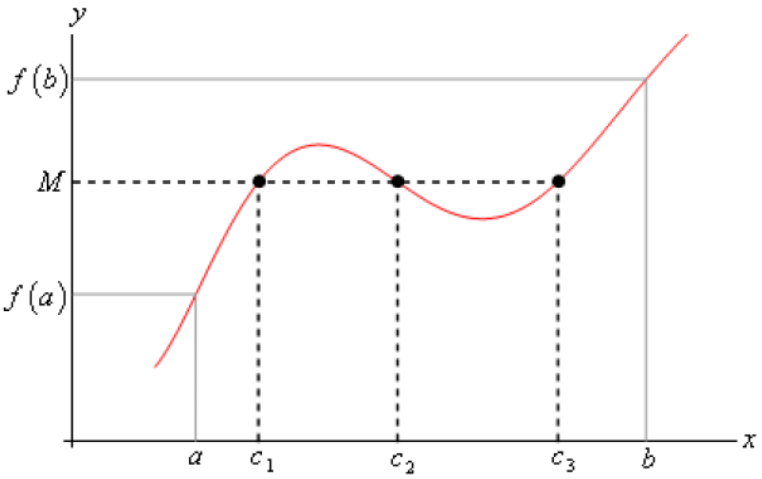
In this case the function is said to have a **removable discontinuity** at the point x = 1.

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that,

1. $a<c<b$
2. $f(c)=M$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between *f(a)* and *f(b)*. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



So, the Intermediate Value Theorem tells us that a function will take the value of *M* somewhere between *a* and *b* but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value. There are important idea to remember about the Intermediate Value Theorem.

Example 5 If possible, determine if $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

(a) Does $f(x) = 10$?

(b) Does $f(x) = -10$?

Solution

Now, for each part we will let *M* be the given value for that part and then we'll need to show that *M* lives between $f(0)$ and $f(5)$. If it does then we can use the Intermediate Value Theorem to prove that the function will take the given value.

So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad\qquad f(5) = 19.7436$$

Now, let's take a look at each part.

(a) Okay, in this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

$$f(c) = 10$$

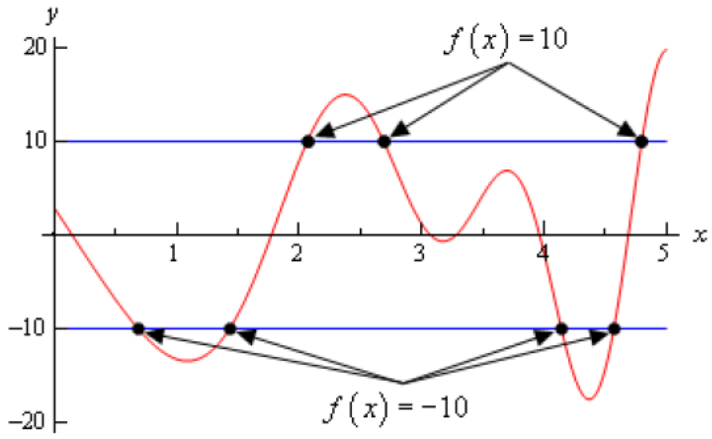
(b) In this part we'll define $M = -10$. We now have a problem. In this part M does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \neq -10$ in [0,5]?

Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \neq -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that c's will exist. The theorem will NOT tell us that c's don't exist.

In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

Sometimes we can use it to verify that a function will take some value in a given interval and in other cases we won't be able to use it.

For completeness sake here is the graph of $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ in the interval [0,5].



From this graph we can see that not only does $f(x) = -10$ in [0,5] it does so a total of 4 times! Also note that as we verified in the first part of the previous example $f(x) = 10$ in [0,5] and in fact it does so a total of 3 times.

Problems: ***Sheet No. 3***

8- ***Problems.***

a- ***Computing Limits***

For problems 1 – 9 evaluate the limit, if it exists.

1. $\lim\limits_{x\to 2}\left(8-3x+12x^2\right)$

2. $\lim\limits_{t\to -3}\dfrac{6+4t}{t^2+1}$

3. $\lim\limits_{x\to -5}\dfrac{x^2-25}{x^2+2x-15}$

4. $\lim\limits_{z\to 8}\dfrac{2z^2-17z+8}{8-z}$

5. $\lim\limits_{y\to 7}\dfrac{y^2-4y-21}{3y^2-17y-28}$

6. $\lim\limits_{h\to 0}\dfrac{(6+h)^2-36}{h}$

7. $\lim\limits_{z\to 4}\dfrac{\sqrt{z}-2}{z-4}$

8. $\lim\limits_{x\to -3}\dfrac{\sqrt{2x+22}-4}{x+3}$

9. $\lim\limits_{x\to 0}\dfrac{x}{3-\sqrt{x+9}}$

10. Given the function

$$f(x)=\begin{cases}7-4x & x<1\\ x^2+2 & x\geq 1\end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to -6} f(x)$ (b) $\lim\limits_{x\to 1} f(x)$

11. Given

$$h(z)=\begin{cases}6z & z\leq -4\\ 1-9z & z>-4\end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{z\to 7} h(z)$ (b) $\lim\limits_{z\to -4} h(z)$

b- ***Infinite Limits***

1. For $f(x)=\dfrac{9}{(x-3)^5}$ evaluate,

(a) $\lim\limits_{x\to 3^-} f(x)$ **(b)** $\lim\limits_{x\to 3^+} f(x)$ **(c)** $\lim\limits_{x\to 3} f(x)$

2. For $h(t)=\dfrac{2t}{6+t}$ evaluate,

(a) $\lim\limits_{t\to -6^-} h(t)$ **(b)** $\lim\limits_{t\to -6^+} h(t)$ **(c)** $\lim\limits_{t\to -6} h(t)$

3. For $g(z)=\dfrac{z+3}{(z+1)^2}$ evaluate,

(a) $\lim\limits_{z\to -1^-} g(z)$ **(b)** $\lim\limits_{z\to -1^+} g(z)$ **(c)** $\lim\limits_{z\to -1} g(z)$

4. For $g(x)=\dfrac{x+7}{x^2-4}$ evaluate,

(a) $\lim\limits_{x\to 2^-} g(x)$ **(b)** $\lim\limits_{x\to 2^+} g(x)$ **(c)** $\lim\limits_{x\to 2} g(x)$

Problems: ***Sheet No. 3***

5. For $h(x)=\ln(-x)$ evaluate,

(a) $\lim\limits_{x\to 0^-} h(x)$ **(b)** $\lim\limits_{x\to 0^+} h(x)$ **(c)** $\lim\limits_{x\to 0} h(x)$

6. For $R(y)=\tan(y)$ evaluate,

(a) $\lim\limits_{y\to \frac{3\pi}{2}^-} R(y)$ **(b)** $\lim\limits_{y\to \frac{3\pi}{2}^+} R(y)$ **(c)** $\lim\limits_{y\to \frac{3\pi}{2}} R(y)$

For problems 7 & 8 find all the vertical asymptotes of the given function.

7. $f(x)=\dfrac{7x}{(10-3x)^4}$

8. $g(x)=\dfrac{-8}{(x+5)(x-9)}$

C- ***Limits At Infinity***

1. For $f(x)=4x^7-18x^3+9$ evaluate each of the following limits.

(a) $\lim\limits_{x\to -\infty} f(x)$ **(b)** $\lim\limits_{x\to \infty} f(x)$

2. For $h(t)=\sqrt[3]{t}+12t-2t^2$ evaluate each of the following limits.

(a) $\lim\limits_{t\to -\infty} h(t)$ **(b)** $\lim\limits_{t\to \infty} h(t)$

For problems 3 – 10 answer each of the following questions.

(a) Evaluate $\lim\limits_{x\to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x\to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

3. $f(x)=\dfrac{8-4x^2}{9x^2+5x}$

4. $f(x)=\dfrac{3x^7-4x^2+1}{5-10x^2}$

5. $f(x)=\dfrac{20x^4-7x^3}{2x+9x^2+5x^4}$

Problems: ***Sheet No. 3***

6. $f(x)=\dfrac{x^3-2x+11}{3-6x^5}$

7. $f(x)=\dfrac{x^6-x^4+x^2-1}{7x^6+4x^3+10}$

8. $f(x)=\dfrac{\sqrt{7+9x^2}}{1-2x}$

9. $f(x)=\dfrac{x+8}{\sqrt{2x^2+3}}$

10. $f(x)=\dfrac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}$

d- Limits of Exponential at infinity.

For problems 1 – 6 evaluate **(a)** $\lim\limits_{x\to-\infty} f(x)$ and **(b)** $\lim\limits_{x\to\infty} f(x)$.

1. $f(x)=\mathbf{e}^{8+2x-x^3}$

2. $f(x)=\mathbf{e}^{\frac{6x^2+x}{5+3x}}$

3. $f(x)=2\mathbf{e}^{6x}-\mathbf{e}^{-7x}-10\mathbf{e}^{4x}$

4. $f(x)=3\mathbf{e}^{-x}-8\mathbf{e}^{-5x}-\mathbf{e}^{10x}$

5. $f(x)=\dfrac{\mathbf{e}^{-3x}-2\mathbf{e}^{8x}}{9\mathbf{e}^{8x}-7\mathbf{e}^{-3x}}$

6. $f(x)=\dfrac{\mathbf{e}^{-7x}-2\mathbf{e}^{3x}-\mathbf{e}^{x}}{\mathbf{e}^{-x}+16\mathbf{e}^{10x}+2\mathbf{e}^{-4x}}$

Problems: ***Sheet No. 3***

e- Limits of Logarithm Functions.

For problems 7 – 12 evaluate the given limit.

7. $\lim\limits_{t\to-\infty} \ln\left(4-9t-t^3\right)$

8. $\lim\limits_{z\to-\infty} \ln\left(\dfrac{3z^4-8}{2+z^2}\right)$

9. $\lim\limits_{x\to\infty} \ln\left(\dfrac{11+8x}{x^3+7x}\right)$

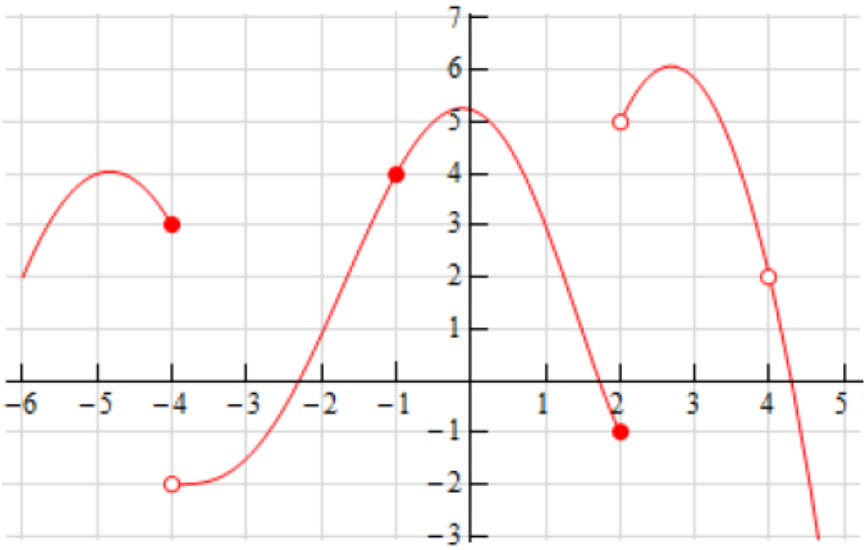
10. $\lim\limits_{x\to-\infty} \tan^{-1}\left(7-x+3x^5\right)$

11. $\lim\limits_{t\to\infty} \tan^{-1}\left(\dfrac{4+7t}{2-t}\right)$

12. $\lim\limits_{w\to\infty} \tan^{-1}\left(\dfrac{3w^2-9w^4}{4w-w^3}\right)$

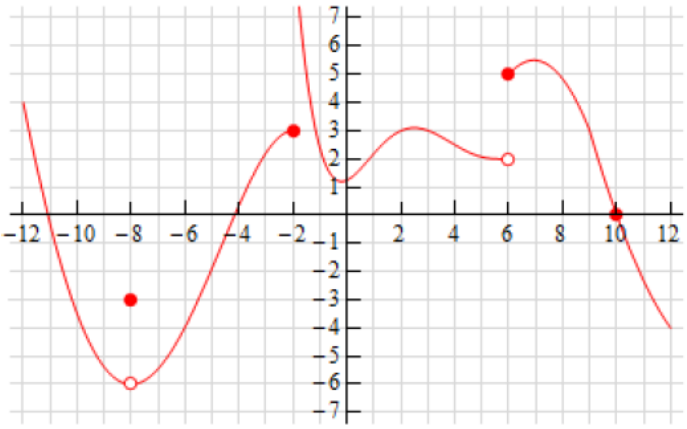
f- Continuity.

1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Problems: ***Sheet No. 3***

2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



For problems 3 – 7 using only Properties 1 – 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the given function is continuous or discontinuous at the indicated points.

3. $f(x)=\dfrac{4x+5}{9-3x}$

(a) $x=-1$, **(b)** $x=0$, **(c)** $x=3$?

4. $g(z)=\dfrac{6}{z^2-3z-10}$

(a) $z=-2$, **(b)** $z=0$, **(c)** $z=5$?

5. $g(x)=\begin{cases} 2x & x<6 \\ x-1 & x\geq 6 \end{cases}$

(a) $x=4$, **(b)** $x=6$?

6. $h(t)=\begin{cases} t^2 & t<-2 \\ t+6 & t\geq -2 \end{cases}$

(a) $t=-2$, **(b)** $t=10$?

7. $g(x)=\begin{cases} 1-3x & x<-6 \\ 7 & x=-6 \\ x^3 & -6<x<1 \\ 1 & x=1 \\ 2-x & x>1 \end{cases}$

(a) $x=-6$, **(b)** $x=1$?