2) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

where $$\rho(x)=e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(x)$.

iii. Find the integrating factor $\rho(x)=e^{\int P(x)dx}$.

iv. Find y using the following equation

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example

Solve the equation $x\dfrac{dy}{dx}-3y=x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions P and Q.*** To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx}-\frac{3}{x}y=x \qquad \Rightarrow \qquad P(x)=-\frac{3}{x}, \qquad Q(x)=x.$$

Step 2: ***Find an anti-derivative of*** $P(x)$.

$$\int P(x)dx = \int -\frac{3}{x}dx = -3\int \frac{1}{x}dx = -3\ln(x)$$

Step 3: ***Find the integrating factor*** $\rho(x)$.

$$\rho(x) = e^{\int P(x)dx} = e^{-3\ln x} = e^{\ln x^{-3}} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

Step 4: ***Find the solution***.

$$y = \frac{1}{\rho(x)}\int \rho(x)Q(x)dx = \frac{1}{(1/x^3)}\int \left(\frac{1}{x^3}\right)(x)dx$$

$$= x^3\int \frac{1}{x^2}dx = x^3\left(-\frac{1}{x}+C\right) = Cx^3 - x^2$$

The solution is the function $y = Cx^3 - x^2$.

Example

Solve the equation $(1+x^2)dy + (y - \tan^{-1}(x))dx = 0$.

Solution

Dividing the two sides by $(1+x^2)dx$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}(x)}{1+x^2} = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}(x)}{1+x^2} \quad \Rightarrow \quad P(x) = \frac{1}{1+x^2}, \quad Q = \frac{\tan^{-1}(x)}{1+x^2}$$

$$\int P(x)dx = \int \frac{1}{1+x^2}dx = \tan^{-1}(x)$$

$$\rho(x) = e^{\tan^{-1}(x)}$$

$$e^{\tan^{-1}(x)}y = \int e^{\tan^{-1}(x)}\frac{\tan^{-1}(x)}{1+x^2}dx + C$$

$$z = \tan^{-1}(x) \quad \Rightarrow \quad dz = \frac{1}{1+x^2}dx$$

$$
\begin{aligned}
e^{\tan^{-1}(x)}y &= \int e^{z} \times z\,dz + C \\
&= ze^{z} - \int e^{z}\,dz + C \\
&= ze^{z} - e^{z} + C
\end{aligned}
$$

$$e^{\tan^{-1}(x)}y = \tan^{-1}(x)e^{\tan^{-1}(x)} - e^{\tan^{-1}(x)} + C$$

Steps for Solving other Form of Linear First Order Equation

There is another form of differential equation that can be written in the form

$$\frac{dx}{dy}+P(y)x=Q(y)$$

where P and Q are functions of y. The solution is found as follows:

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(y)$.

iii. Find the integrating factor $\rho(y)=e^{\int P(y)dy}$.

iv. Find x using the following equation

$$x=\frac{1}{\rho(y)}\int \rho(y)Q(y)dy$$

Example

Solve the equation $e^{2y}dx+2(xe^{2y}-y)dy=0$.

Solution

Dividing the differential equation by $e^{2y}dy$ to get

$$\frac{dx}{dy}+2x-2ye^{-2y}=0$$

$$\frac{dx}{dy}+2x=2ye^{-2y} \qquad \Rightarrow \qquad P(y)=2, \quad Q(y)=2ye^{-2y}$$

$$\int P(y)dy=\int 2dy=2y, \qquad \rho(y)=e^{\int P(y)dy}=e^{2y}$$

$$x=\frac{1}{e^{2y}}\int \left(e^{2y}\right)\left(2ye^{-2y}\right)dy+C \quad \Rightarrow \quad e^{2y}x=2\int ydy+C$$

$$e^{2y}x=2\frac{y^2}{2}+C \qquad \Rightarrow \qquad e^{2y}x=y^2+C$$

Reducible to Linear

- The general form

$$\frac{dy}{dx} + P(x)y = Q(x)f(y)$$

where the function f is y to any power n.

- Also, it may be in the following form

$$\frac{dy}{dx} + P(x)g(y) = Q(x)h(y)$$

where the function g and h are functions of y.

Example

Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \ln(x)y^2$

Solution

Dividing the two sides of the equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \ln(x)$$

Let $z = \frac{1}{y} \quad \Rightarrow \quad \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -y^2\frac{dz}{dx}$

$$-\frac{dz}{dx} + \frac{1}{x}z = \ln(x)$$

$$\frac{dz}{dx} - \frac{1}{x}z = -\ln(x) \qquad \Rightarrow \quad P = \frac{-1}{x}, \quad Q = -\ln(x)$$

$$\int P(x)dx = \int \frac{-1}{x}dx = -\ln(x)$$

$$\rho(x) = e^{\int P(x)dx} = e^{-\ln(x)} = e^{\ln(x)^{-1}} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$\frac{1}{x}z = -\int \frac{1}{x}\ln(x)dx + C$$

$$\frac{1}{x}\times\frac{1}{y} = -\frac{(\ln(x))^2}{2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{(\ln(x))^2}{2} + C$$

Example

Solve the equation $\dfrac{dy}{dx} + x\sin(2y) = x\cos^2(y)$

Solution

Dividing the two sides of the equation by $\cos^2(y)$

$$\frac{1}{\cos^2(y)}\frac{dy}{dx} + x\frac{\sin(2y)}{\cos^2(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)\cos(y)}{\cos^2(y)} = x$$

$$\sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)}{\cos(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + 2x\tan(y) = x$$

Let $z = \tan(y) \quad \Rightarrow \quad \dfrac{dz}{dx} = \sec^2(y)\dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)}\dfrac{dz}{dx}$

$$\frac{dz}{dx} + 2xz = x \quad \Rightarrow \quad P = 2x, \quad Q = x$$

$$\int P(x)dx = \int 2x dx = x^2 \quad \Rightarrow \quad \rho(x) = e^{\int P(x)dx} = e^{x^2}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$e^{x^2}z = \int e^{x^2}(x)dx + C \quad \Rightarrow \quad e^{x^2}\tan(y) = \frac{e^{x^2}}{2} + C$$

Another Form of Reducible to Linear

- The general form may be as follows

$$\frac{dx}{dy}+P(y)x=Q(y)f(x)$$

where the function f is x to any power n.

- Also, it may be in the following form

$$\frac{dx}{dy}+P(y)g(x)=Q(y)h(x)$$

where the function g and h are functions of x.

Example

Solve the equation $\cos(y)dx=x(\sin(y)-x)dy$

Solution

Dividing the two sides of the equation by $\cos(y)dy$

$$\frac{dx}{dy}=\frac{\sin(y)}{\cos(y)}x-\frac{x^2}{\cos(y)} \quad\Rightarrow\quad \frac{dx}{dy}-x\tan(y)=-x^2\sec(y)$$

Dividing by x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}\tan(y)=-\sec(y)$$

Let $z=\frac{1}{x} \quad\Rightarrow\quad \frac{dz}{dy}=\frac{-1}{x^2}\frac{dx}{dy} \quad\Rightarrow\quad \frac{dx}{dy}=-x^2\frac{dz}{dy}$

$$-\frac{dz}{dy}-z\tan(y)=-\sec(y)$$

$$\frac{dz}{dy}+z\tan(y)=\sec(y) \quad\Rightarrow\quad P=\tan(y),\quad Q=\sec(y)$$

$$\int P(y)dy = \int \tan(y)dy = \int \frac{\sin(y)}{\cos(y)}dy = -\ln(\cos(y))$$

$$\rho(y) = e^{\int P(y)dy} = e^{-\ln(\cos(y))} = e^{\ln(\cos(y))^{-1}} = e^{\ln\left(\frac{1}{\cos(y)}\right)} = \sec(y)$$

$$\rho(y)z = \int \rho(y)Q(y)dy + C$$

$$\sec(y) \times \frac{1}{x} = \int \sec(y)\sec(y)dy + C$$

$$\frac{\sec(y)}{x} = \int \sec^2(y)dy + C \quad \Rightarrow \quad \frac{\sec(y)}{x} = \tan(y) + C$$

Exact Differential Equations

Example

If $f(x, y) = C$ and $f(x, y) = \sin(xy)$ then

$$\frac{df}{dx} = y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 0 \text{, or}$$

$$df = y\cos(xy)dx + x\cos(xy)dy = 0$$

i.e., $y\cos(xy)dx + x\cos(xy)dy = 0$

From the above equation, we see that $M(x, y) = y\cos(xy) = \frac{\partial f}{\partial x}$, and $N(x, y) = x\cos(xy) = \frac{\partial f}{\partial y}$. The solution of this differential equation is $f(x, y) = C$.

Exact Differential Equation Test

A differential equation $M(x, y)dx + N(x, y)dy = 0$ is said to be ***exact*** if for some function $f(x, y)$

$$M(x, y)dx + N(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

is exact if and only if $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example

- The equation $(x^2 + y^2)dx + (2xy + \cos(y))dy = 0$ is exact because the partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy + \cos(y)) = 2y$$

are equal.

- The equation $(x+3y)dx+(x^2+\cos(y))dy=0$ is not exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x+3y)=3, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2+\cos(y))=2x$$

are not equal.

Steps for Solving an Exact Differential Equation

i. Match the equation to the form $M(x,y)dx+N(x,y)dy=0$ to identify M and N.

ii. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$).

iii. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $g'(y)$ (or $g'(x)$).

iv. Integrate to find $g(y)$ (or $g(x)$).

v. Write the solution of the exact equation as $f(x,y)=C$.

Example

Solve the differential equation

$$(x^2+y^2)dx+(2xy+\cos(y))dy=0.$$

Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** M.

$$M(x,y)=x^2+y^2$$

Step 2: ***Integrate*** M ***with respect to*** x, ***writing the constant of integration as*** $g(y)$.

$$f(x,y)=\int M(x,y)dx=\int (x^2+y^2)dx=\frac{x^3}{3}+xy^2+g(y)$$

Step 3: ***Differentiate with respect to*** y ***and set the result equal to*** N ***to find*** $g'(y)$.

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3}+xy^2+g(y)\right)=2xy+g'(y)$$

$$2xy+g'(y)=2xy+\cos(y) \quad \Rightarrow \quad g'(y)=\cos(y)$$

Step 4: ***Integrate to find*** $g(y)$.

$$\int g'(y)dy=\int \cos(y)dy=\sin(y)$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y)=C$.

$$\frac{x^3}{3}+xy^2+\sin(y)=C$$

Another Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** N.

$$N(x,y)=2xy+\cos(y)$$

Step 2: ***Integrate*** N ***with respect to*** y, ***writing the constant of integration as*** $g(x)$.

$$f(x,y)=\int N(x,y)dy=\int (2xy+\cos(y))dy=xy^2+\sin(y)+g(x)$$

Step 3: ***Differentiate with respect to*** x ***and set the result equal to*** M ***to find*** $g'(x)$.

$$\frac{\partial}{\partial x}\left(xy^2+\sin(y)+g(x)\right)=y^2+g'(x)$$

$$y^2+g'(x)=x^2+y^2 \quad \Rightarrow \quad g'(x)=x^2$$

Step 4: ***Integrate to find*** $g(x)$.

$$\int g'(x)dx = \int x^2 dx = \frac{x^3}{3}$$

Step 5: ***Write the solution of the exact equation as*** $f(x, y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Reducible to Exact

A differential equation $M(x, y)dx + N(x, y)dy = 0$ which is not exact can be made exact by multiplying both sides by a suitable integrating factor ρ. In other words, the equation

$$\rho M(x, y)dx + \rho N(x, y)dy = 0$$

is an exact equation for an appropriate choice of ρ.

Method to Find the Integrating Factor

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or ***Constant*** then $\rho(x) = e^{\int f(x)dx}$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ or ***Constant*** then $\rho(y) = e^{\int f(y)dy}$.

Example

Solve the equation $2ydx + xdy = 0$

Solution

$$M(x,y) = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N(x,y) = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

This equation is not exact

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x} = f(x)$$

$$\int f(x)dx = \int \frac{1}{x}dx = \ln(x)$$

$$\rho(x) = e^{\int f(x)dx} = e^{\ln(x)} = x$$

Multiplying both sides of the equation by the integrating factor $\rho(x) = x$, we get

$$x(2ydx + xdy) = 0 \quad \Rightarrow \quad 2xydx + x^2dy = 0$$

which is exact because $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and the solution is

$$f(x,y) = \int 2xydx = x^2y + g(y)$$

$$\frac{\partial}{\partial y}\left(x^2y + g(y)\right) = x^2 + g'(y)$$

$$x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = \int g'(y)dy = C \quad \Rightarrow \quad x^2y = C$$

Non Homogenous 2nd Order D.E (const. Coeff)

Consider the non-homogeneous equ. as

$$a \cdot y'' + b \cdot y' + c \cdot y = F(x)$$

D-operator

Let $\frac{d}{dx} = D \qquad \therefore y' = \frac{dy}{dx} = D \cdot y$

$\frac{d^2}{dx^2} = D^2 \qquad \therefore y'' = \frac{d^2y}{dx^2} = D^2 \cdot y$

The equ. becomes

$$\underbrace{\left((a \cdot D^2 + b \cdot D + c) \cdot y = F(x) \right)}$$

The Solution given by :

$$\boxed{y_{G.S.} = y_{C.F.} + y_{P.I.}}$$

where:

$y_{G.S.}$ = General Solution of D.E.

$y_{C.f.}$ = Complementary function Solution

y = Particular Integral Solution

To get $y_{c.f.}$

$y_{c.f.} \rightarrow$ The Solution of the homogeneous D.E.

To get $y_{P.I.}$

$$\because \ (\underbrace{aD^2 + b.D + c}_{P(D)}) \cdot y = F(x)$$

$P(D) \longrightarrow$ diff. polynomial

$$\therefore \quad P(D) \cdot y = F(x)$$

$$\therefore \quad y_{P.I.} = \frac{1}{P(D)} \cdot F(x)$$

Properties :

① $y = \frac{1}{P(D)} \cdot [c\, F(x)] = c \cdot \frac{1}{P(D)} \cdot F(x)$

② $y = \frac{1}{P(D)} \cdot [F_1(x) + F_2(x)] = \frac{1}{P(D)} \cdot F_1(x) + \frac{1}{P(D)} \cdot F_2(x)$

③ $y = \frac{1}{P_1(D) \cdot P_2(D)} \cdot [F(x)] = \frac{1}{P_1(D)} \cdot \left[\frac{1}{P_2(D)} \cdot F(x) \right]$

1) if $f(x) = e^{mx}$

$$\therefore y_{P.I.} = \frac{1}{P(D)} \cdot e^{mx}$$

put $D \longrightarrow m$

Prove)

$$\because D \cdot e^{mx} = m \cdot e^{mx}$$

$$\because D^2 \cdot e^{mx} = m^2 \cdot e^{mx}$$

$$\because D^3 \cdot e^{mx} = m^3 \cdot e^{mx}$$

$$\therefore D^n \cdot e^{mx} = m^n \cdot e^{mx}$$

Special Case

if $f(x) = \text{constant} = c \cdot e^{0 \cdot x}$

$\therefore D \longrightarrow 0$

Example : ① Find $y_{P.I}$ for the DE.

$$(D^2 - 4D + 4)\cdot y = 2\cdot e^{3x}$$

Solution

$$\because y_{P.I} = \frac{1}{D^2 - 4D + 4} \cdot 2e^{3x}$$

$\therefore$ $\boxed{D \longrightarrow 3}$

$$\therefore y_{P.I} = 2 \cdot \frac{1}{9 - 12 + 4} \cdot e^{3x} =$$

$$\therefore y_{P.I} = 2e^{3x}$$

Example : ② Find $y_{P.I.}$ for the DE

$$(D^2 - 2D + 2)\cdot y = 8$$

Solution

$$\because y_{P.I} = \frac{1}{\cancel{D^2} - 2\cancel{D} + 2} \cdot 8$$

$\boxed{\therefore D \longrightarrow 0}$

$$\therefore y_{P.I} = 4$$

Example : ③ Solve for $y_{P.I.}$

$$(D^2 + 2D) \cdot y = 6$$

Solution

$$\therefore\ y_{P.I.} = \frac{1}{D^2 + 2D} \cdot (6) \rightarrow \text{constant}$$

$D \rightarrow 0$

$$\therefore\ y_{P.I.} = \frac{1}{0} \cdot (6) = \alpha\ \alpha\ \alpha$$

$$\therefore\ y_{P.I.} = \frac{1}{D} \cdot \left[\frac{1}{\cancel{D}+2} (6)\right]$$

$D \rightarrow 0$

$$\boxed{D = \frac{d}{dx}}$$

$$\boxed{\frac{1}{D} = \int dx}$$

$$\therefore\ y_{P.I.} = \frac{1}{D} \cdot (3)$$

$$\therefore\ y_{P.I.} = \int 3 \cdot dx = 3 \cdot x$$

2] if $F(x) = \sin(mx)$ or $\cos(mx)$

$$\because y_{P.I.} = \frac{1}{P(D)} \cdot \begin{pmatrix} \sin(mx) \\ \cos(mx) \end{pmatrix}$$

Put $D^2 \longrightarrow -m^2$

Prove)

$$\because D \cdot \sin(mx) = m \cdot \cos(mx)$$

$$\because D^2 \cdot \sin(mx) = -m^2 \cdot \sin(mx)$$

$$D^2 \longrightarrow -m^2$$

$$\because D^3 \cdot \sin(mx) = -m^3 \cdot \cos(mx)$$

$$\because D^4 \cdot \sin(mx) = m^4 \cdot \sin(mx)$$

$$D^4 \longrightarrow m^4$$

Example : ③ Find $y_{P.I.}$ for the D.E.

$$y'' + 9 \cdot y = \sin(2x)$$

Solution

$$\because \ (D^2+9) \cdot y = \sin(2x)$$

$$\therefore \ y_{P.I.} = \frac{1}{D^2+9} \cdot (\sin 2x) \qquad D^2 \to -(2)^2$$

$$\therefore \ D^2 \to -4$$

$$\therefore \ y_{P.I.} = \frac{1}{-4+9} \cdot \sin 2x = \frac{1}{5} \cdot \sin 2x$$

Example : ④ Find $y_{P.I.}$ for

$$(D^2+4)\, y = 10 \cdot \cos(x)$$

Solution

$$\because \ y_{PI} = \frac{1}{\cancel{D^2}+4} \cdot 10 \cos(x)$$

-1

$$\therefore \ D^2 \to -(1)^2$$

$$\therefore \ D^2 \to -1$$

$$\therefore \ y_{P.I.} = \frac{10}{3} \cdot \cos x$$

Example : ⑤ Find $y_{P.I.}$ for the D.E.

$$(D^2+1)\cdot y = \sinh(2x)$$

Solution

$$\because y_{P.I.} = \frac{1}{D^2+1}\cdot \sinh(2x)$$

$D^2 \to (2)^2$

$\therefore D^2 \to 4$

$$\therefore y_{P.I.} = \frac{1}{4+1}\cdot \sinh(2x) = \frac{1}{5}\cdot \sinh(2x)$$

Example : ⑥ find $y_{P.I.}$ for the D.E.

$$(D^2-4)\, y = 2\cdot \cosh(4x)$$

Solution

$$\because y_{P.I.} = \frac{1}{D^2-4}\cdot 2\cosh(4x)$$

$D^2 \to (4)^2$

$\therefore D^2 \to 16$

$$\therefore y_{P.I.} = \frac{1}{16-4}\cdot 2\cosh(4x)$$

$$\therefore y_{P.I.} = \frac{1}{6}\cdot \cosh(4x)$$

4) if $F(x)$ = Polynomial function

$$\therefore F(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \cdots + a_n$$

كثيرة حدود من الدرجة (n)

$$\because y_{P.I.} = \frac{1}{P(D)} \cdot \left[a_0 \cdot x^n + a_1 \cdot x^{n-1} + \cdots + a_n \right]$$

using binomial Theorem

using long division

Recall

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 + \cdots$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2 \cdot x + 3 \cdot x^2 - 4 \cdot x^3 + \cdots$$

Example : 7 Find $y_{P.I.}$ for the DE.

$$(D^2 - 2D + 1)\cdot y = x^2 + 3x + 4$$

Solution

$$\because \; y_{P.I.} = \frac{1}{D^2 - 2D + 1}\cdot\left(x^2 + 3x + 4\right)$$

كثيرة حدود من الدرجة (2)

using long division

لازم ترتيبها

Stop

$$\begin{array}{r|l} & 1 + 2D + 3D^2 \\ \hline 1 - 2D + D^2 & 1 \\ & \cancel{-1} \pm 2D \mp D^2 \\ \hline & 2D - D^2 \\ & -2D \pm 4D^2 \mp 2D^3 \\ \hline & 3D^2 - 2D^3 \end{array}$$

$\div$

$\div$

$$\therefore \; y_{P.I.} = \left(1 + 2D + 3D^2 + \cdots\right)\cdot\left(x^2 + 3x + 4\right)$$

$$\therefore \; y_{P.I.} = 1\cdot\left(x^2 + 3x + 4\right) + 2\cdot(2x + 3) + 3\cdot(2)$$

Example : (8) Find $y_{P.I.}$ for the DE.

$$(D^2+1)\cdot y = x^4 + 2$$

Solution

$$\because \; y_{P.I.} = \frac{1}{D^2+1}\,(x^4+2)$$

كثيرة حدود من الدرجة (4)

long division

Binomial Theorem

$$\because \; \frac{1}{D^2+1} = (1+D^2)^{-1} = 1 - D^2 + D^4 - D^6 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\therefore \; y_{P.I.} = (1 - D^2 + D^4 - \cdots)\cdot(x^4+2)$$

$$\therefore \; y_{P.I.} = 1\cdot(x^4+2) - (12\cdot x^2) + (24)$$

$$\therefore \; y_{P.I.} = x^4 - 12\cdot x^2 + 26$$

Exercise

Find $y_{P.I.}$ for the D.E.s

① $(D^2 - 2D + 3)\cdot y = 2\cdot e^x$

② $(D^2 - 2D)\cdot y = e^{-4x}$

③ $y'' + 2y' = 4$

④ $y'' + 16y = 4\cdot\cos(3x)$

⑤ $y'' - 8\cdot y = 2\cdot\cosh(2x)$

⑥ $(D^2 - 3D + 1)y = 2x^2 + 5x$

⑦ $(D^2 + 3D + 2)y = x^2 + 8$

⑧ $(D-1)^2\cdot y = x^3 + 8$

Chapter Two

Complex numbers

If the imaginary unit i (where $i^2 = -1$) is combine with two real numbers α, β by the processes of addition and multiplication, we obtain a complex number $\alpha + i\beta$. If $\alpha = 0$, the number is said to be purely imaginary, if $\beta = 0$ it is of course real. Zero is the only number which is at once real and imaginary.

Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

i.e. $\alpha_1 + i\beta_1 = \alpha_2 + i\beta_2 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 \quad and \quad \beta_1 = \beta_2$

Assuming that the ordinary rules of arithmetic apply to complex numbers, we find indeed:-

1. $(\alpha_1 + i\beta_1) \mp (\alpha_2 + i\beta_2) = (\alpha_1 \mp \alpha_2) + i(\beta_1 \mp \beta_2)$
2. $(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)$
   where $i^2 = -1$
3. $\dfrac{\alpha_1 + i\beta_1}{\alpha_2 + i\beta_2} * \dfrac{\alpha_2 - i\beta_2}{\alpha_2 - i\beta_2} = \dfrac{\alpha_1\alpha_2 + \beta_1\beta_2}{\alpha_2^2 + \beta_2^2} + i\dfrac{\alpha_2\beta_1 + \alpha_1\beta_2}{\alpha_2^2 + i\beta_2^2}$

The real number $\alpha_2 - i\beta_2$ that is used as multiplier to clear the i out of the denominator is called the complex conjugate of $\alpha_2 + i\beta_2$. It is customary to use $\bar{z}$ to denote the complex conjugate of z, thus $z = \alpha + i\beta$ and $\bar{z} = \alpha - i\beta$.

We note that i^n has only four possible values 1 , i , -1 , $-i$. They correspond to values of n which divided by 4 leave the reminders 0 , 1 , 2 , 3 .

<u>*EX-1*</u> – Find the values of :

$$1)\ (1+2i)^3 \qquad 2)\ \frac{5}{-3+4i} \qquad 3)\ \left(\frac{2+i}{3-2i}\right)^2$$

<u>*Sol.*</u> –

1) $(1+2i)^3 = 1+6i+12i^2+8i^3 = 1+6i-12-8i = -11-2i$

2) $$\frac{5}{-3+4i} * \frac{-3-4i}{-3-4i} = \frac{-15-20i}{9+16} = -\frac{3}{5} - i\frac{4}{5}$$

3) $$\left(\frac{2+i}{3-2i} * \frac{3+2i}{3+2i}\right)^2 = \left(\frac{6+7i-2}{9+4}\right)^2 = \left(\frac{4+7i}{13}\right)^2$$

$$= \frac{16+56i-49}{169} = -\frac{33}{169} + \frac{56}{169}i$$

EX-2- If $z=x+iy$ where x and y are real, find the real and imaginary parts of:-

1) z^4 2) $\frac{1}{z}$ 3) $\frac{z-1}{z+1}$ 4) $\frac{1}{z^2}$

Sol.-

1) $$z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$
$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$$

2) $$\frac{1}{z} = \frac{1}{x+iy} * \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

3) $$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} * \frac{(x+1)-iy}{(x+1)-iy} = \frac{x^2-1-2iy+y^2}{(x+1)^2+y^2}$$
$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} - i\frac{2y}{(x+1)^2+y^2}$$

4) $$\frac{1}{z^2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2xyi} * \frac{x^2-y^2-2xyi}{x^2-y^2-2xyi}$$
$$= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2+4x^2y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

EX-3- Show that $\left(\frac{-1 \mp i\sqrt{3}}{2}\right)^3 = 1$ for all combination of signs.

Sol.-

$$L.H.S. = \left(\frac{-1 \mp i\sqrt{3}}{2}\right)^3 = \frac{1}{8}\left[(-1)^3 + 3(-1)^2(\mp i\sqrt{3}) + 3(-1)(\mp i\sqrt{3})^2 + (\mp i\sqrt{3})^3\right]$$

$$= \frac{1}{8}\left[-1 \cancel{\mp i3\sqrt{3}} + 9 \cancel{\pm i3\sqrt{3}}\right] = 1 = R.H.S.$$

EX-4- **Solve the following equation for the real numbers x and y.**

$$(3+4i)^2 - 2(x - iy) = x + iy$$

Sol.-

$$9 + 24i + 16i^2 = 2x - 2iy + x + iy$$

$$-7 + 24i = 3x - iy \;\Rightarrow\; -7 = 3x \;\Rightarrow\; x = -\frac{7}{3}$$

$$\Rightarrow\; 24 = -y \;\Rightarrow\; y = -24$$

Argrand Diagrams:- **There are two geometric representation of the complex number $z = x + iy$:-**

a) as the point $P(x,y)$ in the xy-plane , and

b) as the vector $\overrightarrow{op}$ from the origin to P.

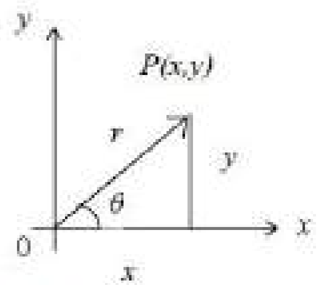
In each representation, the x-axis is called the real axis and the y-axis is the imaginary axis, as following figure.

In terms of the polar coordinates of x and y , we have:-

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad \tan\theta = \frac{y}{x}$$

and $z = r(\cos\theta + i\sin\theta)$

(polar representation)



The length r of a vector $\overrightarrow{op}$ from the origin to $P(x,y)$ is:

$$|x + iy| = \sqrt{x^2 + y^2}$$

The polar angle θ is called the argument of z and is written

$$\theta = \arg z$$

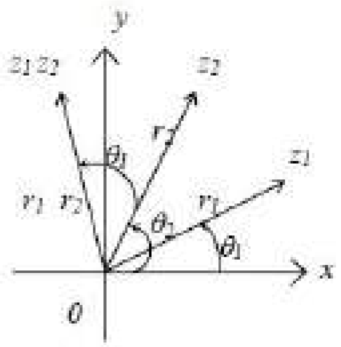
The identity $e^{i\theta} = cos\theta + i\,sin\theta$ is used for calculating products, quotients, powers, and roots of complex numbers. Then $z = re^{i\theta}$ exponential representation.

a) Product: To multiply two complex numbers (figure below):

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ so that $\begin{array}{ll} |z_1| = r_1 & , \; arg\, z_1 = \theta_1 \\ |z_2| = r_2 & , \; arg\, z_2 = \theta_2 \end{array}$

Then $z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$

hence $\begin{array}{l} |z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2| \\ arg(z_1 z_2) = \theta_1 + \theta_2 = arg\, z_1 + arg\, z_2 \end{array}$



b) Quotients: $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

hence $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$ and $arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = arg\, z_1 - arg\, z_2$

<u>*EX-5*</u>- Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find:

1) the exponential representation for z_1 and z_2.
2) the values of $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in exponential and polar representations.

<u>*Sol.*</u>-

1) $z_1 = 1 + i \;\Rightarrow\; x_1 = 1 \;,\; y_1 = 1 \;\Rightarrow\; r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1+1} = \sqrt{2}$

$$tan\,\theta = \frac{y}{x} \;\Rightarrow\; \theta_1 = tan^{-1}\frac{y_1}{x_1} = tan^{-1} 1 = \frac{\pi}{4} \qquad \therefore z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$z_2 = \sqrt{3} - i \;\Rightarrow\; x_2 = \sqrt{3} \;,\; y_2 = -1 \;\Rightarrow\; r_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{3+1} = 2$$

$$\Rightarrow\; \theta_2 = tan^{-1}\frac{y_2}{x_2} = tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{\pi}{6} \qquad \therefore z_2 = 2\, e^{-i\frac{\pi}{6}}$$

$$2)\quad z_1 z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}} \cdot 2\, e^{-i\frac{\pi}{6}} = 2\sqrt{2} e^{i\frac{\pi}{12}} \qquad \textit{exponential representation}$$

$$r = 2\sqrt{2} \quad , \quad \theta = \frac{\pi}{12} \quad \Rightarrow$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12} \right) \qquad \textit{polar representation}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2\, e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}} e^{i\frac{5}{12}\pi} \qquad \textit{exponential representation}$$

$$r = \frac{1}{\sqrt{2}} \quad , \quad \theta = \frac{5}{12}\pi \quad \Rightarrow$$

$$\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\left(\cos(\frac{5}{12}\pi) + i \sin(\frac{5}{12}\pi) \right) \qquad \textit{polar representation}$$

c) **Powers: If n is a positive integer, then:**

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \qquad \textit{hence} \quad \left| z^n \right| = r^n \quad \textit{and} \quad \arg z^n = n\theta$$

$$\textit{DeMoivres Theorem :} \quad (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$$

<u>*EX-6*</u>- Find: $\left(\sqrt{3} - i\right)^{10}$

<u>*Sol.-*</u>

$$\sqrt{3} - i \underset{y=-1}{\overset{x=\sqrt{3}}{\Longrightarrow}} \quad r = \sqrt{3+1} = 2 \quad \textit{and} \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\sqrt{3} - i = 2\left(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}) \right) = 2\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6} \right)$$

$$\left(\sqrt{3} - i\right)^{10} = 2^{10}\left(\cos 10\frac{\pi}{6} - i \sin 10\frac{\pi}{6} \right) = 512 + 512\sqrt{3} i$$

d) **Roots: If $z = re^{i\theta}$ is a complex number different from zero and n is a positive integer, then there are precisely n different complex numbers $w_0, w_1, w_2, \ldots\ldots, w_{n-1}$, that are *nth* roots of z given by:**

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \quad , \quad k = 0,1,2,\ldots\ldots\ldots,n-1$$

EX-7- **Find the four forth roots of (-16)**

Sol.-

$$z=-16 \Rightarrow r=\sqrt{(-16)^2+0}=16 \quad \& \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{0}{-16}=\pi$$

$$\sqrt[4]{-16}=\sqrt[4]{16}\, e^{i(\frac{\pi}{4}+k\frac{2\pi}{4})}=2e^{i(\frac{\pi}{4}+\frac{\pi}{2}k)} \quad , \quad k=0,1,2,3$$

$$at \quad k=0 \Rightarrow 1st\ root=w_0=2e^{i\frac{\pi}{4}}=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}+\sqrt{2}i$$

$$at \quad k=1 \Rightarrow 2nd\ root=w_1=2e^{i\frac{3\pi}{4}}=2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-\sqrt{2}+\sqrt{2}i$$

$$at \quad k=2 \Rightarrow 3rd\ root=w_2=2e^{i\frac{5\pi}{4}}=2\left(\cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\right)=-\sqrt{2}-\sqrt{2}i$$

$$at \quad k=3 \Rightarrow 4th\ root=w_3=2e^{i\frac{7}{4}\pi}=2\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)=\sqrt{2}-\sqrt{2}i$$

EX-8- **Find the four solutions of the equation:-** $z^4-2z^2+4=0$

Sol.-

$$z^4-2z^2+4=0 \Rightarrow z^2=\frac{2\mp\sqrt{4-4*1*4}}{2*1}=1\mp\sqrt{3}i \Rightarrow z=\mp\sqrt{1\mp i\sqrt{3}}$$

$$\left\{ for \quad ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\mp\sqrt{b^2-4ac}}{2a} \right\}$$

$$for \quad \sqrt{1+i\sqrt{3}} \quad \Rightarrow \quad r=\sqrt{1+3}=2 \quad and \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}$$

$$1st\ root=w_0=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3})}=\sqrt{2}e^{i\frac{\pi}{6}}=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$$

$$2nd\ root=w_1=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3}+2\pi)}=\sqrt{2}e^{i\frac{7}{6}\pi}=\sqrt{2}\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

$$=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i$$

$$for \quad \sqrt{1-i\sqrt{3}} \Rightarrow r=\sqrt{1+3}=2 \quad and \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{-\sqrt{3}}{1}=-\frac{\pi}{3}$$

$$3rd\ root\ = w_2 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3})} = \sqrt{2}e^{i(-\frac{\pi}{6})} = \sqrt{2}\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$$

$$= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

$$4th\ root\ = w_3 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3}+2\pi)} = \sqrt{2}e^{i\frac{5}{6}\pi} = \sqrt{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right)$$

$$= -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

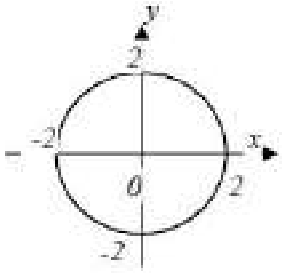
EX-9- **Graph the points $z = x + iy$ that satisfy the given conditions:-**

1) $|z| = 2$ *2)* $|z| < 2$ *3)* $|z| > 2$ *4)* $|z + 1| = |z - 1|$

Sol.-

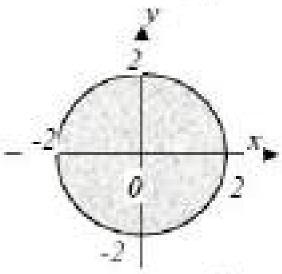
1) $|z| = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$

The points on the circle with center at origin, and radius 2.



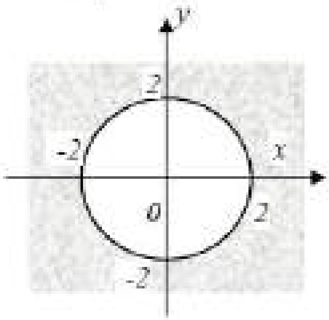
2) $|z| < 2 \Rightarrow \sqrt{x^2 + y^2} < 2 \Rightarrow x^2 + y^2 < 4$

The interior points of the circle with center at origin, and radius 2.



3) $|z| > 2 \Rightarrow \sqrt{x^2 + y^2} > 2 \Rightarrow x^2 + y^2 > 4$

The exterior points of the circle with center at origin, and radius 2.

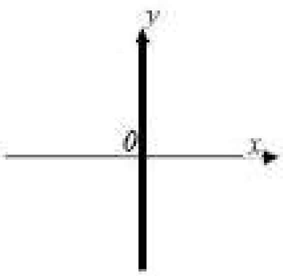


4) $|z + 1| = |z - 1| \Rightarrow |x + iy + 1| = |x + iy - 1|$

$\Rightarrow \sqrt{(x + 1)^2 + y^2} = \sqrt{(x - 1)^2 + y^2} \Rightarrow$

$x^2 + 2x + 1 + y^2 = x^2 - 2x + 1 + y^2 \Rightarrow x = 0$

The points on the y - axis.



Problems

1) **Find the values of:-**

a) $(2+3i)(4-2i)$ *(ans. : $14+8i$)*

b) $(2-i)(-2+3i)$ *(ans. : $-1+8i$)*

c) $(-1-2i)(2+i)$ *(ans. : $-5i$)*

2) **Show that** $\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$ **for all combination of signs.**

3) **Solve the following equation for the real numbers** x **and** y **:-**

$(3-2i)(x+iy) = 2(x-2iy)+2i-1$ *(ans. : $x=-1$; $y=0$)*

4) **Show that** $|\bar{z}| = |z|$.

5) **Let** $Re(z)$ **and** $Im(z)$ **denote respectively the real and imaginary parts of** z **, show that:-**

a) $z+\bar{z} = 2\,Re$

b) $z-\bar{z} = 2i\,Im(z)$

c) $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,Re(z_1\bar{z}_2)$

6) **Graph the points** $z = x+iy$ **that satisfy the given conditions:-**

a) $|z-1| = 2$ *(ans. : on the circle with center (1,0), radius 2)*

b) $|z+1| = 1$ *(ans. : on the circle with center (−1,0), radius 1)*

c) $|z+i| = |z-1|$ *(ans. : on the line $y=-x$)*

7) **Express the following complex number in exponential form with** $r \geq o$ *and* $-\pi < \theta < \pi$:-

a) $(1+\sqrt{-3})^2$ (ans. : $4e^{i\frac{2}{3}\pi}$)

b) $\dfrac{1+i}{1-i}$ (ans. : $e^{i\frac{\pi}{2}}$)

c) $\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$ (ans. : $e^{i\frac{\pi}{2}}$)

d) $(2+3i)(1-2i)$ (ans. : $\sqrt{65}e^{i\tan^{-1}(-0.125)}$)

8) Find the three cube roots of 1. (ans. : $-\dfrac{1}{2}\mp i\dfrac{\sqrt{3}}{2}$)

9) Find the two square roots of i. (ans. : $\mp\dfrac{1}{\sqrt{2}}\mp i\dfrac{1}{\sqrt{2}}$)

10) Find the three cube roots of $(-8i)$.

(ans. : $-2i$; $\mp\sqrt{3}-i$)

11) Find the six sixth roots of (64).

(ans. : ∓ 2 ; $1\mp i\sqrt{3}$; $-1\mp i\sqrt{3}$)

12) Find the six solutions of the equation: $z^6+2z^3+2=0$

(ans. : $\sqrt[3]{2}\left(\cos\dfrac{2}{9}\pi\mp i\sin\dfrac{2}{9}\pi\right)$;

$\sqrt[3]{2}\left(-\cos\dfrac{\pi}{9}\mp i\sin\dfrac{\pi}{9}\right)$; $\sqrt[3]{2}\left(\cos\dfrac{4}{9}\pi\mp i\sin\dfrac{4}{9}\pi\right)$)

13) Find all solutions of the equation: $x^4+4z^2+16=0$

(ans. : $1\mp i\sqrt{3}$; $-1\mp i\sqrt{3}$)

14) Solve the equation: $x^4+1=0$

(ans. : $\dfrac{1}{\sqrt{2}}\mp\dfrac{i}{\sqrt{2}}$; $-\dfrac{1}{\sqrt{2}}\mp\dfrac{i}{\sqrt{2}}$)

Chapter five

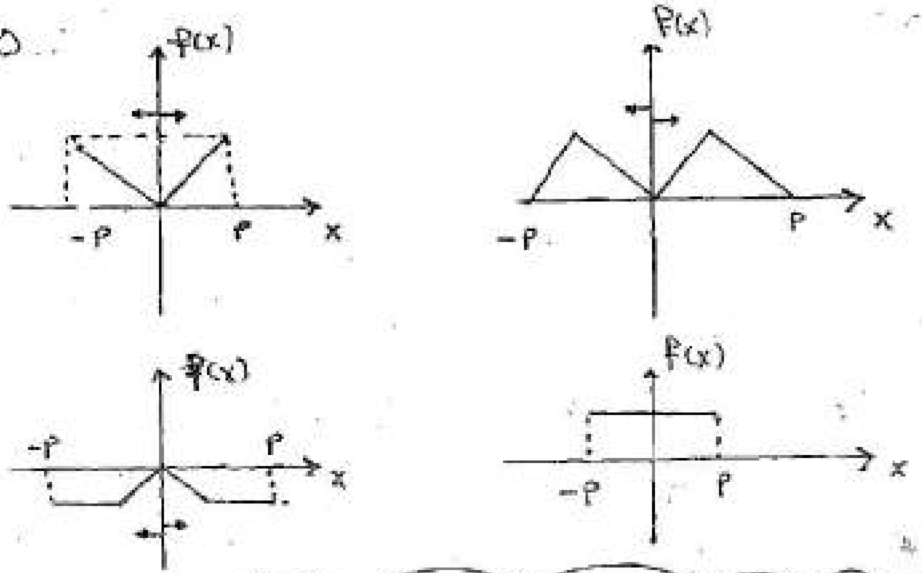
Fourier Series

Some Important Notes

Even function : $f(x) = f(-x)$ الداله الزوجيه

الداله الزوجيه تكون متماثله حول محور (y)

Examples.

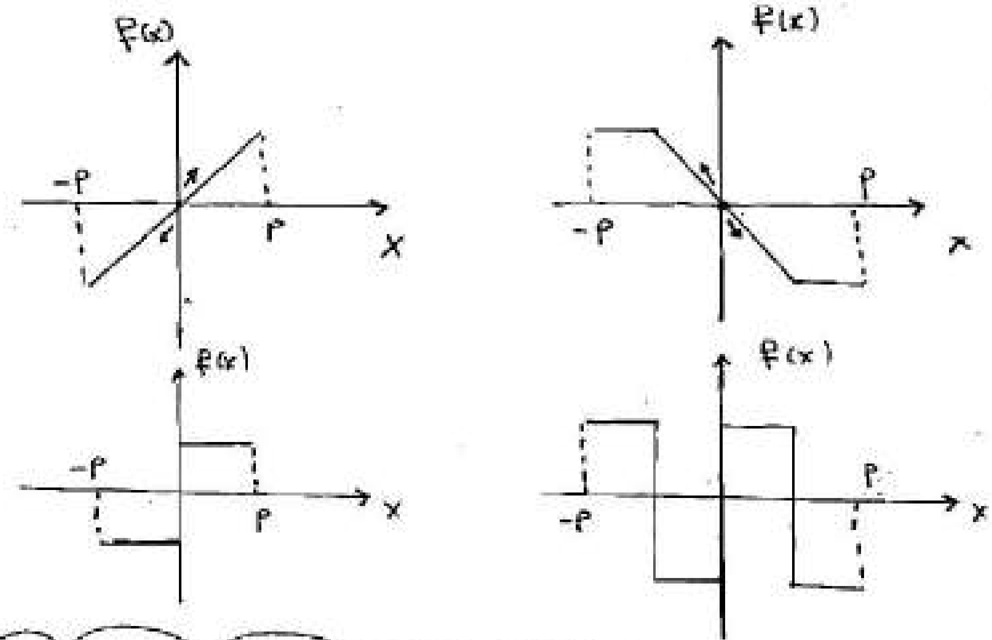


$$\int_{-P}^{P} f(x)\,dx = 2\int_{0}^{P} f(x)\,dx$$

Odd function :- $f(-x) = -f(x)$ الداله الفرديه

الداله الفرديه تكون متماثله حول نقطه الاصل (0)

Examples

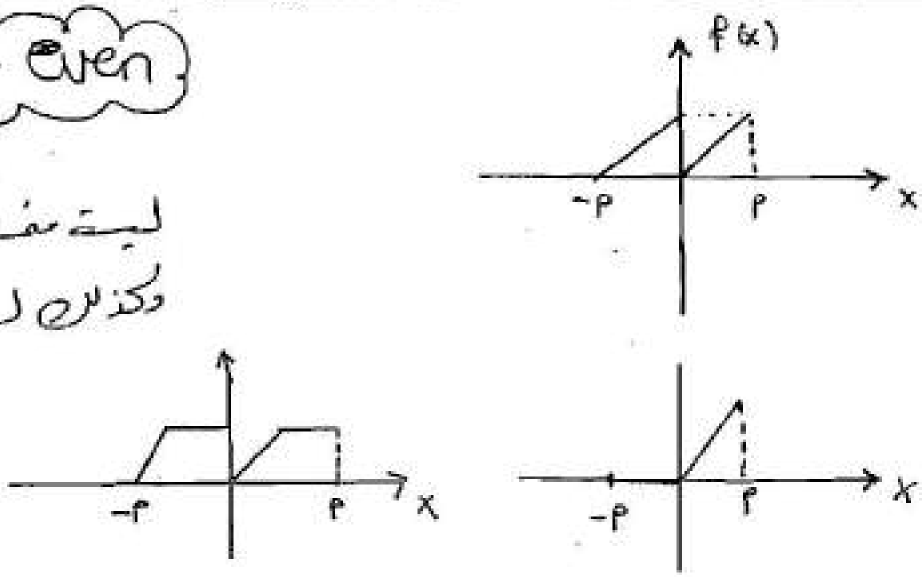


$$\int_{-p}^{p} f(x)\,dx = 0$$

Not odd / Not even

ليست متماثله حول محور (y)

وكذلك ليست متماثله حول نقطة الاصل



Note That

$$\boxed{\sin n\pi = 0}$$

$n = 0, \pm 1, \pm 2, \ldots$

i.e.

$$\boxed{\cos n\pi = (-1)^n} \quad \text{for all } n$$

$$\cos n\pi = \begin{cases} 1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases}$$

even

$$\boxed{\cos 2n\pi = 1}$$

Periodic Function :- $f(x) = f(x+\omega)$

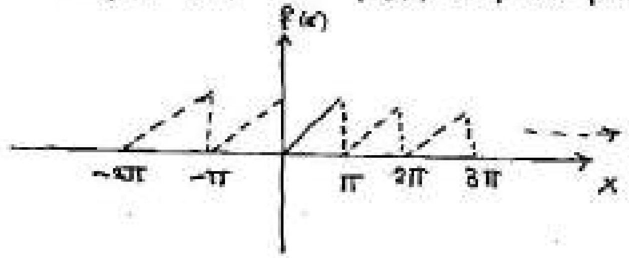
الداله التكراريه (تتكرر كل (ω))

اى ω هى الفتره الكامله (الدوره كامله)

ω = full period

Examples

① $f(x) = x \quad 0 < x < \pi \qquad f(x) = f(x+\pi)$

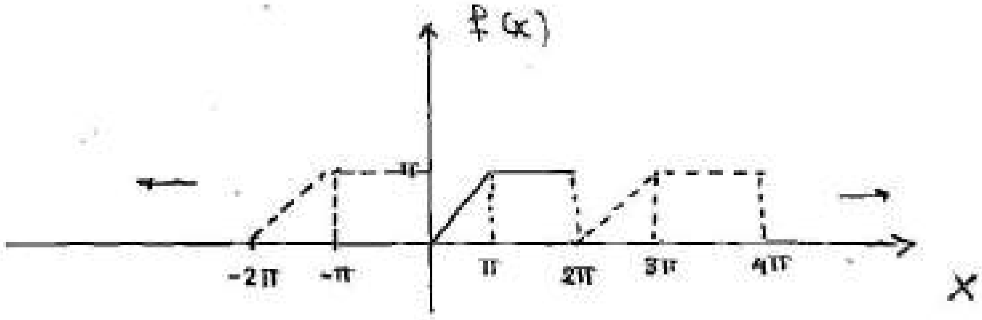


نلاحظ انها ليست زوجيه وليست فرديه

انقل محور (y) عند نهايه الفتره الاخيره كأنه نقطه الاصل وأرسم الداله مره اخرى

② $f(x) = x \quad 0 < x < \pi$

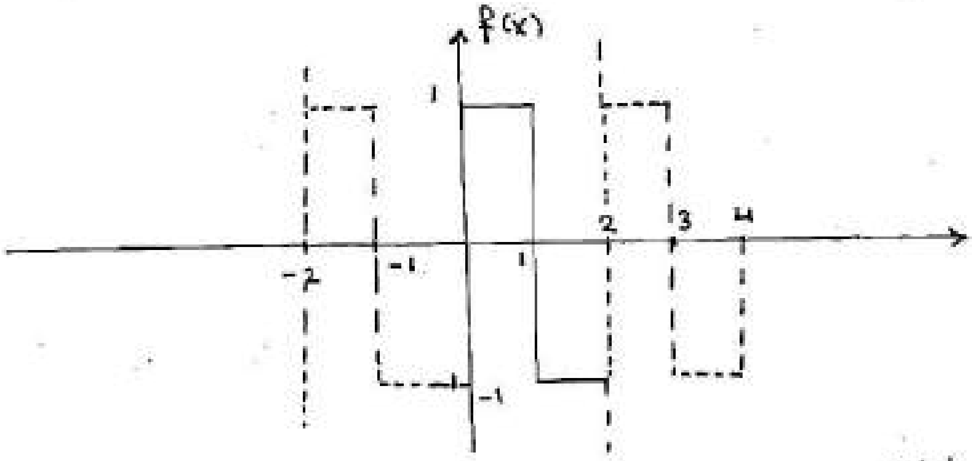
$\quad\;\; = \pi \quad \pi \leq x < 2\pi \qquad f(x) = f(x+2\pi)$



not odd
not even

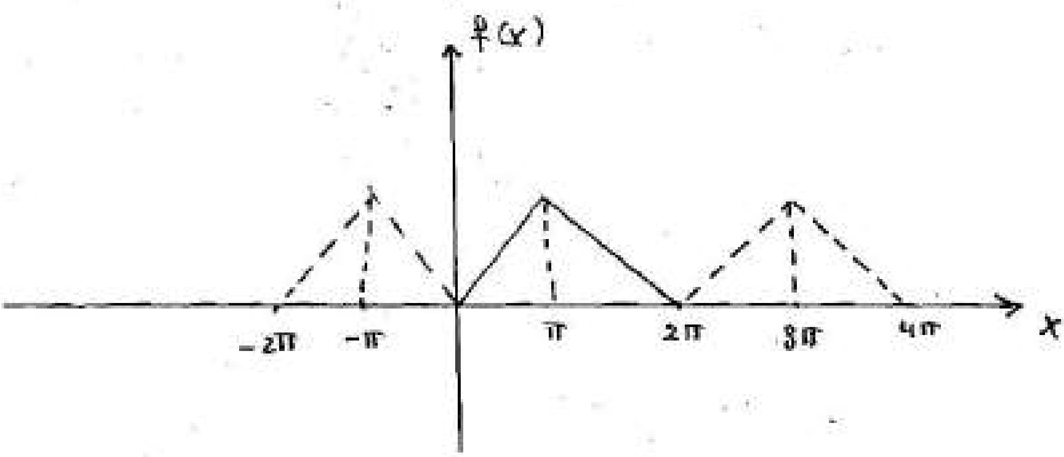
نلاحظ انها ليست فرديه وليست زوجيه

③ $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ -1 & 1 \le x < 2 \end{cases}$ $\quad f(x) = f(x+2)$



odd $\underline{f_n}$

④ $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$ $\quad f(x) = f(x+2\pi)$



even $\underline{f_n}$

Fourier Expansion

If $f(x) = f(x+2p)$ periodic function of period $2p$

$2p =$ الدوره الكامله
$=$ full period

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos \frac{n\pi}{p}x + b_n \sin \frac{n\pi}{p}x\right]$$

$\frac{a_0}{2}$: Average area in one period

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$\int_{-p}^{p} f(x)\,dx$: [area under the Curve in one period

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$f(x)$ odd fn	$f(x)$ Even fn	Not odd / Not even
$\boxed{\text{Sines only}}$ $a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$ او التكامل على نصف فتره	$\boxed{\text{Cosines only}}$ $b_n = 0$ $a_0 = \frac{2}{p}\int_0^p f(x)\,dx$ $a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$ او التكامل على نصف فتره	$a_0 = \frac{1}{p}\int$ $a_n = \frac{1}{p}\int$ $b_n = \frac{1}{p}\int$ التكامل على الفتره المعطاه في المسائل (الفتره الكامله)

Steps of Solution

① ارسم الدالة (الرسم مهم جدا)

② حدد الفترة الكاملة $2p$

③ حدد نوع الدالة زوجية او فردية
او لا زوجية / لا فردية

④ حدد a_0, a_n, b_n

⑤ نعوض في القانون الاصلي

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$$

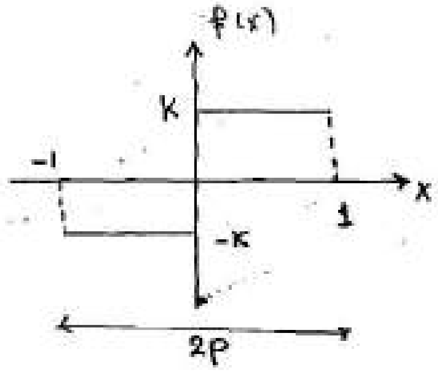
$f(x)$ odd	$f(x)$ even	neither even nor odd
$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$
Sines only	Cosines only	

Examples

① Expand in Fourier Series the f_n

$$f(x) = \begin{cases} -k & -1 < x < 0 \\ k & 0 \leq x < 1 \end{cases} \qquad f(x) = f(x+2)$$

Solution



$2p = 2$

$\boxed{p = 1}$

نلاحظ أن الدالة متماثلة حول نقطة الأصل (o)

$\therefore f(x)$ is odd f_n

$\therefore a_0 = a_n = 0 \qquad \therefore f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx$$

$$b_n = 2 \int_0^1 k \sin n\pi x \, dx$$

$$= -2k \frac{1}{n\pi} \left(\cos n\pi x \right)_0^1$$

$$b_n = \frac{-2k}{n\pi} \left(\cos n\pi - 1 \right) = \frac{-2k}{n\pi} \left((-1)^n - 1 \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{P} x \qquad \boxed{P=1}$$

$$= \sum_{n=1}^{\infty} \frac{-2K}{n\pi}\left((-1)^n - 1\right) \sin n\pi x$$

$$f(x) = \frac{-2K}{\pi}\left(-2\sin \pi x + 0 - \frac{2}{3}\sin 3\pi x + \cdots\cdots\right)$$

OR we can say $b_n = \frac{-2K}{n\pi}\left((-1)^n - 1\right)$

$$b_n = \begin{cases} \frac{4K}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\therefore f(x) = \sum_{n=\text{odd}} \frac{4K}{n\pi} \sin n\pi x$$

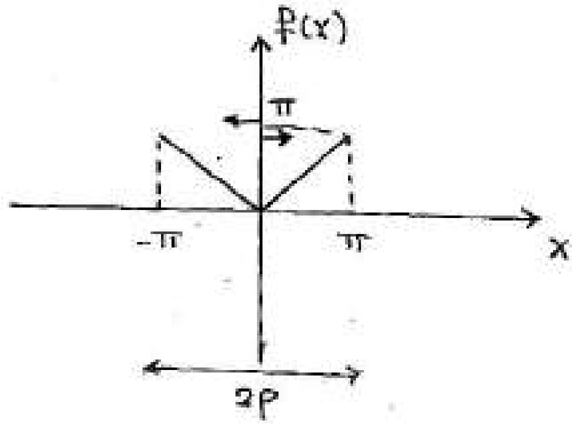
$$f(x) = \frac{4K}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x \cdots\right)$$

② Expand in Fourier series $f(x) = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x \leq 0 \end{cases}$

Soln

$2p = 2\pi$

$\boxed{p = \pi}$



f(x) is even $f_{\underline{n}}$

$\boxed{b_n = 0}$ $a_0 = \checkmark$ $a_n = \checkmark$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$

* $\underline{a_0}$ $a_0 = \frac{2}{p} \int_0^p f(x)\, dx$

$a_0 = \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{\cancel{2}}{\pi} \left(\frac{x^2}{\cancel{2}} \right)_0^{\pi}$

$a_0 = \frac{\pi^{\cancel{2}}}{\cancel{\pi}}$ $\boxed{a_0 = \pi}$

* $\underline{a_n}$ $a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x\, dx$

$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx$

d	$\int$
x	$\cos nx$
1	$\frac{1}{n}\sin nx$
0	$-\frac{1}{n^2}\cos nx$

$$a_n = \frac{2}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi}{n}\underbrace{\cancel{\sin n\pi}}_{0} + \frac{1}{n^2}\underbrace{\cancel{\cos n\pi}}_{(-1)^n}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$a_n = \frac{2}{\pi}\left(\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right)$$

$$\therefore \quad a_n = \frac{2}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad \boxed{p=\pi}$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right)\cos nx$$

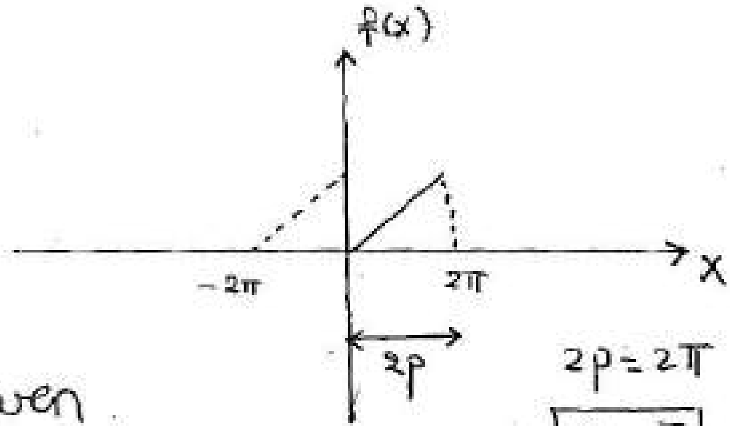
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left(-2\cos x + 0 - \frac{2}{9}\cos 3x \cdots\right)$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x \cdots\right)$$

(5) Find Fourier Series for

$$f(x) = x \qquad 0 \leq x \leq 2\pi$$

Solution



$2p = 2\pi$

$\boxed{p = \pi}$

f(x) not odd / not even

$a_0 = ✓,\ a_n = ✓,\ b_n = ✓$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

* a_0

$$a_0 = \frac{1}{p}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}$$

$$= \frac{1}{\cancel{2\pi}}\cancel{4}^{2}\pi^{\cancel{2}} \qquad \boxed{a_0 = 2\pi}$$

$*\ \underline{a_n}$

$$a_n = \frac{1}{P}\int_0^{2\pi} x\cos nx\, dx$$

d	∫
x	$\cos nx$
1	$\frac{1}{n}\sin nx$
0	$-\frac{1}{n^2}\cos nx$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{1}{n}2\pi\,\underbrace{\sin 2n\pi}_{0} + \frac{1}{n^2}\underbrace{\cos 2n\pi}_{1}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$a_n = \frac{1}{\pi}\underbrace{\left(\frac{1}{n^2} - \frac{1}{n^2}\right)}_{0}$$

$$\boxed{a_n = 0}$$

$*\ \underline{b_n}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\, dx$$

x	$\sin nx$
1	$-\frac{1}{n}\cos nx$
0	$-\frac{1}{n^2}\sin nx$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

$$b_n = \frac{1}{\cancel{\pi}}\left[\left(-\frac{1}{n}2\cancel{\pi}\,\underbrace{\cos 2n\pi}_{1} + \frac{1}{n^2}\underbrace{\sin 2n\pi}_{0}\right) - 0\right]$$

$$b_n = -\frac{2}{n}$$

$$f(x) = \frac{2\pi}{2} + \sum \cancel{0} + \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$$

$$\boxed{f(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots\right)}$$

Laplace Transform

* Recall

In case of $f(t) = t^n$

اذا كانت n ← عدد صحيح موجب

$$\boxed{\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

* (اذا كانت n كسر)

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$\Gamma(n+1) \longrightarrow$ is called Gamma function

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(n+1) = n! \longrightarrow$ n +ve integer

$\Gamma(1) = 1 \quad , \quad \Gamma(1/2) = \sqrt{\pi} \Rightarrow$ حفظ

Ex find $\mathcal{L}\{t^{3/2}\}$

Soln $\mathcal{L}\{t^{3/2}\} = \dfrac{\Gamma(5/2)}{s^{5/2}}$

$$\Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$$

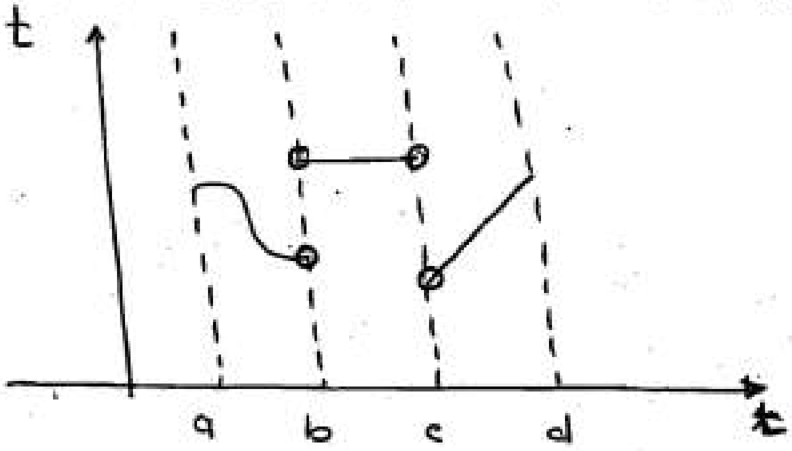
$$\therefore \mathcal{L}\{t^{3/2}\} = \frac{3\sqrt{\pi}/4}{s^{5/2}}$$

Ex $\mathcal{L}\left\{\dfrac{1}{\sqrt{t}}\right\}$

Soln $\mathcal{L}\{t^{-1/2}\} = \dfrac{\Gamma(1/2)}{s^{1/2}}$

$$= \frac{\sqrt{\pi}}{s^{1/2}}$$

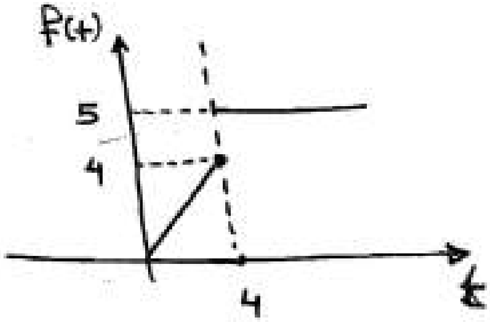
* Piecewise Continous function



$a \leqslant t \leqslant d$

Example :- Find laplace Transform for the Function

$$f(t) = \begin{cases} t & 0 \leqslant t < 4 \\ 5 & t \geqslant 4 \end{cases}$$



Soln

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^4 e^{-st}\, t\, dt + \int_4^\infty e^{-st} (5)\, dt$$

d	$\int$
t	e^{-st}
1	$-\frac{e^{-st}}{s}$
0	$\frac{e^{-st}}{s^2}$

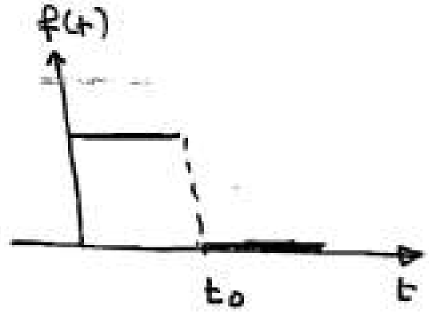
$$= \left[-\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^4 + 5 \left[-\frac{e^{-st}}{s} \right]_4^\infty$$

$$= \left[-\frac{4 e^{-4s}}{s} - \frac{e^{-4s}}{s^2} - \left(0 - \frac{1}{s^2}\right) \right] + 5 \left[0 + \frac{e^{-4s}}{s} \right]$$

$$F(s) = \frac{-4}{s} e^{-4s} - \frac{1}{s^2} e^{-4s} + \frac{1}{s^2} + \frac{5}{s} e^{-4s}$$

Ex:- Find L.T For $f(t) = 2 \qquad 0 \leq t < t_0$

$\qquad\qquad\qquad\qquad\quad = 0 \qquad t \geq t_0$

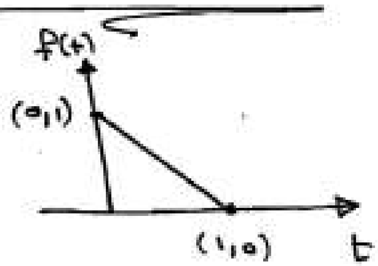


Soln $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^{t_0} e^{-st}(2)\, dt + \underbrace{\int_{t_0}^{\infty} e^{-st}(0)\, dt}_{0}$$

$$= \frac{-2}{s}\left[e^{-st}\right]_0^{t_0} = \frac{-2}{s}\left(e^{-st_0} - 1\right)$$

Ex:- Find L.T for



Soln First we must Find the equation of straight line
معادلة خط مستقيم

$$\Rightarrow \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \rightsquigarrow \text{معادلة الخط المستقيم } (x_1, y_1), (x_2, y_2)$$

$$\frac{y-1}{x-0} = \frac{0-1}{1-0} \qquad y - 1 = -x \qquad \boxed{y = 1 - x}$$

$$\therefore f(t) = \begin{cases} 1 - t & 0 \leq t < 1 \\ 0 & t > 1 \end{cases}$$

Some Important Rules

* $\ln A + \ln B = \ln AB$
* $\ln A - \ln B = \ln A/B$
* $n \ln A = \ln A^n$
* $-\ln A = \ln \frac{1}{A}$
* $\ln 1 = 0$

* $\tan^{-1} \infty = \pi/2$
* $\tan^{-1} 0 = 0$
* $\tan^{-1} 1 = \pi/4$
* $\frac{1}{\infty} = 0$
* $\cot^{-1} x = \tan^{-1} \frac{1}{x} = \pi/2 - \tan^{-1} x$

Note هام

* $\int_{4}^{\infty} \left(\frac{1}{x-3} - \frac{1}{x+1} \right) dx$

$$= \Big[\ln(x-3) - \ln(x+1) \Big]_{4}^{\infty} = \left[\ln \frac{x-3}{x+1} \right]_{4}^{\infty} \div \frac{x}{x}$$

$$= \left[\ln \frac{\infty}{\infty} - \ln \frac{1}{5} \right]$$

كمية غير معرفة

في هذه الحالة (كمية غير معرفة) نقسم البسط والمقام على أعلى أس (power) لـ (x)

$$I = \left[\ln \frac{1 - 3/x}{1 + 1/x} \right]_{4}^{\infty} = \left[\underset{0}{\ln 1} - \ln 1/5 \right] = -\ln 1/5$$

* Properties of Laplace Transform

[II] Derivative of The transform (Multiply by t)

If $\mathcal{L}\{f(t)\} = F(s)$

prove That $\mathcal{L}\{t\,f(t)\} = (-1)\dfrac{dF(s)}{ds}$

Soln:- $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty (-t)\, e^{-st} f(t)\,dt$$

$$\frac{dF(s)}{ds} = -\int_0^\infty e^{-st}\, t\,f(t)\, dt$$

$$\therefore \frac{dF(s)}{ds} = -\mathcal{L}\{t\,f(t)\}$$

$$\therefore \boxed{\begin{aligned} \mathcal{L}\{t\,f(t)\} &= -\frac{dF(s)}{ds} \\ \mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^nF(s)}{ds^n} \end{aligned}}$$

Ex: Find L.T For $F(t) = t\cos 3t$

Soln

$$\mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}F(s)$$

$$\cos 3t \xrightarrow{\mathcal{L}} \frac{s}{s^2+9}$$

$$\therefore \mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}\frac{s}{s^2+9}$$

Note $d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$$\therefore \mathcal{L}\{t\cos 3t\} = -\frac{(s^2+9) - s(2s)}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}$$

Note

ثابت

$$\frac{A}{u} \xrightarrow[\text{تفاضل}]{d/dx} \frac{-A}{u^2}u'$$

$$\frac{u}{v} \xrightarrow[\text{تفاضل}]{d/dx} \frac{vu' - v'u}{v^2}$$

Ex Find $\mathcal{L}.T$ for $f(t) = t\sin^2 2t$

Soln

$$\mathcal{L}\{t\sin^2 2t\} = -\frac{d}{ds}F(s)$$

$$\sin^2 2t = \frac{1}{2}(1-\cos 4t) \xrightarrow{\mathcal{L}.T} \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$\mathcal{L}\{t\sin^2 2t\} = -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right)$$

Ex Find $\mathcal{L}.T$ for $f(t) = te^{2t}\cosh t$

Soln $f(t) = t\,e^{2t}\frac{1}{2}(e^{t}+e^{-t}) = \frac{1}{2}t(e^{3t}+e^{t})$

$$\frac{1}{2}(e^{3t}+e^{t}) \xrightarrow{\mathcal{L}.T} \frac{1}{2}\left(\frac{1}{s-3} + \frac{1}{s-1}\right)$$

$$\frac{1}{2}t(e^{3t}+e^{t}) \xrightarrow{\mathcal{L}.T} -\frac{1}{2}\left(\frac{-1}{(s-3)^2} - \frac{1}{(s-1)^2}\right)$$

$$\therefore \mathcal{L}\{te^{2t}\cosh t\} = -\frac{1}{2}\left(-\frac{1}{(s-3)^2} - \frac{1}{(s-1)^2}\right)$$

Examples

① $\mathcal{L}\{e^{t}\cosh^2 2t\}$

Soln $\mathcal{L}\left\{e^{t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)^2\right\}$

$$=\frac{1}{4}\mathcal{L}\{e^{t}\cdot(e^{4t}+e^{-4t}+2)\}$$

$$=\frac{1}{4}\mathcal{L}\{e^{5t}+e^{-3t}+2e^{t}\}$$

$$=\frac{1}{4}\left[\frac{1}{s-5}+\frac{1}{s+3}+2\,\frac{1}{s-1}\right]$$

② $\mathcal{L}\{t^2\sin 2t\}$

① $\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$

② $\mathcal{L}\{t^2\sin 2t\} = (-1)^2\dfrac{d^2}{ds^2}\left(\dfrac{2}{s^2+4}\right)$

2 Integration of transform (Division by t)

If $\mathcal{L}\{f(t)\} = F(s)$

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds}$$

Take Care

* $\tan^{-1}\infty = \frac{\pi}{2}$
* $\int \frac{1}{s^2+a^2}\, ds = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + C$
* $\int \frac{s}{s^2+a^2}\, ds = \frac{1}{2}\ln(s^2+a^2) + C$

Note That

$$\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_{s}^{\infty} F(s)\, ds$$

Under Condition

$$\lim_{t \to 0} \frac{f(t)}{t} = \text{Exist}$$

Ex $f(t) = \frac{\sin t}{t}$ $\lim_{t \to 0} \frac{\sin t}{t} = 1$

Exist

Ex $f(t) = \frac{\cos t}{t}$ $\lim_{t \to 0} \frac{\cos t}{t} = \infty$

doesn't exist

Ex Find $\mathcal{L}.T$ for $f(t) = \dfrac{e^{3t} - e^{-t}}{t}$

Soln check $\lim_{t \to 0} \dfrac{e^{3t} - e^{-t}}{t} = \dfrac{0}{0}$

$$\lim_{t \to 0} \frac{3e^{3t} + e^{-t}}{1} = 4 \rightarrow \text{Exist}$$

$$\therefore \mathcal{L}\left\{\frac{e^{3t} - e^{-t}}{t}\right\} = \int_s^\infty F(s)\, ds$$

$$\mathcal{L}\{e^{3t} - e^{-t}\} = \frac{1}{s-3} - \frac{1}{s+1}$$

$$\mathcal{L}\left\{\frac{e^{3t} - e^{-t}}{t}\right\} = \int_s^\infty \frac{1}{s-3} - \frac{1}{s+1}\, ds$$

$$= \Big[\ln(s-3) - \ln(s+1)\Big]_s^\infty = \left[\ln \frac{s-3}{s+1}\right]_s^\infty$$

$$= \left[\ln \frac{1 - 3/s}{1 + 1/s}\right]_s^\infty = \left[\cancel{\ln 1}_{\to 0} - \ln \frac{s-3}{s+1}\right]$$

$$= \ln \frac{s+1}{s-3} \quad \#$$

* $F(t) = \dfrac{1-\cos 2t}{t}$

Check: $\lim_{t\to 0} \dfrac{1-\cos 2t}{t} = \dfrac{0}{0}$

$= \lim_{t\to 0} \dfrac{+2\sin 2t}{1} = 0$ Exist

Soln

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty F(s)\,ds'$$

$$1-\cos 2t \xrightarrow{\mathcal{L}} \frac{1}{s} - \frac{s}{s^2+4}$$

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_{s'}^{\infty}\left(\frac{1}{s} - \frac{s}{s'^2+4}\right)ds'$$

$$= \left[\ln s - \frac{1}{2}\ln(s'^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[2\ln s - \ln(s^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[\ln s^2 - \ln(s^2+4)\right]_{s'}^{\infty}$$

$$= \frac{1}{2}\left[\ln\frac{s^2}{s^2+4}\right]_{s'}^{\infty} =$$

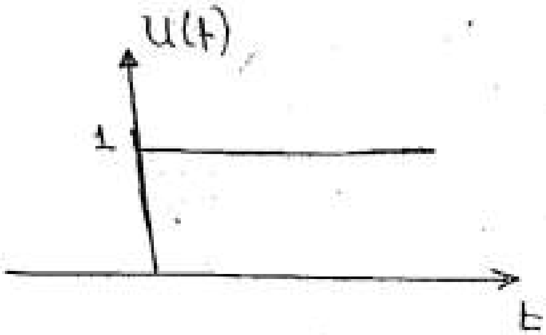
بالقسمة على اعلى اس لـ (s) وهو s^2

$$= \frac{1}{2}\left[\ln\frac{1}{1+4/s^2}\right]_{s}^{\infty} = \frac{1}{2}\left[\underbrace{\ln 1}_{0} - \ln\frac{s^2}{s^2+4}\right]$$

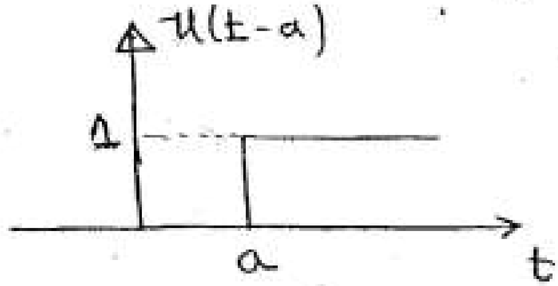
$$= -\frac{1}{2}\ln\frac{s'^2}{s^2+4} = \frac{1}{2}\ln\frac{s^2+4}{s^2}$$

* The Unit Step Function

* $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



* $u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$



* Prove That $\mathcal{L}\{u(t-a)\} = \frac{1}{s} e^{-as}$

Soln $\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$

$$= \int_a^a (0) + \int_a^\infty e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_a^\infty$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right] = \frac{1}{s} e^{-as}$$

$$\therefore \boxed{\mathcal{L}\{u(t-a)\} = \frac{1}{s} e^{-as}}$$

* Special case for $a = 0$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

Expressing any Function as a unit Step Function

كيفية تحويل اى داله بداله unit step

* If $f(t) = g(t) \quad t < a$

$\quad\quad = h(t) \quad t > a$

$$\boxed{f(t) = g(t) + [h(t) - g(t)]\, u(t-a)}$$

* If $f(t) = g_1(t) \quad t > a$

$\quad\quad = g_2(t) \quad a < t < b$

$\quad\quad = g_3(t) \quad t > b$

$$\boxed{f(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t-a) + [g_3(t) - g_2(t)]\, u(t-b)}$$

Ex:- Express the following functions as a unit step

1) $f(t) = 2 \quad 0 < t < 3$

$\quad\quad = -2 \quad t > 3$

2) $f(t) = 0 \quad 0 \leq t < \frac{3\pi}{2}$

$\quad\quad = \sin t \quad t > \frac{3\pi}{2}$

3) $f(t) = t^2 \quad 0 < t < 3$

$\quad\quad = 9 \quad 3 < t < 5$

$\quad\quad = 0 \quad t > 5$

Soln

1) $f(t) = 2 \quad 0 < t < 3$

$\quad = -2 \quad t > 3$

$f(t) = 2 + (-2-2)u(t-3)$

$f(t) = 2 - 4u(t-3)$

2) $f(t) = 0 \quad 0 < t \leqslant 3\pi/2$

$\quad = \sin t \quad t > 3\pi/2$

$f(t) = 0 + (\sin t - 0)u(t - \frac{3\pi}{2})$

$f(t) = \sin t \, u(t - \frac{3\pi}{2})$

3) $f(t) = t^2 \quad 0 < t < 3$

$\quad = 9 \quad 3 < t < 5$

$\quad = 0 \quad t > 5$

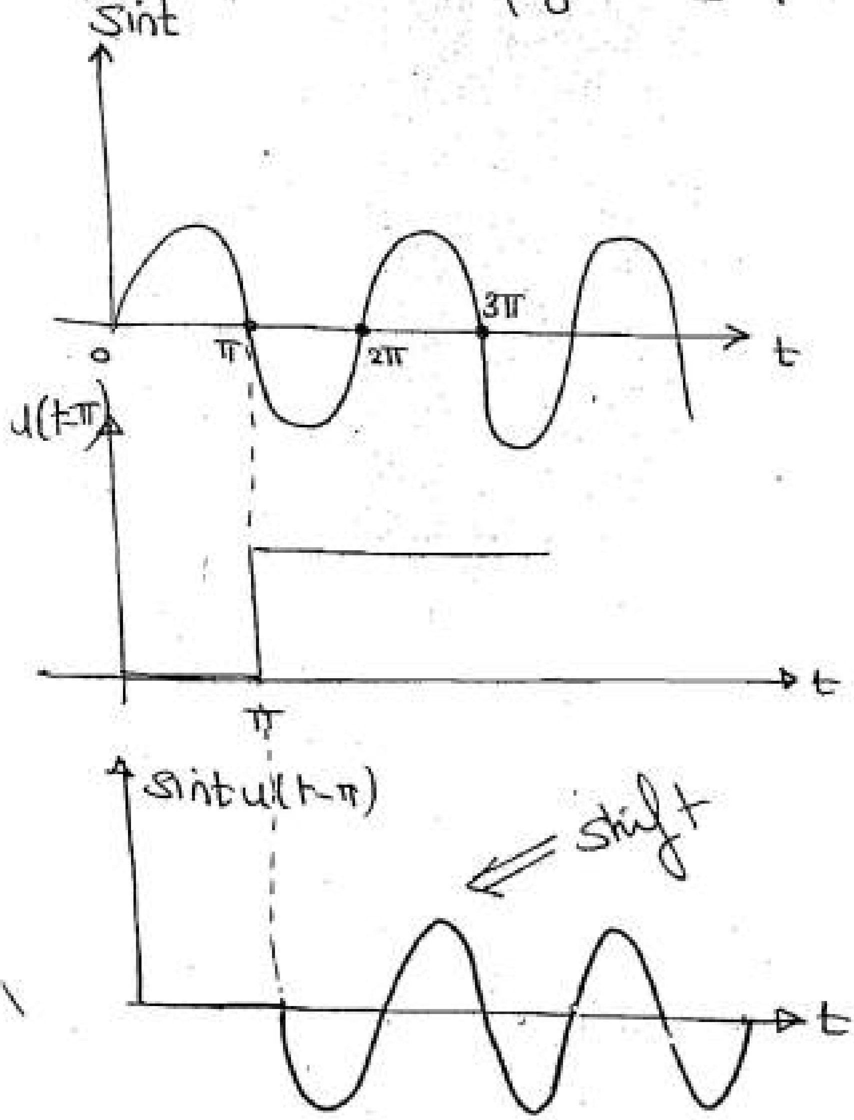
$t) = t^2 + (9 - t^2)u(t-3) + (0-9)u(t-5)$

$f(t) = t^2 + (9 - t^2)u(t-3) - 9u(t-5)$

* Ex:- Draw the given $\underline{fn}$

$$f(t) = \sin t\, u(t-\pi)$$

$$u(t-\pi) = \begin{cases} 1 & t > \pi \\ 0 & t < \pi \end{cases}$$



6 2nd shift Theorm

If $\mathcal{L}\{f(t)\} = F(s)$, then prove

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = F(s)\,e^{-as}$$

Soln

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

$$\therefore \mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^a \cancel{(0)}\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

0

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

put $t-a = x$ $\quad dt = dx \quad t = a+x$ $\quad$ at $t=a$ $\boxed{x=0}$ at $t=\infty$ $\boxed{x=\infty}$

$$\mathcal{L}\{\} = \int_0^\infty e^{-s(x+a)}\,f(x)\,dx$$

$$= \int_0^\infty e^{-sx}\,e^{-sa}\,f(x)\,dx$$

$$= e^{-as}\int_0^\infty e^{-sx}\,f(x)\,dx = e^{-as}F(s)$$

$$\boxed{\cdot a)\,u(t-a)\} = e^{-as}F(s)}$$

* Steps of Solution of 2nd shift

Ex

$\mathcal{L}\{(t-3)\,u(t-3)\}$

① التأكد ان كل t عبارة عن (t-3) وهي نفس الموجوده داخل (u)

$$\mathcal{L}\{(t-3)\,u(t-3)\} = \frac{1}{s^2}\,e^{-3s}$$

Ex: $\mathcal{L}\{(t-1)^3\,u(t-1)\}$

$\mathcal{L}\{t^3\}$

Soln $\mathcal{L}\{(t-1)^3\,u(t-1)\} = \frac{3!}{s^4}\,e^{-s}$

$\therefore \mathcal{L}\{e^{3t-3}\,u(t-1)\}$

$\mathcal{L}\{e^{3t}\}$

$$\mathcal{L}\{e^{3(t-1)}\,u(t-1)\} = \frac{1}{s-3}\,e^{-s}$$

Ex: $\mathcal{L}\{e^{2-t}\,u(t-2)\}$

Soln $\mathcal{L}\{e^{-(t-\cancel{2})}\,u(t-\cancel{2})\} = \dfrac{1}{s+1}\,e^{-2s}$

$\dfrac{1}{s+1} \rightarrow \mathcal{L}\{e^{-t}\}$

Ex: $\mathcal{L}\{\cos 2t\; u(t-\pi)\}$

Soln لازم كل t تكون على صورة $t-\pi$

$\therefore \cos 2t = \cos 2(t-\pi+\pi)$

$= \cos[(2t-2\pi)+2\pi]$

$= \cos(2t-2\pi)\underbrace{\cos 2\pi}_{1} - \underbrace{\sin 2\pi}_{0}\sin(2t-2\pi)$

$= \cos 2(t-\pi)$

$\mathcal{L}\{\cos 2t\, u(t-\pi)\} = \mathcal{L}\{\cos 2(t-\cancel{\pi})\, u(t-\cancel{\pi})\}$

$= \dfrac{s}{s^2+4}\,e^{-\pi s}$

$\hookrightarrow \mathcal{L}\{\cos 2t\}$

Ex: $\mathcal{L}\{\sin t\, u(t-2\pi)\}$

Soln $\sin t = \sin[(t-2\pi)+2\pi]$

$= \sin(t-2\pi)\underbrace{\cos 2\pi}_{1} + \cos(t-2\pi)\underbrace{\sin 2\pi}_{0}$

$= \sin(t-2\pi)$

$\therefore \mathcal{L}\{\sin t\, u(t-2\pi)\} = \mathcal{L}\{\sin(t-\cancel{2\pi})\, u(t-\cancel{2\pi})\}$

$= \mathcal{L}\{\sin t\}\, e^{-2\pi s}$

$= \dfrac{1}{s^2+1}\, e^{-2\pi s}$

---

x: $\mathcal{L}\{\sin t\, u(t-\pi/2)\}$

ln $\sin t = \sin[(t-\pi/2)+\pi/2]$

$= \sin(t-\pi/2)\underbrace{\cos \pi/2}_{0} + \underbrace{\sin \pi/2}_{1}\cos(t-\pi/2)$

int $u(t-\pi/2)\} = \mathcal{L}\{\cos(t-\cancel{\pi/2})\, u(t-\cancel{\pi/2})\}$

$= \dfrac{s}{s^2+1}\, e^{-\pi/2\, s}$

Ex: $\mathcal{L}\{(t-1)^3 e^{t-1} u(t-1)\}$

Soln $\mathcal{L}\{e^{t-\cancel{1}} (t-\cancel{1})^3 u(t-\cancel{1})\}$

$= \mathcal{L}\{e^{t} t^3\} e^{-s}$

$= \dfrac{3!}{(s-1)^4} e^{-s}$

Ex: $\mathcal{L}\{t u(t-3)\}$

Soln $\mathcal{L}\{t u(t-3)\} = \mathcal{L}\{[t-3+3] u(t-3)]$

$= \mathcal{L}\{(t-\cancel{3}) u(t-\cancel{3}) + 3 u(t-\cancel{3})\}$

$= \dfrac{1}{s^2} e^{-3s} + 3 \dfrac{1}{s} e^{-3s}$

Ex: $\mathcal{L}\{t^2 u(t-2)\}$

Soln $\quad t^2 = [(t-2)+2]^2 = (t-2)^2 + 4(t-2) + 4$

$$\mathcal{L}\{t^2 u(t-2)\} = \mathcal{L}\{[(t-2)^2 + 4(t-2) + 4]\, u(t-2)\}$$

$$= \mathcal{L}\{(t-\cancel{2})^2 u(t-\cancel{2}) + 4(t-\cancel{2}) u(t-\cancel{2}) + 4\, u(t-\cancel{2})\}$$

$$= \frac{2!}{s^3} e^{-2s} + 4\, \frac{1}{s^2} e^{-2s} + \frac{4\,1}{s} e^{-2s}$$

Ex: $\mathcal{L}\{(3t+1)u(t-3)\}$

Soln

$$3t = 3[(t-3)+3] = 3(t-3) + 9$$

$$\mathcal{L}\{(3t+1)u(t-3)\} = \mathcal{L}\{[3(t-3)+9+1]\, u(t-3)\}$$

$$= \mathcal{L}\{[3(t-3)+10]\, u(t-3)\}$$

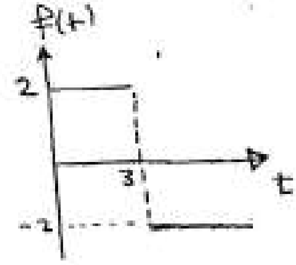
$$\mathcal{L}\{3(t-\cancel{3})u(t-\cancel{3})\} + \mathcal{L}\{10\, u(t-\cancel{3})\}$$

$$\frac{3}{s^2} e^{-3s} + \frac{10}{s} e^{-3s}$$

Important examples

Example (1) Express $f(t)$ in terms of a unit step, Then Find its $\mathcal{L}.T$

$$f(t) = 2 \quad 0 \leq t \leq 3$$
$$= -2 \quad t > 3$$



Soln:- $f(t) = 2 + (-2-2)\,u(t-3)$

$$f(t) = 2 - 4u(t-3)$$

$$\mathcal{L}\{2 - 4u(t-3)\} = \frac{2}{s} - \frac{4}{s} e^{-3s}$$

Ex(2) $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4 & t > 2 \end{cases}$

Sln $f(t) = t^2 + (4 - t^2)\,u(t-2)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 4u(t-2) - t^2 u(t-2)\}$$

$$t^2 = [(t-2)+2]^2 = (t-2)^2 + 4(t-2) + 4$$

$$= \mathcal{L}\{t^2 + \cancel{4u(t-2)} - (t-2)^2 u(t-2) - 4(t-2)u(t-2) - \cancel{4u(t-2)}\}$$

$$\frac{2!}{s^3} - \frac{2!}{s^3} e^{-2s} - \frac{4}{s^2} e^{-2s}$$

Ex(3) $f(t) = \begin{cases} e^{-t} & 0<t<2 \\ 0 & t>2 \end{cases}$

Soln $f(t) = e^{-t} + (0 - e^{-t})\,u(t-2)$

$$f(t) = e^{-t} - e^{-t}\,u(t-2)$$

$$t = (t-2)+2$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t} - e^{-[(t-2)+2]}\,u(t-2)\}$$

ثابت

$$= \mathcal{L}\{e^{-t} - e^{-\cancel{(t-2)}}\, e^{-2}\, u(\cancel{t-2})\}$$

$$= \frac{1}{s+1} - e^{-2}\,\frac{1}{s+1}\,e^{-2s}$$

---

xample $\mathcal{L}\{e^{2t}\,u(t-3)\}$

ln $t = (t-3)+3$

$$e^{2t}\,u(t-3)\} = \mathcal{L}\{e^{2[(t-3)+3]}\,u(t-3)\}$$

$$= \mathcal{L}\{e^{6}\, e^{\cancel{2(t-3)}}\, u(\cancel{t-3})\}$$

$$e^{6}\,\frac{1}{s-2}\,e^{-3s}$$

Example: Find $\mathcal{L}.T$ for $f(t) = e^{t}\sin t\; u(t-\pi)$

Soln.

$$t = (t-\pi)+\pi$$

$$\sin t = \sin[(t-\pi)+\pi]$$

$$= \sin(t-\pi)\underbrace{\cos\pi}_{-1} + \underbrace{\cancel{\sin\pi}}_{0}\cos(t-\pi)$$

$$= -\sin(t-\pi)$$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\left\{ e^{(t-\pi)+\pi} \left(-\sin(t-\pi)\right) u(t-\pi)\right\}$$

$$= \mathcal{L}\left\{ -e^{\pi}\, e^{t-\cancel{\pi}} \sin(t-\cancel{\pi})\, u(t-\cancel{\pi})\right\}$$

$$= -e^{\pi}\, \mathcal{L}\{e^{t}\sin t\}.\, e^{-\pi s}$$

$$= -e^{\pi} \; \frac{1}{(s-1)^2+1} \; e^{-\pi s}$$

Chapter four

Laplace Transform

Definition

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

$s > 0$

This transformation transforms from Domain to another Domain

$$\begin{tikzcd} \text{Time Domain } T \arrow[r, bend left, "L.T"] & \text{Frequency Domain } S \arrow[l, bend left, "(L.T)^{-1}"] \end{tikzcd}$$

Laplace Transform of Elementery f_n :-

②

1. $f(t) = K \longrightarrow$ Constant

$$F(s) = \int_0^\infty e^{-st} K \, dt = K\left(\frac{-1}{s}\right)\left(e^{-st}\right)_0^\infty$$

$$= -\frac{K}{s}\left(e^{-s(\infty)} - 1\right)$$

$$s > 0 \quad \therefore e^{-(s)\infty} = 0$$

$$F(s) = -\frac{K}{s}(0 - 1) = \frac{K}{s}$$

$$\mathcal{L}\{K\} = \frac{K}{s} \quad s > 0$$

---

2. $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{-1}{s-a}\left(e^{-(s-a)t}\right)_0^\infty = \frac{-1}{s-a}(0 - 1) \quad s > a$$

$$= \frac{1}{s-a}, \; s > a$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \; s > a$$

$\boxed{3}$ $f(t) = \cosh at$ $\qquad \cosh at = \frac{e^{at} + e^{-at}}{2}$

$$F(s) = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} + e^{-(s+a)t} \right) dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2}\left[\underset{s>a}{\frac{1}{s-a}} + \underset{s>-a}{\frac{1}{s+a}} \right]$$

$$= \frac{1}{2}\,\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \quad s > |a|$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \; s>|a|$$ بالمثل،

$\boxed{4}$ $f(t) = \sin at$ $\boxed{4}$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \frac{-1}{a} \int_0^\infty \underbrace{e^{-st}}_{u} \, d \underbrace{\cos at}_{v} = -\frac{1}{a} \left(e^{-st} \cos at \right)_0^\infty + \frac{1}{a} \int_0^\infty \cos at \, d e^{-st}$$

$$= -\frac{1}{a} (0 - 1) + \frac{1}{a} \int_0^\infty (-s) \, e^{-st} \cos at \, dt$$

$$= \frac{1}{a} - \frac{s}{a^2} \int_0^\infty \underbrace{e^{-st}}_{u} \, d \underbrace{\sin at}_{v}$$

$$= \frac{1}{a} - \frac{s}{a^2} \left(e^{-st} \cancel{\sin at} \right)_0^\infty + \frac{s}{a^2} \int_0^\infty \sin at \, d e^{-st} \rightarrow 0$$

$$= \frac{1}{a} + \frac{s}{a^2} \int_0^\infty (-s) \, e^{-st} \sin at \, dt$$

$$I = \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt \rightarrow I$$

$$I \left(1 + \frac{s^2}{a^2} \right) = \frac{1}{a} \quad \therefore I \left(\frac{a^2 + s^2}{a^2} \right) = \frac{1}{a}$$

$$\therefore I = \frac{a}{s^2 + a^2}$$

$$\therefore \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

وبالمثل

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

Summary

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Time domain $f(t)$	Laplace s-domain $F(s)$	Region of Convergence
K	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	عدد صحيح موجب n +ve integer $s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
e^{-at}	$\frac{1}{s+a}$	$s > -a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$

Note That

① $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

② $\mathcal{L}\{K\,f(t)\} = K\,\mathcal{L}\{f(t)\}$

③ $\mathcal{L}\{f(t)\,g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$

لا يساوي   هام جداً

* Take Care

* $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

* $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

---

* $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

* $\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$

* $\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

---

* $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

* $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

* $\cos(x \pm y)$
$= \cos x \cos y \mp \sin x \sin y$

* $\sin(x \pm y)$
$= \sin x \cos y \pm \sin y \cos x$

Examples

Find Laplace transform for

1) $f(t) = e^{-3t} + t^5 + 7t^2 - 3\sin 2t$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+3} + \frac{5!}{s^6} + 7\,\frac{2!}{s^3} - 3 * \frac{2}{s^2+4}$$

---

2) $f(t) = 3\cosh 2t - 7\sin 3t + 4 - t^3$

$$\mathcal{L}\{f(t)\} = 3\,\frac{s}{s^2-4} - 7 * \frac{3}{s^2+9} + 4/s - \frac{3!}{s^4}$$

---

3) $f(t) = (t^2+3)^2$

$$f(t) = t^4 + 6t^2 + 9$$

$$\mathcal{L}\{f(t)\} = \frac{4!}{s^5} + 6\,\frac{2!}{s^3} + 9/s$$

---

4) $f(t) = (e^{-2t}+1)^2$

$$f(t) = e^{-4t} + 2e^{-2t} + 1$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+4} + 2\,\frac{1}{s+2} + 1/s$$

$\boxed{5}$ $f(t) = \cos 3t + \sin 2t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 3t + \sin 2t\}$$

$$= \frac{s}{s^2+9} + \frac{2}{s^2+4}$$

$\boxed{6}$ $f(t) = \cos^2 3t$

$$\cos 6t = 2\cos^2 3t - 1$$

$$\cos^2 3t = \left(\frac{1+\cos 6t}{2}\right)$$

$$\therefore f(t) = \frac{1}{2}(1+\cos 6t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$\boxed{7}$ $f(t) = \cos 3t \sin t = \sin t \cos 3t$

We Know $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$$f(t) = \frac{1}{2}\left[\sin(4t) + \sin(-2t)\right] = \frac{1}{2}\left[\sin(4t) - \sin(2t)\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{4}{s^2+16} - \frac{2}{s^2+4}\right]$$

$\boxed{8}$ $f(t) = \sin^3 t$

$$f(t) = \sin t \, \sin^2 t$$

$$= \sin t \left[\tfrac{1}{2}(1-\cos 2t)\right]$$

$(\sin t \cos 2t)$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos 2t \sin t$$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{2}\left(\tfrac{1}{2}(\sin 3t - \sin t)\right)$$

$$= \tfrac{1}{2}\sin t - \tfrac{1}{4}\sin 3t + \tfrac{1}{4}\sin t$$

$$f(t) = \tfrac{3}{4}\sin t - \tfrac{1}{4}\sin 3t$$

$$\mathcal{L}\{f(t)\} = \frac{3}{4}\,\frac{1}{s^2+1} - \frac{1}{4}\,\frac{3}{s^2+9}$$

$\boxed{9}$ $f(t) = \sin(3t+2)$

$$f(t) = \sin 3t \cos(2) + \cos(3t)\sin(2)$$

We know that $\sin(x+y) = \sin x \cos y + \sin y \cos x$

$$\mathcal{L}\{f(t)\} = \cos(2)\,\frac{3}{s^2+9} + \sin(2)\,\frac{s}{s^2+9}$$

ثوابت

Chapter one

Differential Equations

A ***Differential Equation*** is an equation that contains one or more derivatives of a differentiable function. An equation with partial derivatives is called a ***Partial Differential Equation***. While, an equation with ordinary derivatives, that is, derivatives of a function of a single variable, is called an ***Ordinary Differential Equation***.

The ***order*** of a differential equation is the order of the equation's highest order derivative. A differential equation is ***linear*** if it can be put in the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1(x)\frac{dy}{dx}+a_0(x)y=F(x)$$

The ***degree*** of a differential equation is the power (exponent) of the equation's highest order derivative.

Example

First order, first degree, linear $\qquad \frac{dy}{dx}=5y, \qquad 3\frac{dy}{dx}-\sin x=0$

Third order, second degree, nonlinear $\qquad \left(\frac{d^3y}{dx^3}\right)^2+\left(\frac{d^2y}{dx^2}\right)^5-\frac{dy}{dx}=e^x$

Solution of First Order Differential Equations

1) Separable Equations

A first order differential equations is separable if it can be put in the form

$$M(x)dx+N(y)dy=0$$

Steps for Solving a Separable First Order Differential Equation

i. Write the equation in the form $M(x)dx + N(y)dy = 0$.

ii. Integrate M with respect to x and N with respect to y to obtain an equation that relates y and x.

Example

Solve the following differential equations

(a) $\dfrac{dy}{dx} = (1+y^2)e^x$, (b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x+1)}{\sin y + y\cos y}$

Solution

(a) $\dfrac{dy}{dx} = (1+y^2)e^x \quad \Rightarrow \quad e^x dx - \dfrac{1}{1+y^2}dy = 0$

$$\int e^x dx - \int \frac{1}{1+y^2}dy = C \quad \Rightarrow \quad e^x - \tan^{-1} y = C$$

$$\tan^{-1} y = e^x - C \quad \Rightarrow \quad y = \tan(e^x - C)$$

(b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x+1)}{\sin y + y\cos y} \quad \Rightarrow \quad (\sin y + y\cos y)dy = x(2\ln x+1)dx$

$$\int(\sin y + y\cos y)dy - \int x(2\ln x+1)dx = C$$

$$\int \sin(y)dy + \int y\cos(y)dy - 2\int x\ln(x)dx - \int xdx = C$$

$$-\cos(y) + \left[y\sin y - \int \sin(y)dy\right] - 2\left[\ln(x)\times\frac{x^2}{2} - \int \frac{x^2}{2}\times\frac{1}{x}dx\right] - \frac{x^2}{2} = C$$

$$-\cos(y) + y\sin y + \cos y - x^2\ln x + \frac{x^2}{2} - \frac{x^2}{2} = C$$

$$y\sin y - x^2\ln x = C.$$

Notes

1) $f_1(x)g_1(y)dy + f_2(x)g_2(y)dx = 0$ *Separable*

2) $\dfrac{f_1(x)}{g_1(y)}dy + \dfrac{f_2(x)}{g_2(y)}dx = 0$ *Separable*

3) $[f_1(x) \pm g_1(y)]dy + [f_2(x) \pm g_2(y)]dx$ *Not Separable*

Example

$f_1(x) = x, \quad f_2(x) = \sin(x), \quad g_1(y) = y, \quad g_2(y) = \tan(y)$

1) $xydy + \sin(x)\tan(y)dx = 0 \quad \Rightarrow \quad xydy = -\sin(x)\tan(y)dx$

$\dfrac{y}{\tan(y)}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

2) $\dfrac{x}{y}dy + \dfrac{\sin(x)}{\tan(y)}dx = 0 \quad \Rightarrow \quad \dfrac{x}{y}dy = -\dfrac{\sin(x)}{\tan(y)}dx$

$\dfrac{\tan(y)}{y}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

3) $(x + y)dy + (\sin(x) + \tan(y))dx = 0$

$(x + y)dy = -(\sin(x) + \tan(y))dx$ *Not Separable*

Special Type of Separable Equations

If $\frac{dy}{dx} = f(ax+by+c)$; then let $z = ax+by+c$ and the resultant equation may be reduced to a separable equation.

Example

Solve the differential equation $\frac{dy}{dx} = \tan^2(x+y)$

Solution

$$z = x + y \qquad \Rightarrow \qquad \frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \qquad \Rightarrow \qquad \frac{dz}{dx} - 1 = \tan^2(z)$$

$$\frac{dz}{dx} = \tan^2(z) + 1 \qquad \Rightarrow \qquad \frac{dz}{dx} = \sec^2(z)$$

$$\frac{dz}{\sec^2(z)} = dx \qquad \Rightarrow \qquad \cos^2(z)dz = dx$$

$$\int \cos^2(z)dz - \int dx = C \quad \Rightarrow \quad \int \frac{1+\cos(2z)}{2}dz - \int dx = C$$

$$\frac{1}{2}z + \frac{1}{4}\sin(2z) - x = C$$

While $z = x + y$, then the solution is

$$\frac{1}{2}(x+y) + \frac{1}{4}\sin(2(x+y)) - x = C$$

Homogeneous Function

If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then $f(x, y)$ is homogeneous function and n represents the degree of the homogeneous function.

Example

For the function $f(x, y) = x^2 + y^2$ then

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2 \left(x^2 + y^2\right) = \lambda^2 f(x, y)$$

So, the function $f(x, y)$ is homogeneous with degree 2.

Example

For the function $f(x, y) = x + y^2$ then

$$f(\lambda x, \lambda y) = \lambda x + (\lambda y)^2$$
$$= \lambda x + \lambda^2 y^2$$
$$= \lambda\left(x + \lambda y^2\right)$$

So, the function $f(x, y)$ is not homogeneous.

Example

$f(x, y) = x^2 + y^2 + 5$ (Non-homogeneous)

$f(x, y) = x^3 + xy + x$ (Non-homogeneous)

$f(x, y) = \cos(xy)$ (Non-homogeneous)

$f(x, y) = \cos(x^2 \pm y^2)$ (Non-homogeneous)

$f(x, y) = \cos\left(\dfrac{x}{y}\right)$ (Homogeneous)

$f(x, y) = \cos\left(\dfrac{x^2}{y}\right)$ (Non-homogeneous)

Homogeneous Equations

The differential equation $M(x,y)dx + N(x,y)dy$ is homogeneous if M and N are homogeneous functions of the same degree.

Example

1) $(x^2 + y^2)dx + xydy = 0$

This is homogeneous because M and N are both homogeneous with degree 2.

2) $(x^3 + y^3)dx + xydy = 0$

This is not homogeneous because M is homogeneous with degree 3 while N is homogeneous with degree 2.

3) $xdx + (x^2 + y)dy = 0$

This is not homogeneous because N is not homogeneous.

Solution of Homogeneous Equations

A homogeneous first order differential equation can be put in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

This equation can be changed into separable equation with the substitutions

$$v = \frac{y}{x} \qquad \Rightarrow \qquad y = vx \qquad \Rightarrow \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then becomes $$v + x\frac{dv}{dx} = F(v)$$

which can be rearranged algebraically to give

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

with the variables now separated, the equation can now be solved by integrating with respect to x and v. We can then return to x and y by substituting $v = y/x$.

Example

Find the solution of the differential equation

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$$

that satisfies the condition $y(1) = 1$.

Solution

Dividing the numerator and denominator of the right-hand side by x^2 gives

$$\frac{dy}{dx} = -\frac{1+(y/x)^2}{2(y/x)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1+v^2}{2v} = F(v)$$

$$v - F(v) = v + \frac{1+v^2}{2v} = \frac{2v^2+1+v^2}{2v} = \frac{3v^2+1}{2v}$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \quad \Rightarrow \quad \frac{dx}{x} + \frac{2v\,dv}{3v^2+1} = 0$$

The solution of this equation can be written as

$$\int \frac{dx}{x} + \int \frac{2v\,dv}{3v^2+1} = C \quad \Rightarrow \quad \ln x + \frac{1}{3}\ln(1+3v^2) = C$$

$$3\ln x + \ln(1+3v^2) = 3C \quad \Rightarrow \quad \ln x^3 + \ln(1+3v^2) = 3C$$

$$e^{\ln x^3 + \ln(1+3v^2)} = e^{3C} \quad \Rightarrow \quad e^{\ln x^3} \times e^{\ln(1+3v^2)} = e^{3C}$$

$$x^3(1+3v^2) = C' \quad \Rightarrow \quad x^3\left(1 + 3\frac{y^2}{x^2}\right) = C'$$

$$x^3 + 3xy^2 = C'$$

The condition is that when $x=1$ then $y=1$ and the constant C' can be found

$$(1)^3+3(1)(1)^2=C' \qquad \Rightarrow \qquad C'=4$$

The final solution is $x^3+3xy^2=4$.

Reducible to Homogeneous

If the differential equation has the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

Case 1: if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$ then $z=a_1x+b_1y$

Case 2: if $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$ then intersect the two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ to find the intersection point (h,k) and let

$$x=X+h \quad \Rightarrow \quad dx=dX, \text{ and } \quad y=Y+k \quad \Rightarrow \quad dy=dY$$

Example

Solve the differential equation $\frac{dx}{dy}=\frac{4x+6y+5}{3y+2x+4}$

Solution

$$\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}$$

$$a_1=2,\quad a_2=4 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}$$

$$b_1=3,\quad b_2=6 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$$

So $\quad \frac{a_1}{a_2}=\frac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 1***

Let $\quad z=2x+3y \qquad \Rightarrow \qquad \frac{dz}{dx}=2+3\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right) \qquad \Rightarrow \qquad \frac{1}{3}\frac{dz}{dx}-\frac{2}{3}=\frac{z+4}{2z+5} \qquad \Rightarrow \qquad \frac{dz}{dx}=\frac{3z+12}{2z+5}+2$$

$$\frac{dz}{dx}=\frac{3z+12+4z+10}{2z+5} \qquad \Rightarrow \qquad \frac{dz}{dx}=\frac{7z+22}{2z+5} \qquad \Rightarrow \qquad \frac{2z+5}{7z+22}dz=dx$$

$$\int\frac{2z+5}{7z+22}dz-\int dx=C \qquad \Rightarrow \qquad \int\left(\frac{2}{7}-\frac{9}{7}\times\frac{1}{7z+22}\right)dz-\int dx=C$$

$$\int\frac{2}{7}dz-\int\frac{9}{7\times 7}\times\frac{7}{7z+22}dz-\int dx=C$$

$$\frac{2}{7}z-\frac{9}{49}\times\ln(7z+22)-x=C$$

$$\frac{2}{7}(2x+3y)-\frac{9}{49}\times\ln\big(7(2x+3y)+22\big)-x=C$$

Example

Solve the differential equation $(2x+y-3)dy=(x+2y-3)dx$

Solution

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

$$a_1=1,\quad a_2=2 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{1}{2}$$

$$b_1=2,\quad b_2=1 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=2$$

So $\quad \dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 2***

$$\begin{array}{rrrl} x & +2y & -3 & =0 \quad \ldots\ldots \quad (1) \\ 2x & +y & -3 & =0 \quad \ldots\ldots \quad (2) \\ \hline +2x & +4y & -6 & =0 \\ \mp 2x & \mp y & \pm 3 & =0 \\ \hline & 3y & -3 & =0 \end{array}$$

$$\Rightarrow \quad y=1$$

Substituting into (2), we get

$$2x+1-3=0 \quad \Rightarrow \quad x=1$$

The intersection point $(h,k)=(1,1)$.

Let $\quad x=X+1 \quad \Rightarrow \quad dx=dX$

$$y=Y+1 \quad \Rightarrow \quad dy=dY$$

$$\frac{dY}{dX}=\frac{(X+1)+2(Y+1)-3}{2(X+1)+(Y+1)-3}$$

$$=\frac{X+1+2Y+2-3}{2X+2+Y+1-3} \qquad \Rightarrow \qquad \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

$$\frac{dY}{dX}=\frac{1+2\frac{Y}{X}}{2+\frac{Y}{X}}$$

Let $\quad v=\frac{Y}{X} \quad \Rightarrow \quad \frac{dY}{dX}=\frac{1+2v}{2+v}=F(v)$

$$\frac{dX}{X}+\frac{dv}{v-F(v)}=0$$

$$v-F(v)=v-\frac{1+2v}{2+v}$$

$$=\frac{2v+v^2-1-2v}{2+v}=\frac{v^2-1}{2+v}$$

$$\frac{dX}{X}+\frac{2+v}{v^2-1}dv=0 \quad \Rightarrow \quad \ln X+\int\frac{2+v}{v^2-1}dv=C$$

$$\frac{2+v}{v^2-1}=\frac{A}{v+1}+\frac{B}{v-1}=\frac{A(v-1)+B(v+1)}{v^2-1}$$

$$2+v=A(v-1)+B(v+1)$$

$$A=\frac{-1}{2} \qquad B=\frac{3}{2}$$

$$\ln X+\int\frac{-1/2}{v+1}dv+\int\frac{3/2}{v-1}dv=C$$

$$\ln X-\frac{1}{2}\ln(v+1)+\frac{3}{2}\ln(v-1)=C$$

$$\ln X-\frac{1}{2}\ln(\frac{Y}{X}+1)+\frac{3}{2}\ln(\frac{Y}{X}-1)=C$$

$$\ln(x-1)-\frac{1}{2}\ln(\frac{(y-1)}{(x-1)}+1)+\frac{3}{2}\ln(\frac{(y-1)}{(x-1)}-1)=C$$

2) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y=\frac{1}{\rho(x)}\int\rho(x)Q(x)dx$$

where $$\rho(x)=e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.
ii. Find an anti-derivative of $P(x)$.
iii. Find the integrating factor $\rho(x)=e^{\int P(x)dx}$.
iv. Find y using the following equation

$$y=\frac{1}{\rho(x)}\int\rho(x)Q(x)dx$$

Example

Solve the equation $x\frac{dy}{dx}-3y=x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions*** P ***and*** Q. To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx}-\frac{3}{x}y=x \qquad \Rightarrow \qquad P(x)=-\frac{3}{x}, \qquad Q(x)=x.$$

Chapter one

Homogenaus 2$^{\underline{nd}}$ order D.E.

Homogenous 2nd order D.E. with Constant Coeff.

Form

$$a \cdot \frac{d^2y}{dx^2} + b \cdot \frac{dy}{dx} + c \cdot y = \text{Zero}$$

OR

$$a \cdot y'' + b \cdot y' + c \cdot y = \text{Zero}$$

where

a, b & c → are constants

Consider the Solution is

$$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 \cdot e^{mx}$$

Sub

$$\boxed{a \cdot m^2 + bm + C = 0}$$

C/C

Characteristic Equation

To solve the c/c :

$$a.m^2 + b.m + c = 0$$

$$(m - \_\_).(m + \_\_) = 0$$

OR Formula

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a.c}}{2a}$$

OR Using Calc. Mode $\boxed{5} \to \boxed{3}$

$\therefore$ $m_1 = \_\_$ & $m_2 = \_\_$

There are 3 cases of sol.

Case I $\quad b^2 - 4a \cdot c > 0$

$\therefore$ We have two different real roots

$m_1 \quad m_2$

$$\therefore y = c_1 \cdot e^{m_1 \cdot x} + c_2 \cdot e^{m_2 \cdot x}$$

Ex: if $m = 2$ & $m = 3$

$$\boxed{\therefore y = c_1 \cdot e^{2x} + c_2 \cdot e^{3x}}$$

This is the Sol. of D.E.

Case II $\quad b^2 - 4a \cdot c < 0$

$\therefore$ We have two imaginary roots

$$m_{1,2} = \alpha \pm i \cdot \beta$$

$$\therefore y = e^{\alpha \cdot x} \left(C_1 \cdot \cos \beta x + C_2 \cdot \sin \beta x \right)$$

Ex: if $m_{1,2} = \underset{\alpha}{2} \pm i \, 3 \rightarrow \beta$

$$\boxed{\therefore y = e^{2x} \cdot \left(C_1 \cdot \cos(3x) + C_2 \cdot \sin(3x) \right)}$$

Case III $\quad b^2 - 4a.c = 0$

$\therefore$ We have two equal real roots

$m_1 = m_2 = m$

$$\therefore \quad y = (C_1 . x + C_2 .) . e^{mx}$$

Ex: if $m = 3$ & $m = 3$

$$\therefore \quad \boxed{y = (C_1 . x + C_2) . e^{3x}}$$

Ex: ① Solve the following D.E.

① $y'' + 5 \cdot y' + 6 \cdot y = 0$

Sol.

Assume

$y = e^{m \cdot x} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 + 5 \cdot m + 6) \cdot \underbrace{e^{mx}}_{\neq 0} = 0$

C/C

$m^2 + 5 \cdot m + 6 = 0$

$(m+2)(m+3) = 0$

$\boxed{m = -2}$ & $\boxed{m = -3}$ Case I

$\therefore \; y = C_1 \cdot e^{-2x} + C_2 \cdot e^{-3x}$

② $y'' + 4 \cdot y' + 13 \cdot y = 0$

Sol.

Assume

$y = e^{mx} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 + 4 \cdot m + 13) \cdot \underbrace{\cancel{e^{mx}}}_{\neq 0} = 0$

c/c

$m^2 + 4m + 13 = 0$

$\therefore m_{1,2} = \dfrac{-4 \pm \sqrt{16 - 4 * 1 * 13}}{2 * 1} = \dfrac{-4 \pm \sqrt{-36}}{2}$

$\therefore m_{1,2} = \dfrac{-4 \pm i \cdot 6}{2} = \underbrace{-2}_{\alpha} \pm i \underbrace{3}_{\beta}$

$\therefore y = e^{-2x} (C_1 \cdot \cos 3x + C_2 \cdot \sin 3x)$

③ $y'' - 4\cdot y' + 4\cdot y = 0$

Sol

Assume

$y = e^{m\cdot x} \longrightarrow y' = m\cdot e^{mx} \longrightarrow y'' = m^2\cdot e^{mx}$

Sub

$(m^2 - 4m + 4)\cdot \cancel{e^{mx}} = 0$

C.E

$m^2 - 4\cdot m + 4 = 0$

$(m - 2)(m - 2) = 0$ Case III

$\boxed{m = 2}$ & $\boxed{m = 2}$

$\therefore \quad y = (c_1\cdot x + c_2)\cdot e^{2x}$

⑤ $y'' - 3 \cdot y' = 0$

Soln

Assume

$y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

sub

$(m^2 - 3m) \cdot \cancel{e^{mx}} = 0$

c/c

$m^2 - 3m = 0$

$m \cdot (m - 3) = 0$

$\boxed{m = 0}$ & $\boxed{m = 3}$

Case I

$\therefore y = c_1 \cdot \cancel{e^{0x}}_{1} + c_2 e^{3x}$

$\therefore y = c_1 + c_2 \cdot e^{3x}$

(6) $y'' + 16 \cdot y = 0$

Sol

Assume

$y = e^{mx} \longrightarrow y' = m \cdot e^{mx} \longrightarrow y'' = m^2 \cdot e^{mx}$

Sub

$(m^2 + 16) \cdot \cancel{e^{mx}} = 0$

c/c $\quad m^2 + 16 = 0$

$m^2 = -16$

$m = \sqrt{-16} = \pm i.4 = \overset{\alpha}{0} \pm i \overset{\beta}{4}$

$\therefore y = \cancel{e^{0x}} \cdot [c_1 . \cos 4x + c_2 . \sin 4x]$

$\therefore y = c_1 . \cos 4x + c_2 . \sin 4x$

⑧ $y''' - 4\cdot y'' + 3\cdot y' = 0$

Soln

Assume

$y = e^{mx} \to y' = m\cdot e^{mx} \to y'' = m^2\cdot e^{mx} \to y''' = m^3\cdot e^{mx}$

sub)

$(m^3 - 4\cdot m^2 + 3m)\cdot \cancel{e^{mx}} = 0$

C/C $\quad m^3 - 4\cdot m^2 + 3\cdot m = 0$

$m\cdot(m^2 - 4\cdot m + 3) = 0$ Case I

$m\cdot(m-1)(m-3) = 0$

$\therefore \boxed{m=0}, \boxed{m=1}, \boxed{m=3}$

$\therefore y = c_1\cdot \cancel{e^{0x}}_{1} + c_2\cdot e^{1\cdot x} + c_3\cdot e^{3\cdot x}$

(10) $y''' - 6 \cdot y'' + 12 \cdot y' - 8 \cdot y = 0$

Sol.

Assume

$y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = \checkmark \rightarrow y''' = \checkmark$

Sub

ممكن نستخدم

$(m^3 - 6 \cdot m^2 + 12 \cdot m - 8) \cdot \cancel{e^{mx}} = 0$

Calc.

Mode

$\downarrow$

$\boxed{5}$

$\downarrow$

$\boxed{4}$

C/C $\quad m^3 - 6m^2 + 12 \cdot m - 8 = 0$

$m^3 - 8 - 6m^2 + 12m = 0$

$(m-2)(m^2 + 2m + 4) - 6 \cdot m \cdot (m-2) = 0$

$(m-2) \cdot (m^2 - 4m + 4) = 0$

$(m-2) \cdot (m-2) \cdot (m-2) = 0$

$\boxed{m=2}$, $\boxed{m=2}$, $\boxed{m=2}$

$\therefore y = (c_1 x^2 + c_2 x + c_3) \cdot e^{2x}$

(11) $\frac{d^4y}{dx^4} - 5\cdot\frac{d^2y}{dx^2} + 4\cdot y = 0$

Soln

Assume

$y = e^{mx} \to \grave{y} = m\cdot e^{mx} \to y'' \longrightarrow$

Sub

$(m^4 - 5\cdot m^2 + 4)\cdot \cancel{e^{mx}} = 0$

C/C $m^4 - 5\cdot m^2 + 4 = 0$

$(m^2 - 1)\cdot(m^2 - 4) = 0$

$(m-1)(m+1)\cdot(m-2)\cdot(m+2) = 0$

$\therefore$ $\boxed{m=1}$, $\boxed{m=-1}$, $\boxed{m=2}$, $\boxed{m=-2}$

$\therefore y = c_1\cdot e^{x} + c_2\cdot e^{-x} + c_3\cdot e^{2x} + c_4\cdot e^{-2x}$

Exercise

↳ Solve the following Homog. D.E

1. $y'' + y' - 2y = 0$
2. $y'' - 3y' + 2y = 0$
3. $y'' + y' = 0$
4. $y'' + 4y' + 5y = 0$
5. $y'' + 4y = 0$
6. $y'' - y = 0$
7. $y''' - 3y'' + 3y' - y = 0$
8. $y^{(4)} - 4y'' = 0$

* Application of Laplace Transform:

Solution of Differential Equation Using Laplace Transform

Here, we use the derivative property as follows:

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Ex: Solve the D.E $x''(t) + 16x(t) = f(t)$

$x(0) = 0$, $x'(0) = 1$

where $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}$

Taking L.T for both sides

$$s^2 X(s) - s\,\cancel{x(0)}_{0} - \cancel{x'(0)}_{0} + 16X(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos t - \cos 2t\ u(t-\pi)\}$$

$$\cos 2t = \cos 2[(t-\pi)+\pi] = \cos(2(t-\pi) + 2\pi)$$

$$= \cos 2(t-\pi)$$

$$X(s)\left[s^2+16\right] = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-\pi s}$$

$$X(s) = \frac{s}{(s^2+16)(s^2+4)} - \frac{s}{(s^2+16)(s^2+4)}e^{-\pi s}$$

$$X(s) = \frac{1}{16-4}\left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right) - \frac{1}{12}\left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right)e^{-\pi s}$$

$$x(t) = \frac{1}{12}\left[\cos 2t - \cos 4t\right]$$
$$- \frac{1}{12}\left[\cos 2(t-\pi) - \cos 4(t-\pi)\right]u(t-\pi)$$

#

Ex: Solve the following D.E

$$\frac{dy}{dt} - 3y = e^{2t} \quad y(0) = 1$$

Soln $\mathcal{L}\{y'(t)\} = SY(S) - y(0)$

$\mathcal{L}\{y'(t)\} = SY(S) - 1$

$$SY(S) - 1 - 3Y(S) = \frac{1}{S-2}$$

$$Y(S)[S-3] = \frac{1}{S-2} + 1$$

$$Y(S) = \frac{1}{(S-2)(S-3)} + \frac{1}{(S-3)}$$

To get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(S)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-3)} + \frac{1}{s-3} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{-3+2}\left(\frac{1}{s-2} - \frac{1}{s-3} \right) + \frac{1}{s-3} \right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s-3} - \frac{1}{s-2} + \frac{1}{s-3} \right\}$$

$$y(t) = 2e^{3t} - e^{2t} \rightsquigarrow \text{Solution}$$

Ex:-

Solve the following D.E

$$y''(t) + 2y'(t) + y(t) = 3t\,e^{-t}, \quad y(0) = 4, \quad y'(0) = 2.$$

<u>Sol</u> $\mathcal{L}\{y'(t)\} = SY(S) - y(0) = SY(S) - 4$

$$\mathcal{L}\{y''(t)\} = S^2Y(S) - Sy(0) - y'(0)$$

$$= S^2Y(S) - 4S - 2$$

$$\mathcal{L}\{t\,e^{-t}\} = \frac{1}{(S+1)^2}$$

$$\therefore S^2Y(S) - 4S - 2 + 2SY(S) - 8 + Y(S) = \frac{3}{(S+1)^2}$$

$$Y(S)\left[S^2 + 2S + 1\right] - 4S - 10 = \frac{3}{(S+1)^2}$$

$$Y(S)\left[(S+1)^2\right] = \frac{3}{(S+1)^2} + 10 + 4S$$

$$Y(S) = \frac{3}{(S+1)^4} + \frac{10}{(S+1)^2} + \frac{4S}{(S+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\{ Y(s) \}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{3}{(s+1)^4} + \frac{10}{(s+1)^2} + 4 \frac{s}{(s+1)^2} \right\}$$

$$y(t) = 3e^{-t}\frac{t^3}{3!} + 10e^{-t}t + 4\mathcal{L}^{-1}\left\{ \frac{(s+1)-1}{(s+1)^2} \right\}$$

$$= 3e^{-t}\frac{t^3}{3!} + 10e^{-t}t + 4\mathcal{L}^{-1}\left\{ \frac{1}{(s+1)} - \frac{1}{(s+1)^2} \right\}$$

$$y(t) = \frac{3}{3!}e^{-t}t^3 + 10e^{-t}t + 4e^{-t} - 4e^{-t}t$$

$$y(t) = \frac{3}{3!}t^3e^{-t} + 6te^{-t} + 4e^{-t}$$

Ex

→ Compute $y(\frac{\pi}{2})$, $y(3+\frac{\pi}{2})$ for the f_n $y(t)$ which satisfies the boundary value Problem $y''(t)+y(t)=(t-3)\,u(t-3)$

$$y(0)=y'(0)=0$$

Soln $s^2Y(s)-s\cancel{y(0)}-\cancel{y'(0)}+Y(s)$

$$=\frac{1}{s^2}e^{-3s}$$

$$Y(s)\left[s^2+1\right]=\frac{1}{s^2}e^{-3s}$$

$$Y(s)=\frac{1}{s^2(s^2+1)}e^{-3s}$$

$$Y(s)=\frac{1}{1-0}\left[\frac{1}{s^2}-\frac{1}{s^2+1}\right]e^{-3s}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = (t-3)u(t-3) - \sin(t-3)u(t-3)$$

$$y(t) = \left[(t-3) - \sin(t-3)\right]u(t-3)$$

$$y(t) = \begin{cases} 0 & 0 < t < 3 \\ (t-3) - \sin(t-3) & t > 3 \end{cases}$$

$$y\left(\frac{\pi}{2}\right) \rightsquigarrow \frac{\pi}{2} < 3 \quad \therefore y\left(\frac{\pi}{2}\right) = 0$$

$$y\left(\frac{\pi}{2}+3\right) \rightsquigarrow \frac{\pi}{2}+3 > 3$$

$$\therefore y\left(\frac{\pi}{2}+3\right) = \left(\frac{\pi}{2} \cancel{+3} \cancel{-3}\right) - \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}$$

$$= \frac{\pi}{2} - 1$$

$\boxed{2}$ Solution of Integro - D.E

Ex Solve

$$f'(t) + 2f(t) + \int_0^t f(x)\,dx = 0 \quad , \; f(0)=2$$

Sol: Taking Laplace for both sides.

$$sF(s) - \underbrace{\cancel{f(0)}}_{2} + 2F(s) + \frac{F(s)}{s} = 0$$

$$F(s)\left[s + 2 + \frac{1}{s}\right] = +2$$

$$F(s)\left[\frac{s^2+2s+1}{s}\right] = 2 \Rightarrow F(s) = \frac{2s}{s^2+2s+1}$$

$$F(s) = 2\,\frac{s}{(s+1)^2}$$

$$F(s) = 2\,\frac{(s+1)-1}{(s+1)^2} = 2\left[\frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$f(t) = 2\left(e^{-t} - te^{-t}\right)$$

Ex $f'(t) = t + \int_0^t f(t-\lambda)\cos\lambda \, d\lambda$

$f(0) = 1$

Soln

$$sF(s) - \cancel{f(0)}^{\,1} = \frac{1}{s^2} + F(s)\frac{s}{s^2+1}$$

$$F(s)\left[s - \frac{s}{s^2+1}\right] = \frac{1}{s^2} + 1$$

$$F(s)\left[\frac{s^3 + \cancel{s} - \cancel{s}}{(s^2+1)}\right] = \frac{1}{s^2} + 1$$

$$F(s) = \frac{s^2+1}{s^5} + \frac{s^2+1}{s^3} = \frac{1}{s^3} + \frac{1}{s^5} + \frac{1}{s} + \frac{1}{s^3}$$

$$F(s) = \frac{2}{s^3} + \frac{1}{s^5} + \frac{1}{s}$$

$$f(t) = 2\frac{t^2}{2!} + \frac{t^4}{4!} + 1$$

Chapter Three

Gamma & Beta Functions

I. Gamma Function

Definition

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$; $n > 0$

& $\Gamma(n) = \Gamma(n+1) / n$; $n \in R - Z^{\leq 0}$

Results:

(1) $\Gamma(n+1) = n\,\Gamma(n)$; $n > 0$, where $\Gamma(1) = 1$

(2) $\Gamma(n+1) = n!$; $n \in \mathbf{N}$ (convention: $0! = 1$)

(3) $\Gamma(n)\,\Gamma(1-n) = \pi/\sin(n\pi)$; $0 < n < 1$

In Particular;

$\Gamma(1/2) = \sqrt{\pi}$

Examples:

Example(1)

Evaluate $\int_0^\infty x^4 e^{-x} dx$

Solution

$\int_0^\infty x^4 e^{-x} dx = \int_0^\infty x^{5-1} e^{-x} dx = \Gamma(5)$

$\Gamma(5) = \Gamma(4+1) = 4! = 4(3)(2)(1) = 24$

Example(2)

Evaluate $_0\int^{\infty} x^{1/2} e^{-x} dx$

$_0\int^{\infty} x^{1/2} e^{-x} dx = {_0\int^{\infty}} x^{3/2-1} e^{-x} dx = \Gamma(3/2)$

$3/2 = ½ + 1$

$\Gamma(3/2) = \Gamma(½+1) = ½ \; \Gamma(½) = ½ \sqrt{\pi}$

Example(3)

Evaluate $_0\int^{\infty} x^{3/2} e^{-x} dx$

$_0\int^{\infty} x^{3/2} e^{-x} dx = {_0\int^{\infty}} x^{5/2-1} e^{-x} dx = \Gamma(5/2)$

$5/2 = 3/2 + 1$

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2 \; \Gamma(3/2) = 3/2 \,.\, ½ \; \Gamma(½) = 3/2 \,.\, ½ \,.\, \sqrt{\pi} = ¾ \sqrt{\pi}$

Example(4)

Find $\Gamma(-½)$

$(-½) + 1 = ½$

$\Gamma(-1/2) = \Gamma(-½ + 1) \; / \; (-½) = -2 \;\; \Gamma(1/2) = -2 \sqrt{\pi}$

Example(5)

Find $\Gamma(-3/2)$

$(-3/2) + 1 = -½$

$\Gamma(-3/2) = \Gamma(-3/2 + 1) \; / \; (-3/2) = \Gamma(-1/2) \; / \; (-2/3) = (-2\sqrt{\pi}) \; / \; (-2/3) = 4\sqrt{\pi} \; /3$

II. Beta Function

Definition

$B(m,n) = {}_0\int^1 x^{m-1} (1-x)^{n-1} dx$; $m > 0$ & $n > 0$

Results:

(1) $B(m,n) = \Gamma(m)\ \Gamma(n) / \Gamma(m+n)$

(2) $B(m,n) = B(n,m)$

(3) ${}_0\int^{\pi/2} \sin^{2m-1}x \, . \, \cos^{2n-1}x \; dx = \Gamma(m)\ \Gamma(n) / 2\,\Gamma(m+n)$; $m>0$ & $n>0$

(4) ${}_0\int^{\infty} x^{q-1} / (1+x) \, . dx = \Gamma q)\ \Gamma(1-q) = \Pi / \sin(q\pi)$; $0<q<1$

Examples:

Example(1)

Evaluate ${}_0\int^1 x^4 (1-x)^3 dx$

Solution

${}_0\int^1 x^4 (1-x)^3 dx = x^{5-1} (1-x)^{4-1} dx$

$= B(5,4) = \Gamma(5)\ \Gamma(4) / \Gamma(9) = 4!\ .\ 3! / 8! = 3!/(8.7.6.5) = 1/(8.7.5) = 1/280$

Example(2)

Evaluate $I = {}_0\int^1 [\ 1 / \sqrt[3]{[x^2 (1-x)]}\ ]\, dx$

Solution

$I = {}_0\int^1 x^{-2/3} (1-x)^{-1/3} dx = {}_0\int^1 x^{1/3-1} (1-x)^{2/3-1} dx$

$= B(1/3, 2/3) = \Gamma(1/3)\ \Gamma(2/3) / \Gamma(1)$

$\Gamma(1/3)\ \Gamma(2/3) = \Gamma(1/3)\ \Gamma(1-1/3) = \pi / \sin(\pi/3) = \pi / (\sqrt{3}/2) = 2\pi / \sqrt{3}$

Example(3)

Evaluate $I = {}_0\int^1 \sqrt{x} \,.\, (1 - x)\, dx$

Solution

$I = {}_0\int^1 x^{1/2} (1 - x)\, dx = {}_0\int^1 x^{3/2 - 1} (1 - x)^{2-1} dx$

$= B(3/2\,,\, 2) = \Gamma(3/2)\, \Gamma(2) \,/\, \Gamma(7/2)$

$\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\Gamma(5/2) = \Gamma(3/2 + 1) = (3/2)\, \Gamma(3/2) = (3/2) \,.\, \frac{1}{2}\sqrt{\pi} = 3\sqrt{\pi} \,/\, 4$

$\Gamma(7/2) = \Gamma(5/2 + 1) = (5/2)\, \Gamma(5/2) = (5/2) \,.\, (3\sqrt{\pi} \,/\, 4) = 15\sqrt{\pi} \,/\, 8$

Thus,

$I = (\frac{1}{2}\sqrt{\pi}) \,.\, 1! \,/\, (15\sqrt{\pi} \,/\, 8) = 4/15$

II. Using Gamma Function to Evaluate Integrals

Example(1)

Evaluate: $I = {}_0\int^\infty x^6\, e^{-2x}\, dx$

Solution:

Letting $y = 2x$, we get

$I = (1/128)\, {}_0\int^\infty y^6\, e^{-y}\, dy = (1/128)\, \Gamma(7) = (1/128)\, 6! = 45/8$

Example(2)

Evaluate: $I = {}_0\int^\infty \sqrt{x}\, e^{-x^3}\, dx$

Solution:

Letting $y = x^3$, we get

$I = (1/3)\, {}_0\int^\infty y^{-1/2}\, e^{-y}\, dy = (1/3)\, \Gamma(1/2) = \sqrt{\pi} \,/\, 3$

Example(3)

Evaluate: $I = {}_0\int^{\infty} x^m \ e^{-k x^n} dx$

Solution:

Letting $y = k x^n$, we get

$I = [1 / (n . k^{(m+1)/n})] {}_0\int^{\infty} y^{[(m+1)/n - 1]} \ e^{-y} dy = [1 / (n . k^{(m+1)/n})] \ \Gamma[(m+1)/n]$

II. Using Beta Function to Evaluate Integrals

Formulas

(1) ${}_0\int^1 x^{m-1} \ (1 - x)^{n-1} dx = B(m,n) = \Gamma(m) \ \Gamma(n) \ / \ 2 \, \Gamma(m+ n)$; $m > 0$ & $n > 0$

(3) ${}_0\int^{\pi/2} \sin^{2m-1}x \ . \ \cos^{2n-1}x \ dx = (1/2) B(m,n)$; $m>0$ & $n>0$

(4) ${}_0\int^{\infty} x^{q-1} \ / \ (1+x) \ . dx = \Gamma(q) \ \Gamma(1-q) = \Pi / \sin(q\pi)$; $0 < q < 1$

Using Formula (1)

Example(1)

Evaluate: $I = {}_0\int^2 x^2 \ / \sqrt{(2 - x)} \ . dx$

Solution:

Letting $x = 2y$, we get

$I = (8/\sqrt{2}) {}_0\int^1 y^2 \ (1 - y)^{-1/2} \ dy = (8/\sqrt{2}) \ . \ B(3 \ , \ 1/2) = 64\sqrt{2} /15$

Example(2)

Evaluate: $I = {}_0\int^a x^4 \sqrt{(a^2 - x^2)} \ . dx$

Solution:

Letting $x^2 = a^2 y$, we get

$$I = (a^6 / 2)\ {}_0\int^1 y^{3/2} (1 - y)^{1/2} dy = (a^6 / 2) . B(5/2 , 3/2) = a^6 / 32$$

Example(3)

Evaluate: $I = {}_0\int^2 x \sqrt{(8 - x^3)} \ . dx$

Hint

Lett $x^3 = 8y$

Answer

$$I = (8/3)\ {}_0\int^1 y^{-1/3} (1 - y)^{1/3} . dy = (8/3) B(2/3 , 4/3) = 16\pi / (9\sqrt{3})$$

Using Formula (3)

Example(3)

Evaluate: $I = {}_0\int^\infty dx / (1+x^4)$

Solution:

Letting $x^4 = y$, we get

$$I = (1/4)\ {}_0\int^\infty y^{-3/4} dy / (1 + y) = (1/4) . \Gamma(1/4) . \Gamma(1 - 1/4)$$

$$= (1/4) . [\pi / \sin(¼ . \pi)] = \pi\sqrt{2} / 4$$

Using Formula (2)

Example(4)

a. Evaluate: $I = {}_0\int^{\pi/2} \sin^3 \ . \ \cos^2 x \ dx$

b. Evaluate: $I = {}_0\int^{\pi/2} \sin^4 \ . \ \cos^3 x \ dx$

Solution:

a. Notice that: $2m - 1 = 3 \rightarrow m = 2$ & $2n - 1 = 2 \rightarrow m = 3/2$

$I = (1/2)\ B(2, 3/2) = 8/15$

b. $I = (1/2)\ B(5/2, 3) = 8/315$

Example(5)

a. Evaluate: $I = {}_0\int^{\pi/2} \sin^6 \ dx$

b. Evaluate: $I = {}_0\int^{\pi/2} \cos^6 x \ dx$

Solution:

a. Notice that: $2m - 1 = 6 \rightarrow m = 7/2$ & $2n - 1 = 0 \rightarrow m = 1/2$

$I = (1/2)\ B(7/2, 1/2) = 5\pi/32$

b. $I = (1/2)\ B(1/2, 7/2) = 5\pi/32$

Example(6)

a. Evaluate: $I = {}_0\int^{\pi} \cos^4x \; dx$

b. Evaluate: $I = {}_0\int^{2\pi} \sin^8 \; dx$

Solution:

a. $I = {}_0\int^{\pi} \cos^4x = 2 \; {}_0\int^{\pi/2} \cos^4x = 2\,(1/2)\,B\,(1/2\;,5/2\;) = 3\pi \,/\, 8$

b. $I = I = {}_0\int^{\pi} \sin^8x = 4 \; {}_0\int^{\pi/2} \sin^8x = 4\,(1/2)\,B\,(9/2\;,\,1/2\;) = 35\pi \,/\, 64$

Details

I.

Example(1)

Evaluate: $I = {}_0\int^{\infty} x^6 \; e^{-2x}\, dx$

$x = y/2$

$x^6 = y^6/64$

$dx = (1/2)dy$

$x^6 \; e^{-2x}\, dx = y^6/64 \;\; e^{-y} \;.\,(1/2)dy$

Example(2)

$I = {}_0\int^{\infty} \sqrt{x} \;\; e^{-x^3}\, dx \;\; x = y^{1/3}$

$\sqrt{x} = y^{1/6}$

$dx = (1/3)y^{-2/3}\, dy$

$\sqrt{x} \;\; e^{-x^3}\, dx = y^{1/6}\, e^{-y} \,.\, (1/3)y^{-2/3}\, dy$

Example(3)

Evaluate: $I = {}_0\int^{\infty} x^m \;\; e^{-k\,x^n}\, dx$

$y = k\,x^n$

$x = y^{1/n} \,/\, k^{1/n}$

$x^m = y^{m/n} \,/\, k^{m/n}$

$dx = (1/n)\, y^{(1/n-1)} \,/\, k^{1/n} \;\; dy$

$x^m \;\; e^{-k\,x^n}\, dx = (\, y^{m/n} \,/\, k^{m/n} \;)\,.\,e^{-y}\,.\,(1/n)\, y^{(1/n-1)} \,/\, k^{1/n} \;\; dy$

$m/n + 1/n - 1 = (m+1)/n - 1$

$-m/n - 1/n = -\,(m+1)/n$

$I = [\,1 \,/\, (\,n\,.\,k^{(m+1)/n})\,] \; {}_0\int^{\infty} y^{[(m+1)/n\,-\,1]} \;\; e^{-y}\, dy$

II.
Examples

Example(1)

$I = {}_0\int^2 x^2 / \sqrt{(2-x)}\ .\ dx$

$x = 2y$

$dx=2dy$

$x^2 = 4y^2$

$\sqrt{(2-x)} = \sqrt{(2-2y)} = \sqrt{2}\ \sqrt{(1-y)}$

$x^2 / \sqrt{(2-x)}\ .\ dx = 4y^2 / \sqrt{2}\ \sqrt{(1-y)}\ 2dy$

y=0 when x=0

y=1 when x=2

Example(2)

Evaluate: $I = {}_0\int^a x^4 \sqrt{(a^2-x^2)}\ .\ dx$

$x^2 = a^2 y$, we get

$x^4 = a^4 y^2$

$x= a\, y^{1/2}$

$dx= (1/2)a\, y^{-1/2}\, dy$

$\sqrt{(a^2-x^2)} = \sqrt{(a^2-a^2 y)} = a\,(1-y)^{1/2}$

$x^4 \sqrt{(a^2-x^2)}\ .\ dx = a^4 y^2\ a\,(1-y)^{1/2}\ (1/2)a\, y^{-1/2}\, dy$

y=0 when x=0

y=1 when x=a

Example(3)

$I = {}_0\int^\infty dx / (1+x^4)$

$x^4 = y$

$x=y^{1/4}$

$dy= (1/4)\, y^{-3/4}\, dy$

$dx / (1+x^4) = (1/4)\, y^{-3/4}\, dy / (1+y)$

Proofs of formulas (2) & (3)

Formula (2)

We have,

$$B(m,n) = \int_0^1 x^{m-1} \ (1-x)^{n-1} \, dx$$

Let $x = \sin^2 y$
Then $dx = 2 \sin y \ \cos^{-} y \ dy$

&

$$x^{m-1} \ (1-x)^{n-1} \, dx \ = (\sin^2 y)^{m-1} \ (\cos^2 y)^{n-1} \ (dy \ / \ 2 \sin x \ \ cox \ )$$

$$= 2 \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy$$

When x=0 , we have $y = 0$
When x=1, we hae $y = \pi/2$

Thus,

$$I = 2 \int_0^{\pi/2} \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy$$

$$I = \int_0^{\pi/2} \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy \ = \ B(m,n) \ / \ 2$$

Formula (3)

We have,

$I = {}_0\int^{\infty} x^{q-1} / (1+x)\ dx$

Let
$y = x / (1+x)$

Hence, $x = y / 1-y$

, $1 + x = 1 + (y / 1-y) = 1/(1-y)$

& $dx = - [(1- y) - y(-1)] / (1-y)^2 \ . dy= 1 / (1-y)^2 \ . dy$

whn $x = 0$, we have $y = 0$
when $x \to \infty$, we have $y = \lim_{x\to\infty} x / (1+x) = 1$

Thus,

$I = {}_0\int^{\infty} [x^{q-1} / (1+x)] dx = {}_0\int^{\infty} [(y / 1-y)^{q-1} / (1/(1-y))] \ . 1 / (1-y)^2 \ . dy$

$= {}_0\int^{1} [y^{q-1} / (1-y)^{-q}]\ dy$

$= B(q , 1-q) = \Gamma(q)\ \Gamma(1-q)$

Proving that $\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(1/2) = {}_0\int^{\infty} x^{1/2-1} e^{-x} dx = {}_0\int^{\infty} x^{-1/2} e^{-x} dx$

Let $y = x^{½}$

$x = y^2$

$dx = 2y\ dy$

$\Gamma(1/2) = {}_0\int^{\infty} y^{-1} e^{-y^2} 2y\ dy$

$= 2\ {}_0\int^{\infty} e^{-y^2}\ dy$

$=2 (\sqrt{\pi} / 2) = \sqrt{\pi}$

Inverse of Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$, Then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{K}{s}$	K
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$ n: positive & integer
$\frac{1}{s \pm a}$	$e^{\mp at}$
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{s^2}$	t
$F(s \mp a)$	$e^{\pm at} f(t)$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$

$\mathcal{L}^{-1}\{\ \}$	$f(t)$
$\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\}$	$-t\,f(t)$
$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$	$\int_0^t f(t)\,dt$
$\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-as}\right\}$	$u(t-a)$
$\mathcal{L}^{-1}\left\{F(s)e^{-as}\right\}$	$f(t-a)\,u(t-a)$

Ex:① $\mathcal{L}^{-1}\left\{\frac{2s-18}{s^2+9}\right\}$

Soln $\mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{18}{s^2+9}\right\}$

$= 2\cos 3t - 6\sin 3t$

Ex② $\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$

Soln $\mathcal{L}^{-1}\left\{\frac{[(s+2)-2+1]^2}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{[(s+2)-1]^2}{(s+2)^4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{(s+2)^2-2(s+2)+1}{(s+2)^4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}\right\}$

$e^{-2t}\left[t - \frac{2}{2!}t^2 + \frac{1}{3!}t^3\right]$

Ex⑤ $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$

Soln

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2-1+5}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$=e^{t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=e^{t}\,\frac{1}{2}\sin 2t$$

Note

لازم معامل x^2 واحد

$\boxed{1}x^2+ax+b$

$\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+b$ اكمال المربع

Ex(3) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-2s+5}\right\}$

Soln $\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+4}\right\}$

$=\mathcal{L}^{-1}\left\{\frac{(s-1)+1}{(s-1)^2+4}\right\}$

$=\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$

$=e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$= e^{t}\cos 2t+\frac{1}{2}e^{t}\sin 2t$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^5}\right\}$

Soln $\mathcal{L}^{-1}\left\{\dfrac{(s+1)-1}{(s+1)^5}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^4} - \dfrac{1}{(s+1)^5}\right\}$

$= e^{-t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^4} - \dfrac{1}{s^5}\right\}$

$= e^{-t}\left[\dfrac{t^3}{3!} - \dfrac{t^4}{4!}\right]$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s+18}{s^2+9} + \dfrac{24-30s}{s^4}\right\}$

Soln

$\mathcal{L}^{-1}\left\{\dfrac{5}{s^2} + \dfrac{4}{s^3} - \dfrac{2s}{s^2+9} - \dfrac{18}{s^2+9} + \dfrac{24}{s^4} - \dfrac{30}{s^3}\right\}$

$= 5t + 4\dfrac{t^2}{2} - 2\cos 3t - 6\sin 3t + \dfrac{24}{3!}t^3 - \dfrac{30}{2!}t^2$

Ex: $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{4}{s^2}\right)\right\}$

Soln $f(t) = \mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+4}{s^2}\right)\right\}$

$= \mathcal{L}^{-1}\left\{\ln(s^2+4) - \ln(s^2)\right\}$

$\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\} = -t\,f(t)$

$F(s) = \ln(s^2+4) - \ln s^2$

$\frac{dF(s)}{ds} = \frac{2s}{s^2+4} - \frac{2}{s}$

$\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\} = -t\,f(t)$

$= \left[2\cos 2t - 2\right]$

$\therefore f(t) = -2\frac{(\cos 2t - 1)}{t}$

$f(t) = 2\cdot\frac{(1-\cos 2t)}{t}$

Ex: Find $\mathcal{L}^{-1}\left\{ \cot^{-1}(s+1) \right\}$

Sol $\mathcal{L}^{-1}\left\{ \cot^{-1}(s+1) \right\} = e^{-t} \mathcal{L}^{-1}\left\{ \cot^{-1} s' \right\}$

* for $\mathcal{L}^{-1}\left\{ \cot^{-1} s' \right\}$

$$F(s') = \cot^{-1} s'$$

$$\frac{dF(s')}{ds'} = -\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{ \frac{dF(s)}{ds'} \right\} = \cancel{-}\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} = \cancel{-}t\, f(t)$$

$$= \sin t = t\, f(t)$$

$$\therefore f(t) = \frac{\sin t}{t}$$

$$\therefore \mathcal{L}^{-1}\left\{ \cot^{-1}(s+1) \right\} = e^{-t} \frac{\sin t}{t}$$

Ex: Find $\mathcal{L}^{-1}\left\{ \tan^{-1}\frac{3}{s} \right\}$

Soln $F(s) = \tan^{-1}\frac{3}{s}$

$$\frac{d\,F(s)}{ds} = \frac{1}{\left(\frac{3}{s}\right)^2 + 1}\left(\frac{-3}{s^2}\right)$$

$$\frac{d\,F(s)}{ds} = \frac{-3}{9 + s^2}$$

$$\mathcal{L}^{-1}\left\{ \frac{d}{ds}F(s) \right\} = \mathcal{L}^{-1}\left\{ \frac{-3}{9+s^2} \right\} = -t\,f(t)$$

$$= \cancel{-}\cancel{3}\ \frac{\sin 3t}{\cancel{3}} = \cancel{-}t\ f(t)$$

$$f(t) = \frac{\sin 3t}{t}$$

$$\therefore \mathcal{L}^{-1}\left\{ \tan^{-1}\frac{3}{s} \right\} = \frac{\sin 3t}{t}$$

Ex: If $f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{4e^{-s}}{s^2} + 4\frac{e^{-3s}}{s^2}\right\}$

Find and sketch $f(t)$, evaluate $f(2)$, $f(7)$

Soln $f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4}{s^2}e^{-3s}\right\}$

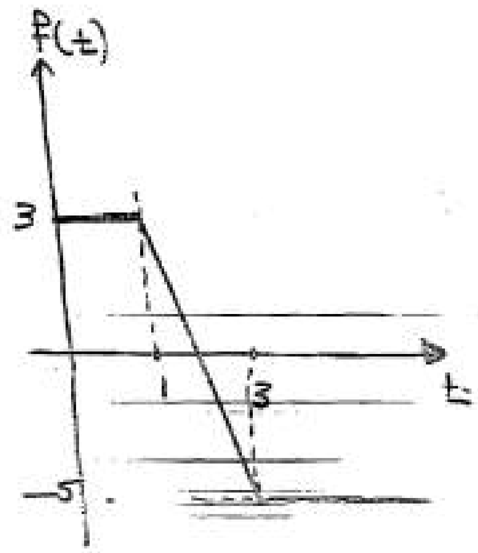
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{4}{s^2}e^{-s} + \frac{4}{s^2}e^{-3s}\right\}$$

$$= 3 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$$

$$f(t) = \begin{cases} 3 & 0 < t < 1 \\ 3 - 4(t-1) & 1 < t < 3 \\ 3 - 4(t-1) + 4(t-3) & t > 3 \end{cases}$$

$$= \begin{cases} 3 & 0 < t < 1 \\ 7 - 4t & 1 < t < 3 \\ -5 & t > 3 \end{cases}$$



$\therefore f(2) = -1$ ⟶ From 2nd Term

$\therefore f(7) = -5$ ⟶ From 3rd Term

$$\therefore f(t) = \begin{cases} \frac{1}{2}t^2 & 0 < t < 2 \\ \frac{1}{2}t^2 - 2(t-2)^2 & 2 < t < 4 \\ \frac{1}{2}t^2 - 2(t-2)^2 + \frac{3}{2}(t-4)^2 & t > 4 \end{cases}$$

$f(1) = \frac{1}{2}(1)^2 = 1/2 \rightsquigarrow$ From 1st Term

$f(3) = \frac{1}{2}(3)^2 - 2(1)^2 = \frac{9}{2} - 2 = 2.5 \rightsquigarrow$ From 2nd Term

$f(5) = \frac{1}{2}(5)^2 - 2(3)^2 + \frac{3}{2}(1)^2 = \checkmark \rightsquigarrow$ From 3rd Term

Ex: $\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^4}\right\}$

Soln $\mathcal{L}^{-1}\left\{\frac{e^{4}\, e^{-3s}}{(s+4)^4}\right\}$ $\longrightarrow u(t-3)$

$$\mathcal{L}^{-1}\left\{\frac{e^4}{(s+4)^4}\right\} = e^4 e^{-4t} \frac{t^3}{3!} = \frac{1}{3!} e^{-4(t-1)}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^4}\right\} = e^{-4(t-1-3)} \frac{1}{3!}(t-3)^3\, u(t-3)$$

$$= e^{-4(t-4)} \frac{1}{3!}(t-3)^3\, u(t-3)$$

Ex: Find $\mathcal{L}^{-1}\left\{ \frac{(s+2)e^{-\pi s}}{s^2+2s+2} \right\}$

Soln $\mathcal{L}^{-1}\left\{ \frac{(s+2)e^{-\pi s}}{s^2+2s+2} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{s+2}{(s+1)^2-1+2} \right\} = \mathcal{L}^{-1}\left\{ \frac{(s+1)+1}{(s+1)^2+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{(s+1)}{(s+1)^2+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+1} \right\}$$

$$= e^{-t}\cos t + e^{-t}\sin t$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{(s+2)e^{-\pi s}}{s^2+2s+2} \right\} = \left[e^{-(t-\pi)}(\cos(t-\pi) + \sin(t-\pi) \right] u(t-\pi)$$