Control Theory I

By

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No	Subject	Week (3 H/W)
1.	**Introduction to control system:** ➢ *Introduction.* ➢ *Open Loop System.* ➢ *Close Loop System.* ➢ *Definitions.* ➢ *The engineering control problem.*	1
2.	**Mathematical Representation of physical systems:** ➢ *Linear system, nonlinear system.* ➢ *Transfer functions, Block diagram.* ➢ *Electrical systems.* ➢ *Mechanical translation system.* ➢ *Mechanical rotational system.* ➢ *Thermal system.* ➢ *Modeling in state space.* ➢ *How to derive transfer function from the state space equations.* ➢ *State space representation of dynamic system.*	3
3.	**Block diagrams Processing:** ➢ *Procedures for drawing a block diagram.* ➢ *Block diagram reduction.* ➢ *Closed loop system subjected to a disturbance.* ➢ *Multivariable Systems, transfer matrices.*	2

4.	**Signal flow graphs:** ➢ *Signal flow graph representation of linear system.* ➢ *Mason's gains formula for signal flow graph.* ➢ *Transfer function process in signal flow graph.*	2
5.	**Transient response analysis:** ➢ *Test signals, impulse response function.* ➢ *First order system, higher order system.* ➢ *Definitions of time constant, damping ratio and natural frequency.* ➢ *Definitions of transient response specifications.* ➢ *Impulse response, dominant poles.*	2
6.	**Steady – state error in unity- feedback control system:** ➢ *Classifications of control systems.* ➢ *static position error coefficients.* ➢ *dynamic error coefficients.*	2
7.	***Routh's Stability Criterion*** ➢ *Introduction.* ➢ *Routh's Criteria Rules.* ➢ *Solved problem for Checking System Stability.*	2
8.	**Root Locus:** ➢ *Root locus plot.* ➢ *general rules for constructing root loci.* ➢ *special cases, conditionally stable system.* ➢ *non-minimum phase systems.*	1

References:

1) *“ Linear Control System Analysis and Design with MATLAB ” by John J. D’Azzo and Constantine H. Houpis , 2003.*

2) *“Classical Control Theory”, By Yousif Al Mashhadany, first Edition, 2014*

3) *“Modern Control Engineering” , By Katsuhiko Ogata, Fifth Edition, 2010*

4) *“Control Systems Engineering” , Sixth Edition, By Norman S. Nise, 2011*

Lecture No. One

Introduction To Control System

This lecture discusses the following topics :

1.1 Introduction.

1.2 Open Loop System.

1.3 Close Loop System.

1.4 Definitions of control system.

1.5 The engineering control problem.

1.6 Solved Examples

1.7 Problems

1.1. Introduction:

The toaster in Fig.1.1 can be set for the desired darkness of the toasted bread. The setting of the ‘‘darkness’’ knob, or timer, represents the input quantity, and the degree of darkness and crispness of the toast produced is the output quantity. If the degree of darkness is not satisfactory, because of the condition of the bread or some similar reason, this condition can in no way automatically alter the length of time that heat is applied. Since the output quantity has no influence on the input quantity, there is no feedback in this system. The heater portion of the toaster represents the dynamic part of the overall system, and the timer unit is the reference selector.

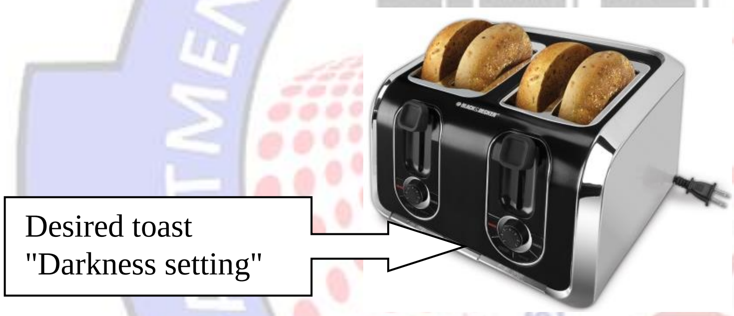
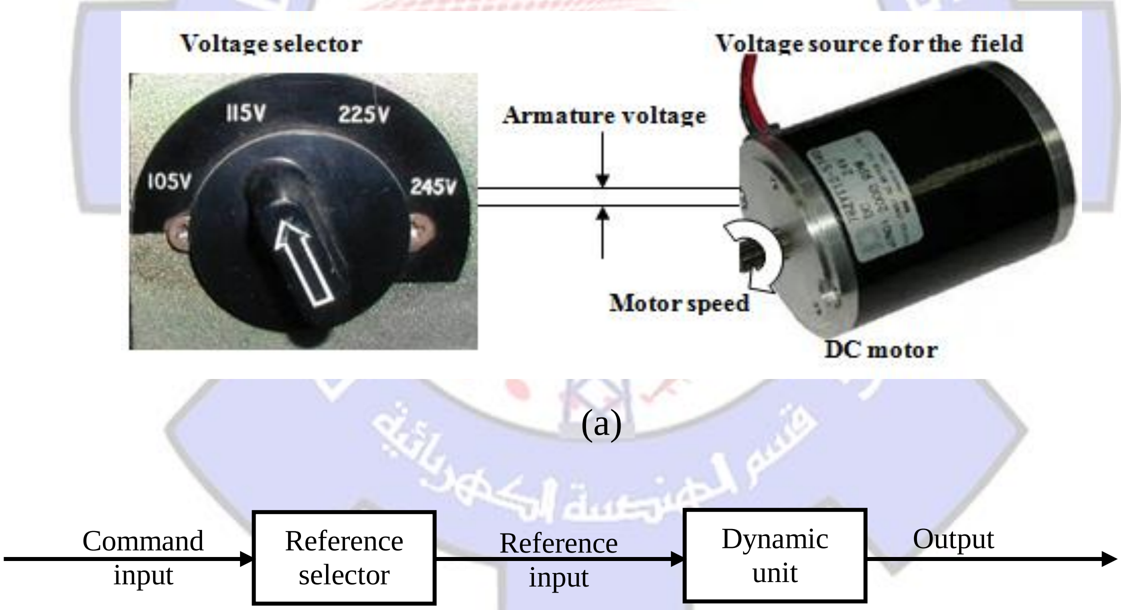


Fig. 1.1 Open-loop control system automatic toaster

The dc shunt motor of Fig. 1.2 is another example. For a given value of field current, a required value of voltage is applied to the armature to produce the desired value of motor speed. In this case the motor is the dynamic part of the system, the applied armature voltage is the input quantity, and the speed of the shaft is the output quantity. A variation of the speed from the desired value, due to a change of mechanical load on the shaft, can in no way cause a change in the value of the applied armature voltage to maintain the desired speed. Therefore, the output quantity has no influence on the input quantity.

1.2. Open Loop System:

Systems in which the output quantity has no effect upon the input quantity are called open-loop control systems. The examples just cited are represented symbolically by a functional block diagram, as shown in Fig. 1.2.C. In this figure, (1) the desired darkness of the toast or the desired speed of the motor is the command input, (2) the selection of the value of time on the toaster timer or the value of voltage applied to the motor armature is represented by the reference-selector block, and (3) the output of this block is identified as the reference input. The reference input is applied to the dynamic unit that performs the desired control function, and the output of this block is the desired output.



(b)

Fig. 1.2. Open-loop control system (a) electric motor; (b) functional block diagram.

A person could be assigned the task of sensing the actual value of the output and comparing it with the command input. If the output does not have the desired value, the person can alter the reference-selector position to achieve this value. Introducing the person provides a means through which the output is feedback and is compared with the input. Any necessary change is then made in order to cause the output to equal the desired value.

1.3. Close Loop System:

The feedback action therefore controls the input to the dynamic unit. Systems in which the output has a direct effect upon the input quantity are called closed loop control systems. To improve the performance of the closed-loop system so that the output quantity is as close as possible to the desired quantity, the person can be replaced by a mechanical, electrical, or other form of a comparison unit. The functional block diagram of a single-input single-output (SISO) closed-loop control system is illustrated in Fig. 1.3. Comparison between the reference input and the feedback signals results in an actuating signal that is the difference between these two quantities. The actuating signal acts to maintain the output at the desired value. This system is called a closed-loop control system..

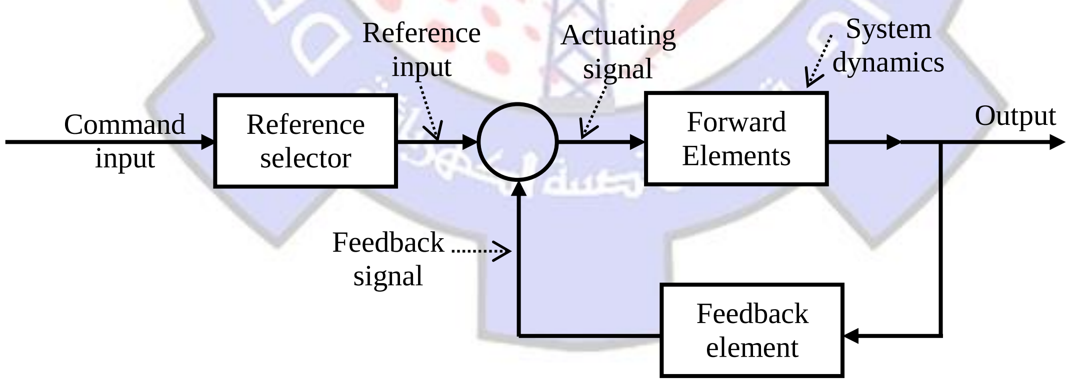


Fig. 1.3. Functional block diagram of a closed-loop system

The designation closed-loop implies the action resulting from the comparison between the output and input quantities in order to maintain the output at the desired value. Thus, the output is controlled in order to achieve the desired value.

Examples of closed-loop control systems are illustrated in Figs. 1.4&1.5 . In a home heating system the desired room temperature (command input) is set on the thermostat in Fig.1.4. (reference selector).A bimetallic coil in the thermostat is affected by both the actual room temperature (output) and the reference-selector setting. If the room temperature is lower than the desired temperature, the coil strip alters its shape and causes a mercury switch to operate a relay, which turns on the furnace to produce heat in the room.

When the room temperature reaches the desired temperature, the shape of the coil strip is again altered so that the mercury switch opens. This deactivates the relay and in turn shuts off the furnace. In this example, the bimetallic coil performs the function of a comparator since the output (room temperature) is fed back directly to the comparator. The switch, relay, and furnace are the dynamic elements of this closed-loop control system.

A closed-loop control system of great importance to all multistory buildings is the automatic elevator of Fig.1.5. A person in the elevator presses the button corresponding to the desired floor. This produces an actuating signal that indicates the desired floor and turns on the motor that raises or lowers the elevator. As the elevator approaches the desired floor, the actuating signal decreases in value and, with the proper switching sequences, the elevator stops at the desired floor and the actuating signal is reset to zero. The closed loop control system for the express elevator in the Sears Tower building in Chicago is designed so that it ascends or descends the 103 floors in just under1min with maximum passenger comfort.

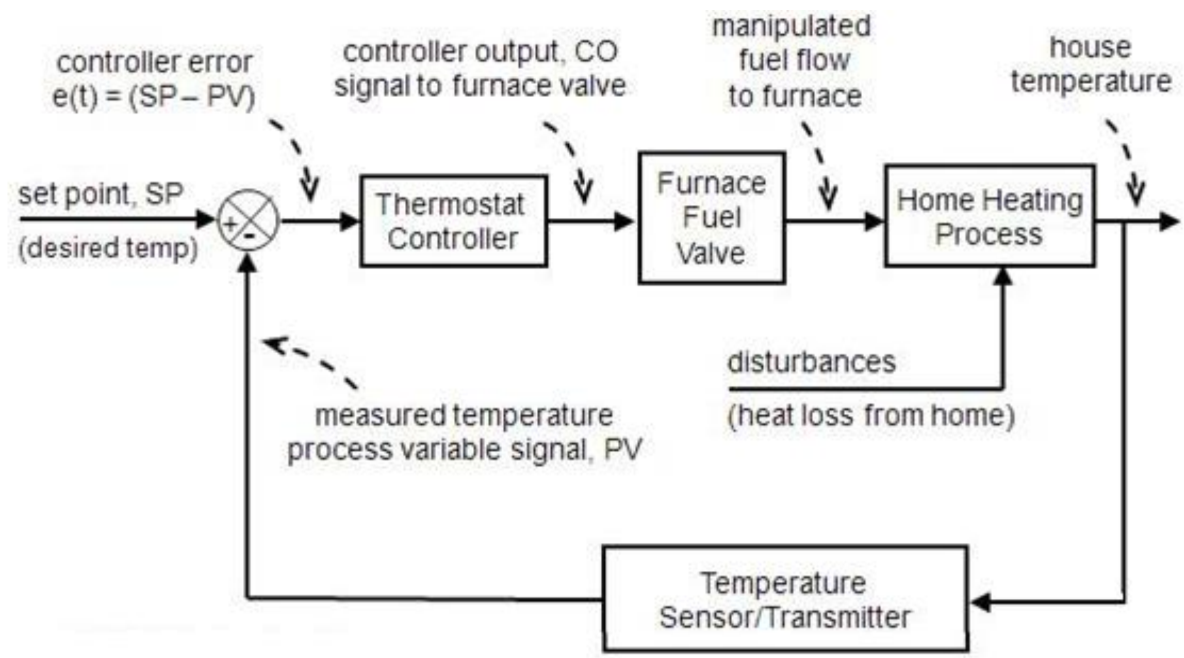


Fig.1.4. Home heating block diagram control system.

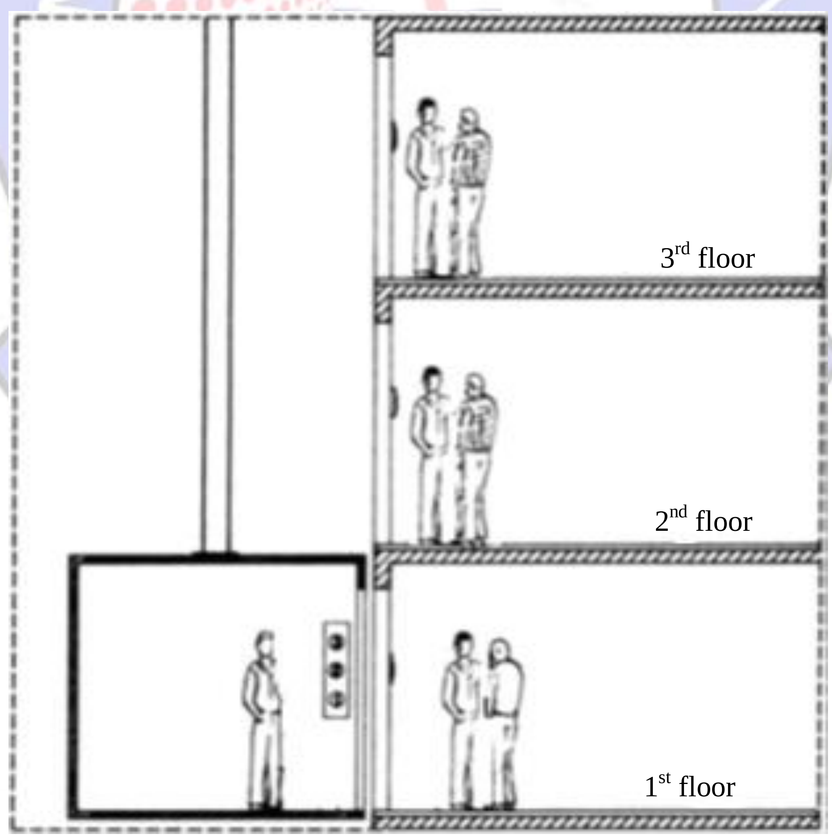


Fig. 1.5. Automatic elevator.

1.4. Definitions of control system.

From the above description there are many terms can be defined as:

Control Theory: It is that part of science which concern control problems.

Control Problem: If we want something to act or vary according to a certain performance specification, then we say that we have a control problem. Ex. We want to keep the temperature in a room at certain level and as we order, then we say that we have temperature control problem.

Plant: A piece of equipment's the purpose of which is to perform a particular operation (we will call any object to be controlled a plant). Ex. Heating furnace, chemical reactor or space craft.

The system: A combination of components that act together to perform a function not possible with any of the individual parts. The word system as used herein is interpreted to include physical, biological, organizational, and other entities, and combinations thereof, which can be represented through a common mathematical symbolism. The formal name systems engineering can also be assigned to this definition of the word system. Thus, the study of feedback control systems is essentially a study of an important aspect of systems engineering and its application.

Process: Progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead towered a particular results or end. In this lectures we call any operation to be controlled a process.

Reference selector (reference input element): The unit that establishes the value of the reference input. The reference selector is calibrated in terms of the desired value of the system output.

Reference input: The reference signal produced by the reference selector, i.e., the command expressed in a form directly usable by the system. It is the actual signal input to the control system.

Disturbance input: An external disturbance input signal to the system that has an unwanted effect on the system output.

Forward element (system dynamics): The unit that reacts to an actuating signal to produce a desired output. This unit does the work of controlling the output and thus may be a power amplifier.

Output (controlled variable): The quantity that must be maintained at a prescribed value, i.e., following the command input without responding the disturbance inputs.

Open-loop control system: A system in which the output has no effect upon the input signal. Ex. heater, light, washing machine.

Feedback element: The unit that provides the means for feeding back the output quantity, or a function of the output, in order to compare it with the reference input.

Actuating signal: The signal that is the difference between the reference input and the feedback signal. It is the input to the control unit that causes the output to have the desired value.

Closed-loop control system: A system in which the output has an effect upon the input quantity in such a manner as to maintain the desired output value.

The fundamental difference between the open- and closed-loop systems is the feedback action, which may be continuous or discontinuous. In one form of discontinuous control the input and output quantities are periodically sampled and discontinuous. Continuous control implies that the output is continuously feedback and compared with the reference input compared; i.e., the control action is discontinuous in time. This is commonly called a digital, discrete-data or sampled-data feedback control system. A discrete data control system may incorporate a digital computer that improves the performance achievable by the system. In another form of discontinuous control system the actuating signal must reach a prescribed value before the system dynamics reacts to it; i.e., the control action is discontinuous in amplitude rather than in time. This type of discontinuous control system is commonly called an on-off or relay feedback control system. Both forms may be present in a system. In this text continuous control systems are considered in detail since they lend themselves readily to a basic understanding of feedback control systems. With the above introductory material, it is proper to state a definition of a feedback control system: ‘‘A control system that operates to achieve prescribed relationships between selected system variables by comparing functions of these variables and using the comparison to effect control.’’ The following definitions are also used.

Servomechanism (often abbreviated as servo): The term is often used to refer to a mechanical system in which the steady-state error is zero for a constant input signal. Sometimes, by generalization, it is used to refer to any feedback control system.

Regulator. This term is used to refer to systems in which there is a constant steady-state output for a constant signal. The name is derived from the early speed and voltage controls, called speed and voltage regulators.

Major advantages of open loop control system:

1. Simple construction and ease of maintenance.
2. Less expensive than the corresponding closed loop system.
3. There is no stability problem.
4. Convenient when output is hard to measure or economically not feasible.

The disadvantages of open loop control systems are as follows:

1. Disturbances and changes in calibration cause errors and the output may be different from what is desired.
2. To maintain the required quality in the output, recalibration is necessary from time to time.

1.5. The engineering control problem:

In general, a control problem can be divided into the following steps:

1. A set of performance specifications is established.
2. The performance specifications establish the control problem.
3. A set of linear differential equations that describe the physical system is formulated or a system identification technique is applied in order to obtain the plant model transfer functions.
4. A control-theory design approach, aided by available computer aided-design (CAD) packages or specially written computer programs, involves the following:
   a. The performance of the basic (original or uncompensated) system is determined by application of one of the available methods of analysis (or a combination of them).

b. If the performance of the original system does not meet the required specifications, a control design method is selected that will improve the system's response.

c. For plants having structured parameter uncertainty, the quantitative feedback theory (QFT) design technique may be used. Parametric uncertainty is present when parameters of the plant to be controlled vary during its operation.

5. A simulation of the designed nonlinear system is performed.
6. The actual system is implemented and tested.

Design of the system to obtain the desired performance is the control problem. The necessary basic equipment is then assembled into a system to perform the desired control function. Although most systems are nonlinear, in many cases the nonlinearity is small enough to be neglected, or the limits of operation are small enough to allow a linear analysis to be used. This textbook considers only linear systems. A basic system has the minimum amount of equipment necessary to accomplish the control function. After a control system is synthesized to achieve the desired performance, final adjustments can be made in a simulation, or on the actual system, to take into account the nonlinearities that were neglected. A computer is generally used in the design, depending upon the complexity of the system. The essential aspects of the control system design process are illustrated in Fig.1.6. Note: The development of this figure is based upon the application of the QFT design technique. A similar figure may be developed for other design techniques. The intent of (Fig.1.6) is to give the reader an overview of what is involved in achieving a successful and practical control system design.

Finally the following design policy includes factors that are worthy of consideration in the control system design problem:

1. Use proven design methods.
2. Select the system design that has the minimum complexity.
3. Use minimum specifications or requirements that yield a satisfactory system response. Compare the cost with the performance and select the fully justified system implementation.
4. Perform a complete and adequate simulation and testing of the system.

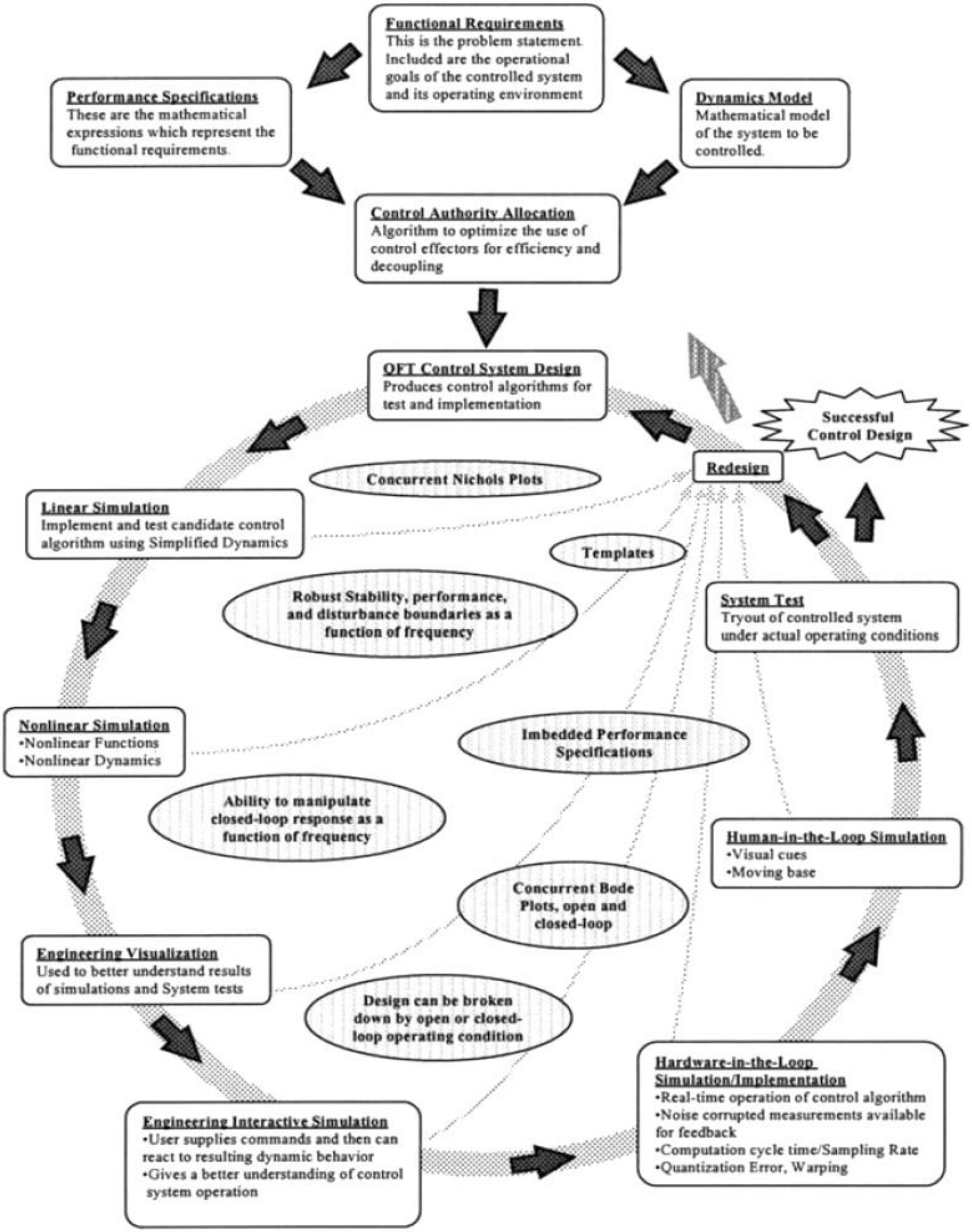


Fig. 1.6. A control system design process: bridging the gap [ref 2].

1.6. Solved Examples

Example 1: Fig.1.7.a. is a schematic diagram of a liquid level control system. Here the automatic controller maintains the liquid level by comparing the actual level with the desired level and correcting any error by adjusting the opening of pneumatic valve. Fig.1.7.b. is the corresponding block diagram of the control system.

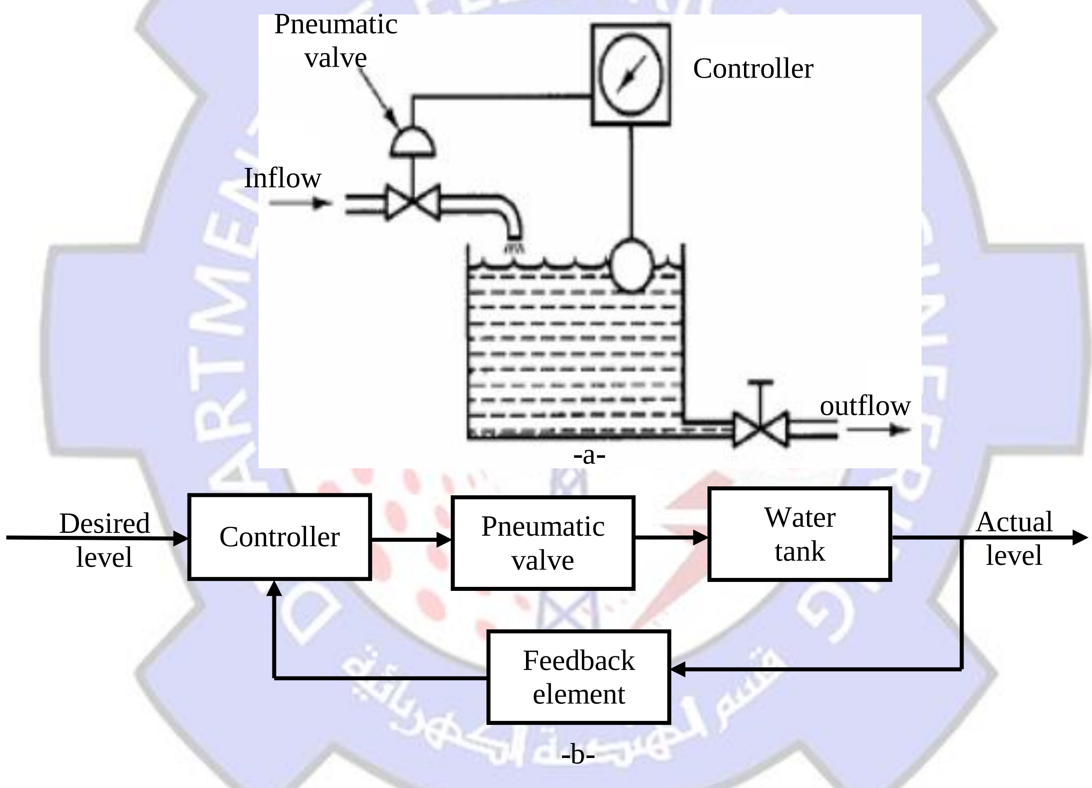


Fig.1.7. (a) Liquid level control system,(b) corresponding block diagram

To draw the block diagram for a human operated liquid level as an example of closed loop control system we will need the following parts (see Fig. 1.8):

1. Eyes as a sensor . 2. Brain as a controller
3. Muscles as a pneumatic valve as in the following block diagram

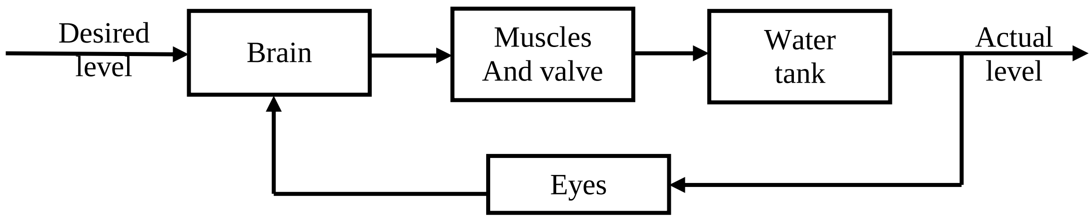


Fig.1.8. Block diagram of human operated liquid level control system

Example 2. An engineering organization system is composed of major groups such as management, research and development, preliminary design, experiment, product design and drafting, fabrication and assembling and testing. These groups are interconnected to make up the whole operation. The system may be organized by reducing it to the most elementary set of components necessary that can provide the analytical detail required and by representing the dynamic characteristics of each component by a set of simple equations. The functional block diagram of the engineering organization can be illustrated as in the block diagram is shown in Fig. 1.9.

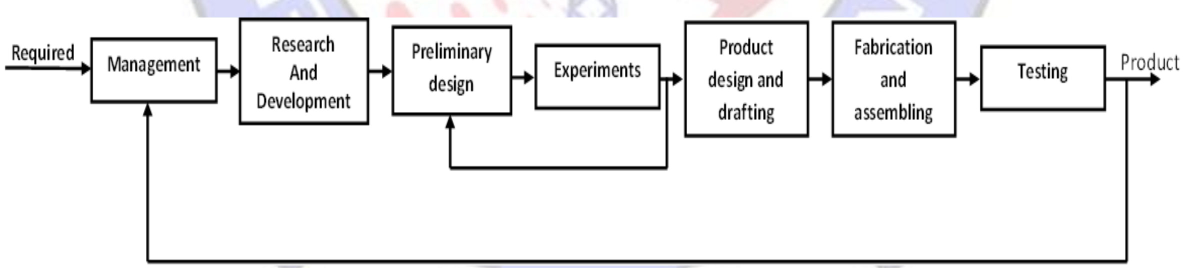


Fig.1.9. Block Diagram of an engineering organization system

1.7. Problems

P.1.1. Give two examples of feedback control systems in which a human acts as a controller?

P.1.2. Explain the open-loop control system by functional diagram and describe the blocks by practical example?

P.1.3. Explain the closed-loop system by functional block diagram and compared it with open-loop control system?

P.1.4. Many closed-loop and open-loop control systems may be found in homes. List several examples and describe them?

P.1.5. Draw the general block diagram of control system and explain each block in the sketch?

Lecture No. Two

Mathematical Representation of Physical Systems

This lecture discusses the following topics :

2.1. Introduction:

2.2. Electrical system.

2.3. Multiloop Electric Circuits.

2.4. State Space Concepts (S.S)

2.5. Transfer Function (T.F):

2.6. Correlation between transfer functions and state-space equations.

2.7. Transfer Function From State-Variable Representation:

2.8. State Variable Representation From Transfer Function:

2.9. Properties of the State Transition Matrix:

2.10. Complex impedances.

2.11. Transfer function of nonloading cascaded system.

2.12. Mecanical Systems.

- **2.12.1. Translational mechanical systems**
- **2.12.2. Rotational mechanical systems**

2.13. Liquid systems.

2.14. Thermal systems.

2.15. Extra systems

- **2.15.1. Gear trains**
- **2.15.2. Potentiometer**
- **2.15.3. Error Detector**
- **2.15.4. First-Order Op-Amp**

2.16. Simulation diagram

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2.1. Introduction:

● Mechanical, electrical, thermal, hydraulic, economic, biological, etc, systems, may be characterized by differential equations.

●The response of dynamic system to an input may be obtained if these differential equations are solved.

●The differential equations can be obtained by utilizing physical laws governing a particular system, for example, Newton's laws for mechanical systems, Kirchhoff's laws for electrical systems, etc.

Mathematical models: The mathematical description of the dynamic characteristic of a system. The first step in the analysis of dynamic system is to derive its model. Models may assume different forms, depending on the particular system and the circumstances. In obtaining a model, we must make a compromise between the simplicity of the model and the accuracy of results of the analysis.

Transfer functions: The transfer function of a linear time-invariant system is define to be the ratio of the Laplace transform (z transform for sampled data systems) of the output to the Laplace transform of the input (driving function), under the assumption that all initial conditions are zero.

Example: Consider the linear time-invariant system

$$a_o y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y^{\cdot} + a_n y = b_o x^{(m)} + b_1 x^{(m-1)} + ... + b_{m-1} x^{\cdot} + b_m x \quad , n \geq m$$

Taking the Laplace transform of y(t)

$$\ell(a_o y^{(n)}) = a_o \ell(y^{(n)}) = a_o [S^n Y(S) - S^{n-1} y(0) - S^{n-2} y'(0) - ...... - y^{(n-1)}(0)$$

$$\ell(a_1 y^{(n-1)}) = a_1 \ell(y^{(n-1)}) = a_o[S^{n-1}Y(S) - S^{n-2}y(0) - S^{n-3}y'(0) - ...... - y^{(n-2)}(0)$$

$$\ell(a_2 y^{(n-2)}) = a_1 \ell(y^{(n-2)}) = a_o[S^{n-2}Y(S) - S^{n-3}y(0) - S^{n-4}y'(0) - ...... - y^{(n-3)}(0).$$

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$$\ell(a_n y) = a_n \ell(y) = a_n Y(s)$$

same thing is applied to obtain the L.T. of x(t).by substitute all initial condition to zero. The transfer function of the system become.

$$\text{Transfer function} = G(s) = \frac{b_o S^m + b_1 S^{(m-1)} + ... + b_{m-1} S + b_m}{a_o S^n + a_1 S^{(n-1)} + ... + a_{n-1} S + a_m} \qquad (2.1)$$

● Transfer function is not provide any information concerning the physical structure of the system (the T.F. of many physically different system can be identical).

● The highest power of s in the denominator of T. F. is equal to the order of the highest derivative term of the output. If the highest power of s is equal to n the system is called an nth order system.

How you can obtain the transfer function (T. F.)?

1- Write the differential equation of the system

2- Take the L. T. of the differential equation, assuming all initial condition to be zero.

3- Take the ratio of the output to the input. This ratio is the T. F.

In general, variables that are functions of time are represented by lowercase letters. These are sometimes indicated by the form x(t), but more often this is written just as x. There are some exceptions, because of established convention, in the use of certain symbols. To simplify the writing of differential equations. The symbols D and1/D are defined by:

$$Dy \equiv \frac{dy(t)}{dt}, D^2 y \equiv \frac{d^2 y(t)}{dt^2} \tag{2.2}$$

$$D^{-1}y \equiv \frac{1}{D} y \equiv \int_0^t y(\tau)d\tau + \int_{-\infty}^0 y(\tau)d\tau = \int_0^t y(\tau)d\tau + Y_0 \tag{2.3}$$

where *Yo* represents the value of the integral at time t = 0, that is, the initial value of the integral.

2.2. Electrical system:

For the series RLC circuit shown in Fig.2.1,

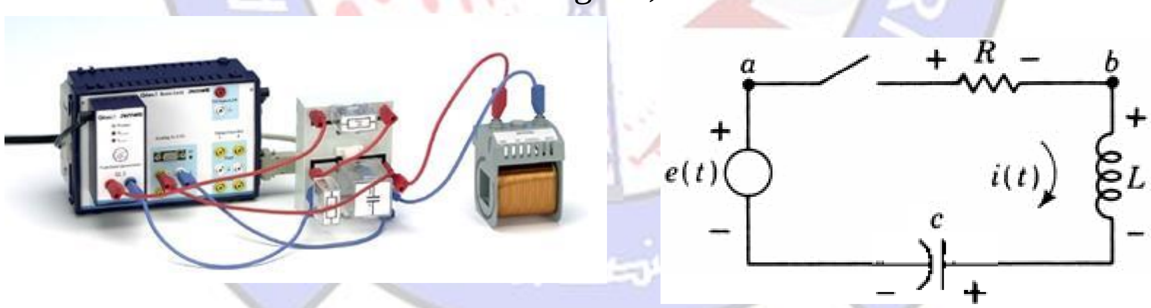


Fig.2.1. Series Resistor–Inductor–Capacitor Circuit

the applied voltage is equal to the sum of the voltage drops when the switch is closed:

$$V_L + V_R + V_C = e \tag{2.4}$$

$$LD_i + Ri + \frac{1}{CD}i = e \qquad (2.5)$$

The circuit equation can be written in terms of the voltage drop across any circuit element. For example, in terms of the voltage across the resistor, *VR* = *Ri*, the equation become:

$$\frac{L}{R}D_{VR} + v_R + \frac{1}{RCD}v_R = e \qquad (2.6)$$

For LRC circuit in Fig.2.2.

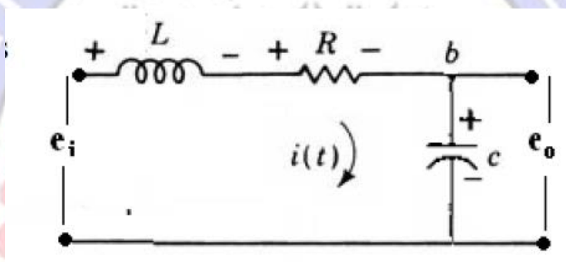


Fig.2.2. Series Resistor–Inductor–Capacitor Circuit

Applying Kirchhoff's voltage law to the system shown. We obtain the following equation;

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \qquad (2.7)$$

$$\frac{1}{C}\int i\,dt = e_o \qquad (2.8)$$

Above two equations give a mathematical model of the circuit. Taking the L.T. of equations, assuming zero initial conditions, we obtain:

$$LSI(S) + RI(S) + \frac{1}{CS} I(S) = E_i(S) \tag{2.9}$$

$$\frac{1}{CS} I(S) = E_o(S) \tag{2.10}$$

The final transfer function of the series RLC circuit will be as in the following equation :

$$\frac{E_o(S)}{E_i(S} = \frac{1}{LCS^2 + RCS + 1} \tag{2.11}$$

2.3. Multi loop Electric Circuits

Multi loop electric circuits (see Fig.2.3) can be solved by either loop or nodal equations.

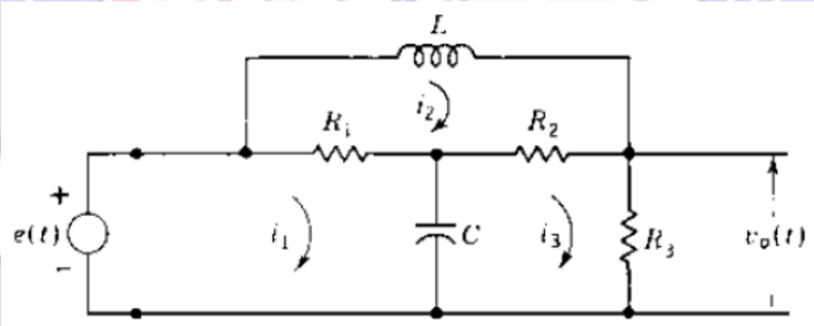


Fig.2.3. Multi loop network

The following example illustrates both methods. The problem is to solve for the output voltage Vo.

Loop Method: A loop current is drawn in each closed loop (usually in a clockwise direction); then Kirchhoff 's voltage equation is written for each loop:

$$(R_1 + \frac{1}{CD})i_1 - R_1 i_2 - \frac{1}{CD} i_3 = e$$

$$-R_1 i_1 + (R_1 + R_2 + LD)i_2 - R_2 i_3 = 0$$

$$-\frac{1}{CD} i_1 - R_2 i_2 + (R_2 + R_3 + \frac{1}{CD})i_3 = 0$$

The output voltage is Vo=R3 i3 These four equations must be solved simultaneously to obtain Vo(t) in terms of the input voltage e(t) and the circuit parameters.

Node Method: The junctions, or nodes, are labeled by letters in Fig.2.4. Kirchhoff's current equations are written for each node in terms of the node voltages, where node **d** is taken as reference. The voltage $\mathbf{V_{bd}}$ is the voltage of node **b** with reference to node **d**. For simplicity, the voltage $\mathbf{V_{bd}}$ is written just as $\mathbf{V_b}$.

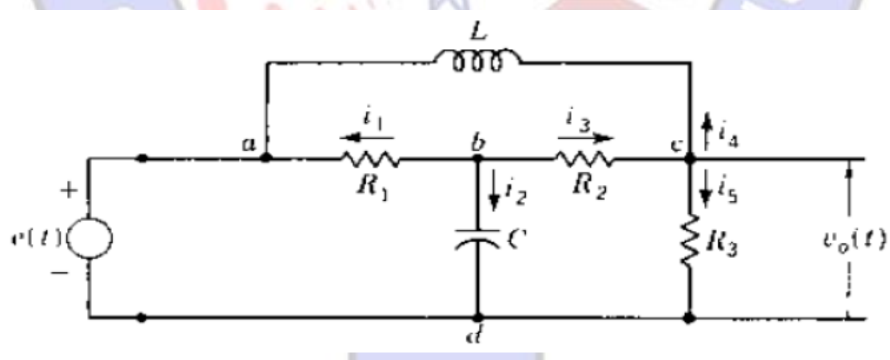


Fig.2.4. Multi node network

$$i_1 + i_2 + i_3 = 0$$

$$-i_3 + i_4 + i_5 = 0$$

Since there is one known node voltage ***Va=e*** two unknown voltages ***Vb*** and ***Vo***, only two equations are required:

For node b and node c:

In terms of the node voltages, these equations are:

$$\frac{v_b - v_a}{R_1} + CDV_b + \frac{v_b - v_0}{R_2} = 0$$

$$\frac{v_0 - v_b}{R_2} + \frac{V_o}{R_3}V + \frac{1}{LD}(v_o - e) = 0$$

Rearranging the terms in order to systematize the form of the equations gives:

$$(\frac{1}{R_1} + CD + \frac{1}{R_2})V_b + \frac{v_0}{R_2} = \frac{e}{R_1}$$

$$\frac{1}{R_1}V_b + (\frac{1}{LD} + \frac{1}{R_2} + \frac{1}{R_3})V_0 = \frac{e}{LD}$$

For this example, only two nodal equations are needed to solve for the potential at node *c*. An additional equation must be used if the current in *R3* is required. With the loop method, three equations must be solved simultaneously to obtain the current in any branch; an additional equation must be used if the voltage across *R3* is required. The method that requires the solution of the fewest equations should be used. This varies with the circuit.

The rules for writing the node equations are summarized as follows:

1. The number of equations required is equal to the number of unknown node voltages.

2. An equation is written for each node.

3. Each equation includes the following:

(a) The node voltage multiplied by the sum of all the admittances that are connected to this node. This term is positive.

(b) The node voltage at the other end of each branch multiplied by the admittance connected between the two nodes. This term is negative.

2.4. State Space Concepts:

Basic matrix properties are used to introduce the concept of state and the method of writing and solving the state equations.

State: The state of a system is a mathematical structure containing a set of n variables $x1(t), x2(t), \dots, xi(t), \dots, xn(t)$, called the state variables, such that the initial values $xi(to)$ of this set and the system inputs $uj(t)$ are sufficient to describe uniquely the system's future response of $t \geq to$. A minimum set of state variables is required to represent the system accurately. The m inputs, $u1(t), u2(t), \dots, uj(t), \dots, um(t)$, are deterministic; i.e., they have specific values for all values of time $t \geq to$.

Generally the initial starting time to is taken to be zero. The state variables need not be physically observable and measurable quantities; they may be purely mathematical quantities. The following additional definitions apply:

State Vector: The set of state variables *xi(t)* represents the elements or components of the n-dimensional state vector *x(t);* that is,

$$X(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ . \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix} \equiv X \qquad (2.12)$$

The order of the system characteristic equation is **n**, and the state equation representation of the system consists of n first-order differential equations. When all the inputs **uj (t)** to a given system are specified for **t> to**, the resulting state vector uniquely determines the system behavior for any **t > to**.

State Space: State space is defined as the n-dimensional space in which the components of the state vector represent its coordinate axes.

State Trajectory: State trajectory is defined as the path produced in the state space by the state vector **x(t)** as it changes with the passage of time. State space and state trajectory in the two-dimensional case are referred to as the phase plane and phase trajectory, respectively.

The first step in applying these definitions to a physical system is the selection of the system variables that are to represent the state of the system.

Note that there is no unique way of making this selection. The three common representations for expressing the system state are the physical, phase, and canonical state variables.

The selection of the state variables for the physical-variable method is based upon the energy-storage elements of the system. Table 1 lists some common energy-storage elements that exist in physical systems and the corresponding energy equations. The physical variable in the energy equation for each energy-storage element can be selected as a state variable of the system. Only independent physical variables are chosen to be state variables.

Independent state variables are those state variables that cannot be expressed in terms of the remaining assigned state variables. In some systems it may be necessary to identify more state variables than just the energy-storage variables. This situation is illustrated in some of the following examples, where velocity is a state variable. When position, the integral of this state variable, is of interest, it must also be assigned as a state variable.

For the circuit of Series RLC Circuit (Fig.2.2). This circuit contains two energy-storage elements, the inductor and capacitor. From Table 1, the two assigned state variables are identified as ***x1=Vc*** (the voltage across the capacitor) and ***x2=i*** (the current in the inductor). Thus two state equations are required.

Fig 2.3 is redrawn in Fig. 2.4 with node ***b*** as the reference node. The node equations for node ***a*** and the loop equations are, respectively,

$$Cx^{\cdot} = x_2$$

$$Lx_2^{\cdot} + Rx_2 + x_1 = u$$

Rearranging terms to the standard state equation format yields:

$$\begin{aligned} x_1^{\cdot} &= \frac{1}{C}x_2 \\ x_2^{\cdot} &= -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \end{aligned} \tag{2.13}$$

Table 2.1.Energy storage elements

System	Elemnet	Energy	Physical variable
Electrical	Capacitor (C)	$\frac{Cv^2}{2}$	Voltage (v)
	Inductor (L)	$\frac{Li^2}{2}$	Current (i)
Mechanical	Mass (M)	$\frac{Mv^2}{2}$	Translational velocity (v)
	Moment of inertia (J)	$\frac{Jw^2}{2}$	Rotational velocity (w)
	Spring (K)	$\frac{Kx^2}{2}$	Displacement (x)
Fulid	Fluid compressibility ($\frac{V}{K_B}$)	$\frac{VP_L^2}{2K_B}$	Pressure (P_L)
	Fluid capacitor C=ρA	$\frac{\rho Ah^2}{2}$	Height (h)
Thermal	Thermal capacitor C	$\frac{C\theta^2}{2}$	Temperature (Θ)

Equation (2.13) represents the state equations of the system containing two independent state variables. Note that they are first-order linear differential equations and are ***n=2*** in number. They are the minimum number of state equations required to represent the system's future performance.

State equation. The state equations of a system are a set of **n** first-order differential equations, where **n** is the number of independent states.

The state equation represented by Eq(2.12)is expressed in matrix notation as:

$$\begin{bmatrix} X_1^{\cdot} \\ X_2^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u \qquad (2.14)$$

The standard form of the state space equation is:

$$X^{\cdot} = Ax + Bu$$

In this case ,the matrix A which is the system matrix will be :

A= $\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}$, nxn plant coefficient matrix

Matrix B ,which is input matrix will be :

B= $\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$, nx1 control matrix

and, in this case**,** u=[u] is a one-dimensional control vector.

In ($X^{\cdot} = Ax + Bu$), matrix **A** and **x** are conformable. If the output quantity y(t) for the circuit of Fig.8 is the voltage across the capacitor v_C, then

$$y(t) = v_c = x_1$$

Thus the matrix system output equation for this example is:

$$y(t) = Cx + Du = [1 \quad 0]\begin{bmatrix} x1 \\ x2 \end{bmatrix} + [0]u \qquad (2.15)$$

Where : C is the output matrix with 1xn dimension for single input single output system (SISO),

D is the forward matrix =0.

For a multiple-input multiple-output (MIMO) system, with m inputs and l outputs, these equations become:

$X^{\cdot} = Ax + Bu$ and $y = Cx + Du$; Where:

A = n × n plant or system matrix ; B = n × m control or input matrix

C = l × n output matrix ; D = l × m feed forward matrix

u = m- dimensional control vector

y = l- dimensional output vector

The block diagram of the state space representation can be shown in the Fig.2.5.

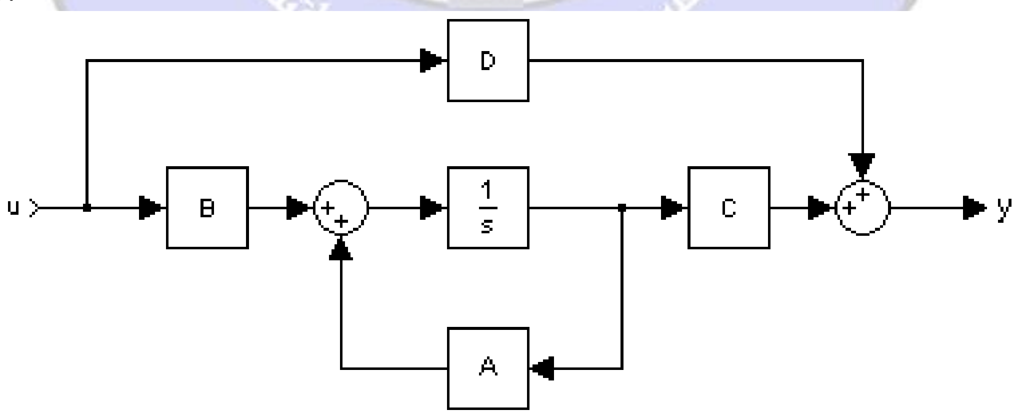


Fig.2.5. Block diagram of the linear, continuous time control system represented in state space.

2.5. Transfer Function (T.F):

If the system differential equation is linear, the ratio of the output variable to the input variable, where the variables are expressed as functions of the D operator, is called the transfer function.

Consider the system output **vC=y** in the RLC circuit of Fig.2.1 substituting **i=CDv$_C$** into Eq(LDi+Ri+1/CD i=e), yields: $(LCD^2+RCD+1)v_c$;(t)

The system transfer function is:

$$G(D) = \frac{y(t)}{u(t)} = \frac{v_c(t)}{e(t)} = \frac{1}{LCD^2 + RCD + 1} \qquad (2.16)$$

The notation G(D) is used to denote a transfer function when it is expressed in terms of the D operator. It may also be written simply as G.

The block diagram representation of this system (Fig.2.6) represents the mathematical operation G(D)u(t)=y(t); that is, the transfer function times the input is equal to the output of the block. The resulting equation is the differential equation of the system.

Fig.2.6. Block diagram representation.

Note: sometime used the symbol (s) instead of (D) and transfer function becomes writing as (G(s)) and D≡s and D2≡s2 ……, and the equation (2.16) will write as:

$$G = \frac{1}{LCS^2 + RCS + 1} \qquad (2.17)$$

The program by using Matlab to change between the two forms for representation of control system (State Space and Transfer Function) can be shown below;

% The change in form from SS to TF is:

```
Cy=1;Ly=10;Ry=100;

A=[ 0  /Cy  -1/Ly -Ry/Ly ];

B=[0  1/Ly];

C=[1 0];

D=[0];

[num,den]=ss2tf(A,B,C,D)
```

% The change in form from TF to SS is:

```
numc=[0 0 1];

denc=[Ly*Cy Ry*Cy 1];

[AA,BB,CC,DD]=tf2ss(numc,denc)
```

In the Matlab/Simulink can be done as in the Fig. 2.7.

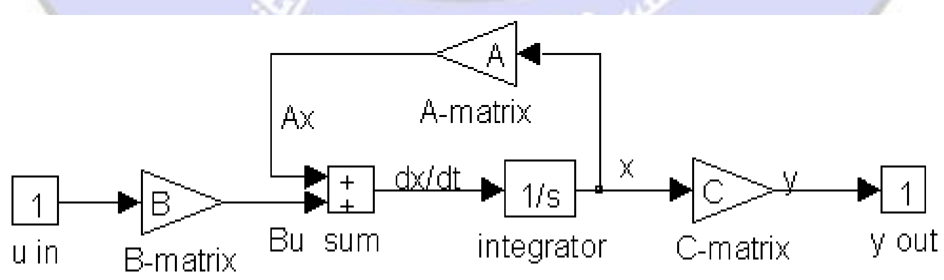


Fig.2.7. Simulink state space representation of control system

H.W. Implement the program and compare between two results?

2.6. Correlation between transfer functions and state-space equations

The following full derivation of transfer function of SISO system from the state-space equations. Let us consider the system whose transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)} \tag{2.18}$$

This system may be represented in state space by the following equations:

$$X^{\cdot} = Ax + Bu \tag{2.19}$$

$$y = Cx + Du$$

Where

x is the state vector ,u is the input and y is the output. The Laplace transform of the equation 2 is given by:

$$\left.\begin{aligned} sx(s) - x(0) &= Ax(s) + Bu(s) \\ Y(s) &= Cx(s) + Du(s) \end{aligned}\right\} \tag{2.20}$$

Since the transfer function is previously defined as Laplace transformation of the output to the input with zero initial conditions, we assume that x(0)=0,then we have

$$sx(s) - Ax(s) = Bu(s) \text{ or } (SI - A)X(s) = BU(s)$$

Multiplying $(SI - A)^{-1}$ to both sides of the last equation we will obtain

$$X(s) = (SI - A)^{-1} BU(S) \tag{2.21}$$

Substitute (2.21) in (2.20) we get

$$Y(s) = [C(SI - A)^{-1}B + D]U(s) \tag{2.22}$$

So that the transfer function of the system represented by state space will be:

$$G(s) = Y(s)/U(s) = [C(SI - A)^{-1}B + D] \tag{2.23}$$

The right hand side of equation (2.23) involves $(SI - A)^{-1}$.Hence G(s) can be written as

$$G(s) = Q(s)/|(SI - A)|$$

Where Q(s) is a polynomial in s .Therefore, $|(SI - A)|$is equal to the characteristic polynomial of G(s).in other words ,the Eigen values of A are identical to the poles of G(s).

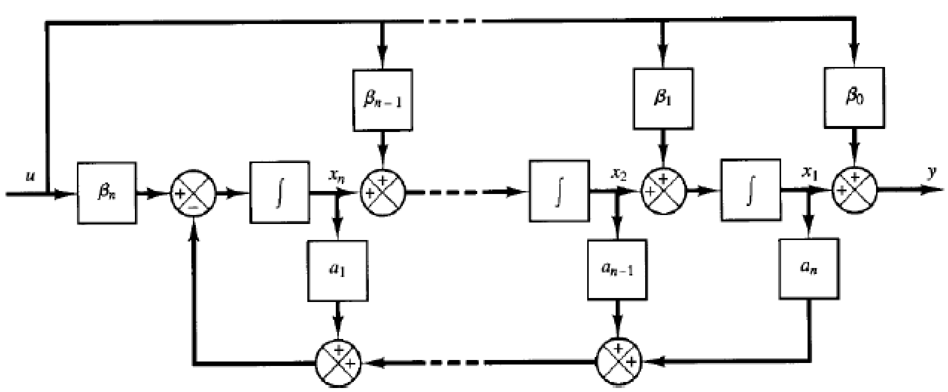


Fig.2.8. Block diagram of state equation and output equation

From Fig.2.8. can be written the following states equations:

$$x_1 = y - \beta_0 u$$

$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$

$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = \dot{x}_2 - \beta_2 u$

$x_n = y^{n-1} - \beta_0 u^{n-1} - \beta_1 u^{n-2} - \ldots\ldots\ldots\ldots - \beta_{n-2} \dot{u} - \beta_{n-1} u \; = \dot{x}_{n-1} - \beta_{n-1} u$

Where $\beta_0 , \beta_1 , \beta_2 , \beta_n$ are determined from

$\beta_0 = b_0$

$\beta_1 = b_1 - a_1 \beta_0$

$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$

$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$

.

.

$\beta_n = b_n - a_1 \beta_{n-1} - a_{n-2} \beta_{n-2} - \ldots\ldots - a_n \beta_0$

With this choice of state variables, the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables). With the present choice of state variable, we obtain

$\dot{x}_1 = x_2 + \beta_1 u$

$\dot{x}_2 = x_3 + \beta_2 u$

$$x^{\cdot}_{n-1} = x_n + \beta_{n-1}u$$

$$x^{\cdot}_{n} = -a_n x_1 - a_{n-1}x_2 - \dots\dots - a_1 x_n + \beta_n u$$

In term of vector matrix equations, the output equation can be written as

$$\begin{bmatrix} x^{\cdot}_1 \\ x^{\cdot}_2 \\ x^{\cdot}_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$[y] = [1 \quad 0 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} + \beta_0 u$$

Or

$$X^{\cdot} = AX + BU$$
$$Y = CX + DU$$

Where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_1 \end{bmatrix}, B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$D = \beta_0 = b_0$$

Note that the state space representation for the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_n}$$

Example (1): Obtain the state equations for the circuit of Fig.2.9. The output is the voltage v_1.The input or control variable is a current source i(t). The assigned state variables

are i_1 , i_2 , i_3 , v_1 , and v_2 , ?

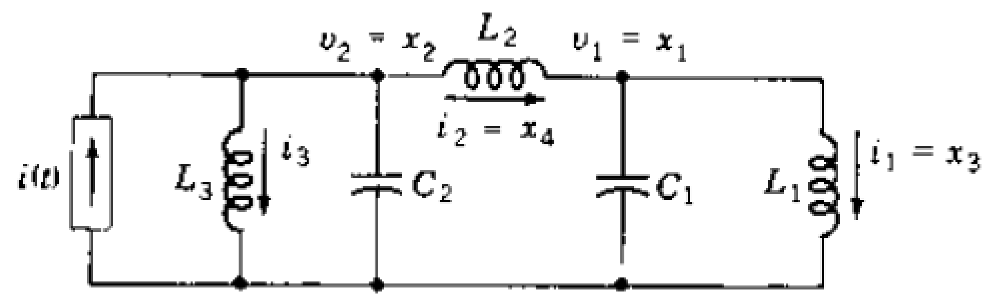


Fig.2.9. Circuit of Example 4.

Solution:

Three loop equations and two node equations are written:

$$v_1 = L_1 D i_1 \text{ or } v_1 = L_1 \frac{di_1}{dt}$$

$$v_2 = L_2 D i_2 + v_1$$

$$v_2 = L_3 D i_3$$

$i_2 = C_1 D v_1 + i_1$

$i = i_3 + C_2 D v_2 + i_2$

$L_3 i_3 = L_2 i_2 + L_1 i_1 + K$

$$X^{\cdot} = \begin{bmatrix} 0 & 0 & -1/C_1 & -1/C_1 \\ 0 & 0 & -L_1/C_2 L_3 & -(L_2+L_3)/C_2 L_3 \\ 1/L_1 & 0 & 0 & 0 \\ -1/L_2 & 1/L_2 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1/C_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$[y] = [1 \quad 0 \quad 0 \quad 0]X$$

2.7. Transfer Function from State-Variable Representation:

Having established the conditions for the equivalence of the state-variable representation with that of the transfer-function, we are interested to find one representation from the other by finding their relationship. Let us consider first the problem of determining the transfer function of a system given the state variable representation

$$X^{\cdot}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

since the transfer-function representation is expressed in the frequency domain, we begin by taking the Laplace transform of both equations, assuming as usual in transfer-function determination that the initial conditions on x are all zero.

$$SX = AX(s) + BU(s) \qquad (2.24)$$

$$Y(s) = CX(s) \qquad (2.25)$$

Grouping the two X(s) terms in Equation (2.24) we have

$(sI - A)X(s) = BU(s)$

where the identity matrix has been introduced to allow the indicated multiplication compatible. Now, pre-multiplying both sides of the above equation by $(sI - A)^{-1}$, we get

$X(s) = (sI - A)^{-1} BU(s)$

We substitute this result in Equation (2.25) to obtain

$Y(s) = C(sI - A)^{-1} BU(s)$

Comparing this relation between Y(s) and U(s) with the Equation

$(Y(s) = G(s)U(s))$

we find that the transfer function matrix G(s) as:

$G(s) = C(sI - A)^{-1} B$

For the single input-single output case, this result reduces to

$G(s) = c'(sI - A)^{-1} b$

The matrix $(sI - A)^{-1}$ is commonly referred to as the resolving matrix and is designated by $\varphi(s)$,

$\varphi(s) = (sI - A)^{-1}$

In terms of this notation the Equations become

G(s) = C φ(s) B and G(s) =c′ φ(s) b (2.26)

Example (2): Consider the system represented by the equations

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

The matrix (*s*I – A) in this example becomes

$$s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 10 & s+7 \end{bmatrix}$$

Its inverse is found as

$$\phi(s) = (sI - A)^{-1} = \frac{adj(sI-A)}{\det(sI-A)} = \frac{\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}}{s^2 + 7s + 10}$$

Hence, transfer function will be as

$$G(s) = c'\phi(s)b = (sI - A)^{-1} = \frac{[1 \;\; 0]\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 7s + 10} = \frac{1}{s^2 + 7s + 10}$$

In the above example, we observe that the determinant of the matrix (*s*I – A) is equal to the denominator polynomial of G(*s*). This is always true for single input - single output systems. Although Equation (2.26) provide a direct method for determining the transfer function of a system from a state-variable representation of the system, it is generally not the most efficient method.

2.8. State Variable Representation from Transfer Function:

In Section 2.6 we have shown how to get the transfer function model of a linear continuous system when its state-variable form is available. We shall now take up the issue of getting the state-variable model when the transfer function model is available. Since the state-variable representation is not unique, there are, theoretically, an infinite number of ways of writing the state equations. We shall present here one method for deriving a set of continuous state variable representation from the transfer function. Analogous procedure may be followed for writing the continuous state equation from pulse transfer function in S domain. The transfer function of single-input-single-output system of the form:

$$G(s)=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\dots\dots+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots\dots.+b_1s+b_0}$$

Can be written ,after introducing an auxiliary variable E(s) as

$$G(s)=\frac{Y(s)}{U(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\dots\dots+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots\dots.+b_1s+b_0}*\frac{E(s)}{E(s)}$$

We let now

$$Y(s)=(a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\dots\dots+a_1s+a_0)E(s)$$

$$U(s)=(s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots\dots.+b_1s+b_0)E(s)$$

From Theorem of Laplace transform, we note the following relations between the variables in the *s* domain and time domain with zero initial conditions

$$E(s)\rightarrow e(t)$$

$sE(s) \rightarrow e^{\cdot}(t)$

$s^2 E(s) \rightarrow e^{\cdot\cdot}(t)$

Under this correspondence we define the state variables

$x_1(t) = e(t)$

$x_2(t) = x_1^{\cdot}(t) = e^{\cdot}(t)$

$x_3(t) = x_2^{\cdot}(t) = e^{\cdot\cdot}(t)$

.

.

$x_n(t) = x_{n-1}^{\cdot}(t) = e^{n-1}(t)$

From above two Equations group we obtain the state equations

$x^{\cdot}_1(t) = x_2(t)$

$x^{\cdot}_2(t) = x_3(t)$

$x^{\cdot}_3(t) = x_4(t)$

.

$x^{\cdot}_n(t) = x_n(t) = -b_0 x_1(t) - b_1 x_2(t) - b_2 x_3(t) - b_{n-1} x_n(t) + u$

In matrix notation this becomes

$$\begin{bmatrix} x^{\cdot}_1 \\ x^{\cdot}_2 \\ x^{\cdot}_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 & -b_3 & -b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^{\cdot}_{n-1} \\ x^{\cdot}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

In compact form, it is written as :

x'(t) = A x(t) + Bu(t) ; The output equation is obtained from as

which may be written compactly a s **y**(t) = C**x**(t)

Hence the last two Equations are a set of state equations for the continuous system described by transfer function. Another convenient and useful representation of the continuous system is the signal flow graph or the equivalent simulation diagram. These two forms can be derived, after dividing both the numerator and denominator of first Equation by *sn* :

$$G(s)=\frac{Y(s)}{U(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+.......+b_1s+b_0}*\frac{E(s)}{E(s)}$$

From this expression we can get two equations

$$Y(s)=(a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0)E(s)$$

$$U(s)=(s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+.......+b_1s+b_0)E(s)$$

The above equation can be rewritten as follows

$$E(s)=U(s)-b_{n-1}s^{-1}E(s)-b_{n-2}s^{-2}E(s)+.......-b_1s^{1-n}E(s)-b_0s^{-n}E(s)$$

Example (3): Let us consider a single input single output system of the last Example which is reproduced below for quick reference:

$$A=\begin{bmatrix}0 & 1\\ -10 & -7\end{bmatrix}, b=\begin{bmatrix}0\\ 1\end{bmatrix}, c=\begin{bmatrix}1 & 0\end{bmatrix}, d=\begin{bmatrix}0 & 0\end{bmatrix}$$

We are interested to find its solution with initial condition $x'(t_0) = x'(0)$ = [0 0] and unity step input $u(t) = us(t)$. The resolving matrix φ(s) given by relation (φ(s) = (sI – A)$^{-1}$) is written as :

$$\phi(s) = (sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)} = \frac{\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}}{s^2 + 7s + 10}$$

$$\phi(s) = \begin{bmatrix} \frac{s+7}{s^2+7s+10} & \frac{1}{s^2+7s+10} \\ \frac{-10}{s^2+7s+10} & \frac{s}{s^2+7s+10} \end{bmatrix}$$

$$\phi(s) = \begin{bmatrix} \frac{1}{3}(\frac{5}{s+2} - \frac{2}{s+5}) & \frac{1}{3}(\frac{1}{s+2} - \frac{1}{s+5}) \\ \frac{10}{3}(\frac{1}{s+2} - \frac{1}{s+5}) & \frac{1}{3}(\frac{5}{s+5} - \frac{2}{s+25}) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} \frac{1}{3}(5e^{-2} - 2e^{-5}) & \frac{1}{3}(1e^{-2} - 1e^{-5}) \\ \frac{10}{3}(1e^{-2} - 1e^{-5}) & \frac{1}{3}(5e^{-5} - 2e^{-2}) \end{bmatrix}$$

Substituting the value of x'(0)=[0 0] and unit step input in the equation we get

$$x(t) = \int_0^t \phi(t-\tau) bu(\tau) d\tau = \begin{bmatrix} \frac{1}{3}\int_0^t (5e^{-2(t-\tau)} - 2e^{-5(t-\tau)}) d\tau & \frac{1}{3}\int_0^t (1e^{-2(t-\tau)} - 1e^{-5(t-\tau)}) d\tau \\ \frac{10}{3}\int_0^t (1e^{-2(t-\tau)} - 1e^{-5(t-\tau)}) d\tau & \frac{1}{3}\int_0^t (5e^{-5(t-\tau)} - 2e^{-2(t-\tau)}) d\tau \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{1}{3}\int_0^t (1e^{-2(t-\tau)} - 1e^{-5(t-\tau)}) d\tau \\ \frac{10}{3}\int_0^t \quad \frac{1}{3}\int_0^t (5e^{-5(t-\tau)} - 2e^{-2(t-\tau)}) d\tau \end{bmatrix} = \begin{bmatrix} \frac{1}{10} - \frac{1}{6}e^{-2t} + \frac{1}{15}e^{-5t} \\ \frac{1}{3}e^{-2t} - \frac{1}{3}e^{-5t} \end{bmatrix}$$

Therefore y(t) is computed as *y(t)=cx(t)+du(t)*

$$y(t) = \frac{1}{10} - \frac{1}{6}e^{-2t} + \frac{1}{15}e^{-5t}, t > 0$$

2.9. Properties of the State Transition Matrix:

Some useful properties of the state transition matrix φ(t) are recorded below :

1. $\phi(0) = e^{A0} = I$ (identity matrix)

2. $\phi(t) = e^{At} = e^{(-At)^{-1}} = \phi(-t)^{-1}$ or $\phi(-t)^{-1} = \phi^{-1}(t) = \phi(-t)$

3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}.e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

4. $(\phi(t))^n = \phi(nt)$

5. $\phi(t_1 - t_2)\phi(t_2 - t_3) = \phi(t_1 - t_3)$ for any t_1, t_2, t_3

6. $\dfrac{\phi(t)}{dt} = A\phi(t)$

2.10. Complex impedances.

Consider the circuit shown in Fig.2.10, then the T.F of this circuit is

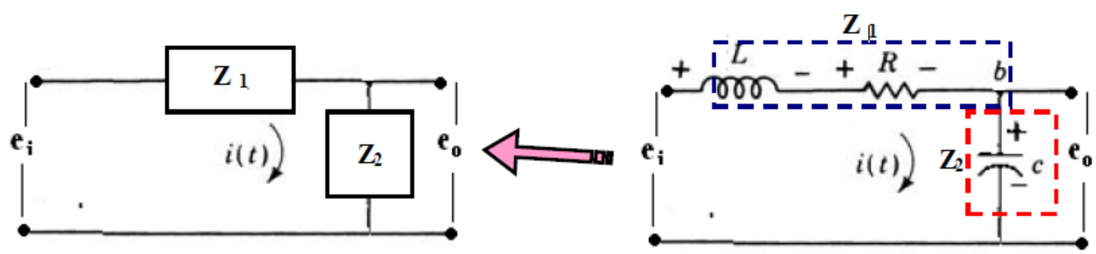


Fig.2.10. circuit diagram of complex impedance

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Where

$$Z_1(s) = Ls + R$$

$$Z_2(s) = \frac{1}{Cs}$$

$$Z(s) = E(s)/I$$

Hence the T.F. Eo(s)/Ei(s) can be found as follows;

$$\frac{E_0(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Example (4): Consider the electrical cct shown in Fig.2.11. Obtain the T.F Eo(s)/Ei(s) by use of the block diagram approach?

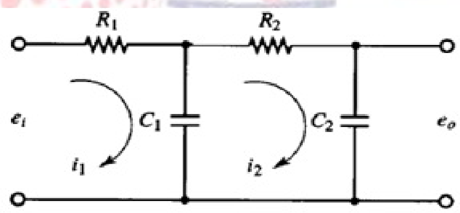


Fig. 2.11. Circuit diagram of Example 5.

Solution:

Equations of the circuit in Fig.2.11 are:

$$e_i = \frac{1}{C_1}\int (i_1 - i_2)dt + i_1 R_1$$

$$0 = \frac{1}{C_1}\int (i_2 - i_1)dt + \frac{1}{C_2}\int i_2 dt + i_2 R_2$$

$$e_0 = \frac{1}{C_2}\int i_2 dt$$

By taking the L.T. for the above equation:

$$E(s)_i = \frac{1}{C_1 s}(I_1(s) - I_2(s) + I_1(s)R_1 \quad (2.27)$$

$$0 = \frac{1}{C_1 s}(I_2(s) - I_1(s) + \frac{1}{C_2 s} I_2(s) I_2(s) R_2 \quad (2.28)$$

$$E_0(s) = \frac{1}{C_2 s} I_2(s)$$

By using Eq(2.27), we get:

$$C_1 s[E(s)_i - I_1(s)R_1] = I_1(s) - I_2(s) \quad (2.29)$$

From Eq's(2.28)&(2.29) we get:

$$I_2(s) = \frac{C_2 s}{R_2 C_2 s + 1} * \frac{1}{C_1 s}[I_1(s) - I_2(s)]$$

The transfer function of Eo(s)/Ei(s) can written in term of I2(s) as follow:

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s} \quad (2.30)$$

The term $R_1 C_2 s$ in the denominator of the transfer function represents the interaction of two simple RC circuits .since $(R_1 C_1 + R_2 C_2 + R_1 C)^2 > (4R_1 C_1 R_2 C_2)$ the two roots of the denominator of equation (2.30) are real. The present analysis show that if two RC circuits are connected cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1 C_1 s+1)$ and $1/(R_2 C_2 s+1)$.

The reason for this is that when we derive the transfer function for an isolated circuit, we implicitly assume to be infinite which means that no power is being withdrawn at the output. when the second circuit is connected to the output of the first , however, a certain amount of power is with –drawn and thus assumption of no loading then violated therefore if the transfer function of this system is obtained under the assumption of no loading then it is not valid . The degree of the loading effect determines the amount of modification of the transfer function.

2.11. Transfer functions of non-loading cascaded elements

The transfer function of a system consisting of two no loading cascaded elements can be obtained by eliminating the intermediate input and output. For example consider the system shown in Fig.2.12.a. The transfer functions of the elements are: $G_1(s)=\frac{X_2(s)}{X_1(s)}$ and $G_2(s)=\frac{X_3(s)}{X_2(s)}$

If the input impedance of the second elements is infinite, the input of the first element is not affected by connecting it to second element. Then transfer function of whole system becomes: $G(s)=\frac{X_3(s)}{X_1(s)}=\frac{X_2(s)}{X_1(s)}\frac{X_3(s)}{X_2(s)}$

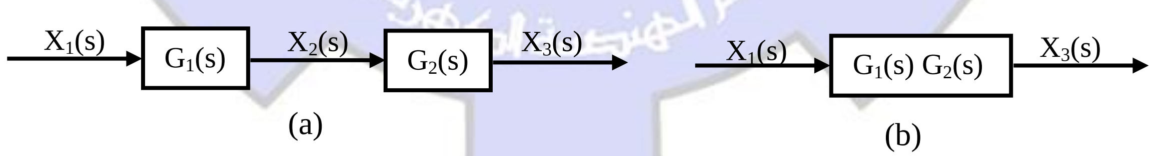


Fig. 2.12. (a) System consisting of two non-loading cascaded elements; (b) an equivalent system.

The transfer function of whole system is thus the product of transfer functions of the individual elements. This is shown in Fig.2.12.b.as an example, consider the system shown in Fig.2.13,the insertion of an isolating amplifier between the circuits to obtain non-loading characteristics is frequently used in combining circuits. Since the amplifier has very high input impedance, an isolation amplifier inserted between two circuits justifies the non-loading assumption. The two simple RC circuit, isolated by an amplifier as shown in Fig.2.13. Have negligible effects and the transfer function of the entire circuit is equal to product of the individual transfer functions, thus in this case

$$\frac{E_0(s)}{E_i(s)} = (\frac{1}{R_1C_1s+1})K(\frac{1}{R_2C_2s+1})$$

$$\frac{E_0(s)}{E_i(s)} = (\frac{K}{(R_1C_1s+1)(R_2C_2s+1)}$$

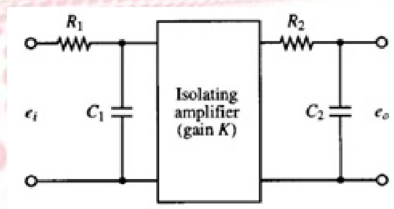


Fig.2.13. Electrical System.

Example (5): Armature-Controlled dc motors

The dc motors have separately excited fields. They are either armature controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals. Consider the armature-controlled dc motor shown in the following Fig.2.14.

R_a = armature-winding resistance, ohms

L_a = armature-winding inductance, henrys

i_a = armature-winding current, amperes

i_f = field current, a-pares ; ea = applied armature voltage, volt

e_b = back emf, volts; θ = angular displacement of the motor shaft, radians

T = torque delivered by the motor, Newton*meter

J = equivalent moment of inertia of the motor and load referred to the motor shaft kg.m2

f = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton*m/rad/s

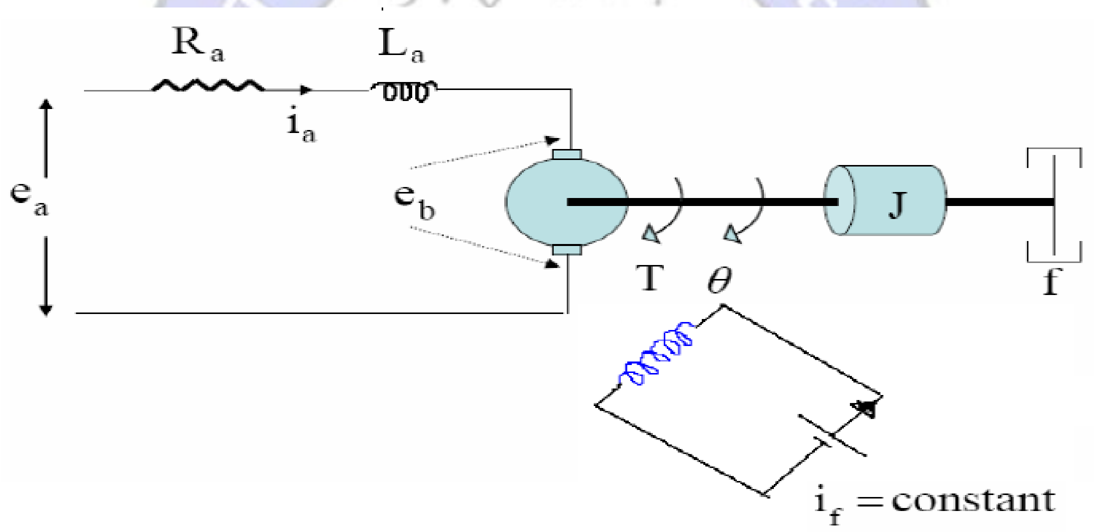


Fig.2.14. Circuit Diagram of armature-controlled dc motor

$T = K_1 i_a \psi$; Where

ψ is the air gap flux, $\psi = K_f i_f$,k_f is constant

$T = K_1 i_a K_f i_f$

For a constant field current

$T = Ki_a$,k is a motor torque constant

$e_b = k_2 \psi \omega$

For constant flux

$e_b = k_b \dfrac{d\vartheta}{dt}$, k_b is back emf constant

The armature circuit equation is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \qquad (2.31)$$

The armature current produces torque which is applied to the both inertia and friction to rotate the motor, hence

$$J \frac{d^2\vartheta}{dt} + f \frac{d\vartheta}{dt} = T = ki_a \qquad (2.32)$$

Taking the Laplace transform of the above three equation with assuming all initial condition is zero

$$E_b(s) = K_b s\theta(s)$$

$$L_a sI_a(s) + R_a I_a(s) + E_b(s) = E_a(s)$$

$$(L_a s + R_a) I_a(s) + E_b(s) = E_a(s)$$

$$Js^2\theta + fs\theta = T = KI_a(s)$$

$$(Js^2 + fs)\theta = T = KI_a(s)$$

The transfer function can be obtained as

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s(L_a J s^2 + (L_a f + R_a J)s + R_a f + K K_b}$$

Try to check this equation

Example (6): Field-Controlled dc motor

Find the T.F **(Θ(s)/Ef(s))**For the field-controlled dc motor shown in Fig.2.15 below

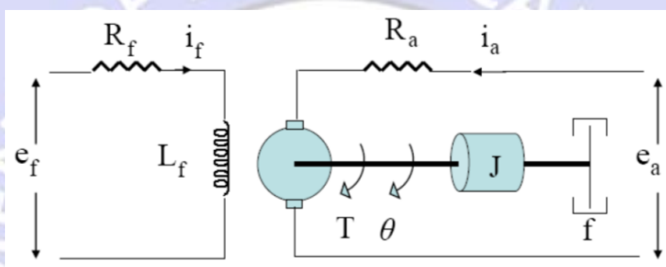


Fig.2.15. Circuit Diagram of field-controlled dc motor

The torque T developed by the motor is proportional to the product of the air gap flux ψ and armature current i_a so that

$T = K_1 i_a \psi$, k_1 is constant

$T = K_2 i_f$, k_2 is constant

$$L_f \frac{di_f}{dt} + R_f i_f = e_f \tag{2.33}$$

$$J\frac{d^2\vartheta}{dt} + f\frac{d\vartheta}{dt} = T = k_f i_f \tag{2.34}$$

Taking the Laplace transform of the above three equation with assuming all initial condition is zero

$$(L_f s+R_f)I_f(s)=E_f(s)$$

$$(Js^2+fs)\theta=T=K_2 I_f(s)$$

The transfer function can be obtained as

$$\frac{\theta(s)}{E_a(s)}=\frac{K_2}{s(L_f s+R_f)(Js+f)}$$ Try to check this equation

H.W. Find the transfer function $\frac{\theta(s)}{I_f(s)}$ and $\frac{E_f(s)}{I_f(s)}$

2.12. Mechanical Systems:

Mechanical systems obey Newton's law that the sum of the forces equals zero; that is, the sum of the applied forces must be equal to the sum of the reactive forces. The three qualities characterizing elements in a mechanical translation* system are mass, elastic, and damping. The following analysis includes only linear functions. Static friction, Coulomb friction, and other nonlinear friction terms are not included. Basic elements entailing these qualities are represented as network elements, and a mechanical network is drawn for each mechanical system to facilitate writing the differential equations. The mass M is the inertial element. A force applied to a mass produces an acceleration of the mass. The reaction force ***fM*** is equal to the product of mass and acceleration and is opposite in direction to the applied force. In terms of displacement ***x***, velocity ***v***, and acceleration ***a***, the force equation is

$$f_m=M_a=MD_v=MD^2x \tag{2.35}$$

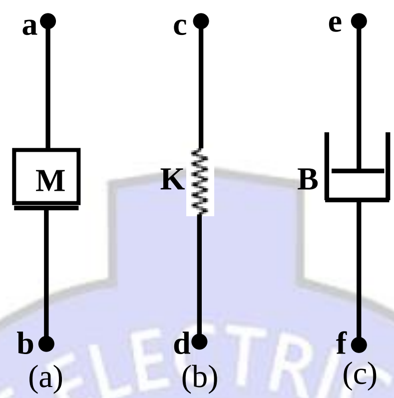


Fig. 2.16. Network elements of mechanical translate.

The network representation of mass is shown in Fig. 2.16.a.One terminal, ***a***, has the motion of the mass; and the other terminal, b, is considered to have the motion of the reference. The reaction force ***fM*** is a function of time and acts ‘‘through ’’*M*. The elastance, or stiffness, K provides a restoring force as represented by a spring. Thus, if stretched, the string tries to contract; if compressed, it tries to expand to its normal length. The reaction force ***fk*** on each end of the spring is the same and is equal to the product of the stiffness K and the amount of deformation of the spring. The network representation of a spring is shown in Fig.16b. The displacement of each end of the spring is measured from the original or equilibrium position. End c has a position xc, and end d has a position xd, measured from the respective equilibrium positions. The force equation, in accordance with Hooke’s law, is

$$f_K = K(x_c - x_d)$$

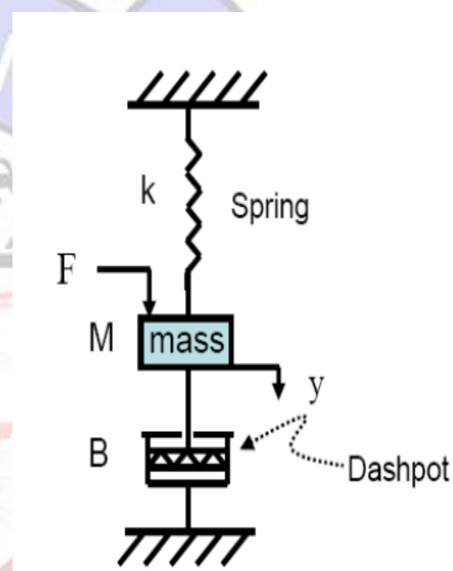
If the end d is stationary, then xd =0 and the preceding equation reduces to

$$f_K = Kx_c$$

The plot ***fk*** vs. **xc** for a real spring is not usually a straight line, because the spring characteristic is nonlinear. However, over a limited region of operation, the linear approximation, i.e., a constant value for K, gives satisfactory results.

2.12.1. Translational mechanical system:

Example (7): find the transfer function of the following system



$$\sum F = ma$$

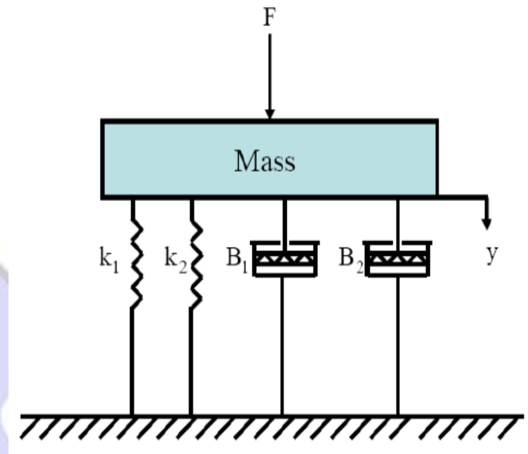
$F = Ky + mD^2 y + BDy$,K is spring constant,

B viscous friction coefficient

$$F = mD^2 y + BDy + ky$$

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + Bs + k}$$

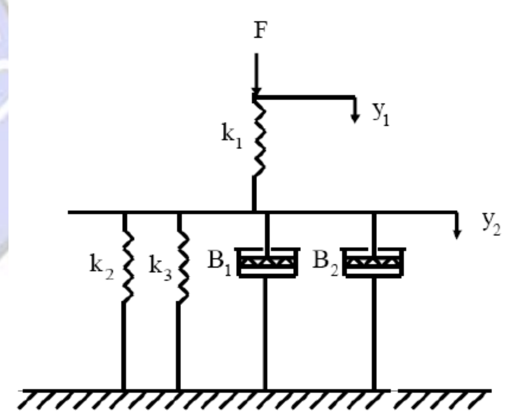
Example (8): Find the transfer function of the following system



$$F = mD^2 y + (B_1 + B_2)Dy + (k_1 + k_2)y$$

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + (B_1 + B_2)s + (k_1 + k_2)}$$

Example (9): Find the transfer function of the following system

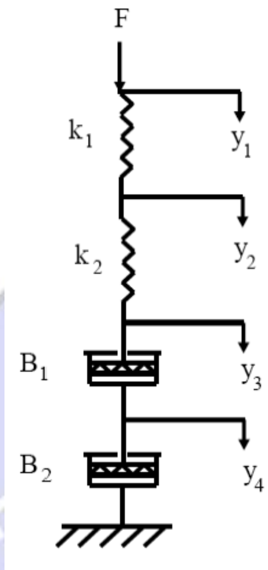


$$F = k_1(y_1 - y_2)$$

$$k_1(y_1 - y_2) = (k_2 + k_3)y_2 + (B_1 + B_2)Dy_2$$

Example (10): Find the transfer function of the following system

$$F = k_1(y_1 - y_2) = k_2(y_2 - y_3) = B_1 D(y_3 - y_4) = B_2 D y_4$$



2.12.2. Rotational mechanical systems:

$J\alpha = \sum T$

Where

J is the moment of inertia

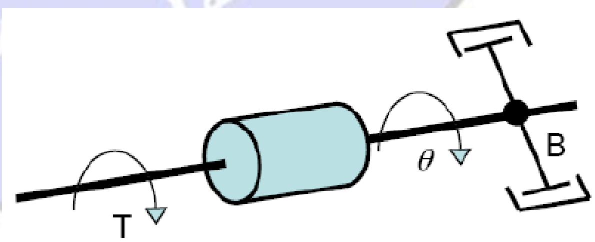
α is the rotational acceleration

T is the toque

Example(11): Write the mathematical model of the following system

$T = JD^2\theta + BD\theta$

$T = JD\omega + B\omega$



Where

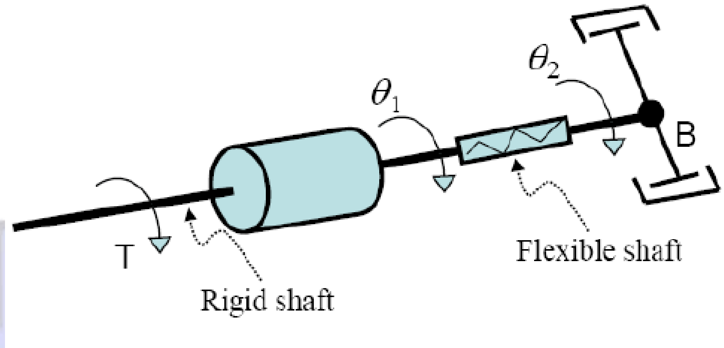
$\omega = \theta^{\cdot} = D\theta$

$\alpha = \omega^{\cdot} = \theta^{\cdot\cdot} = D^2\theta$

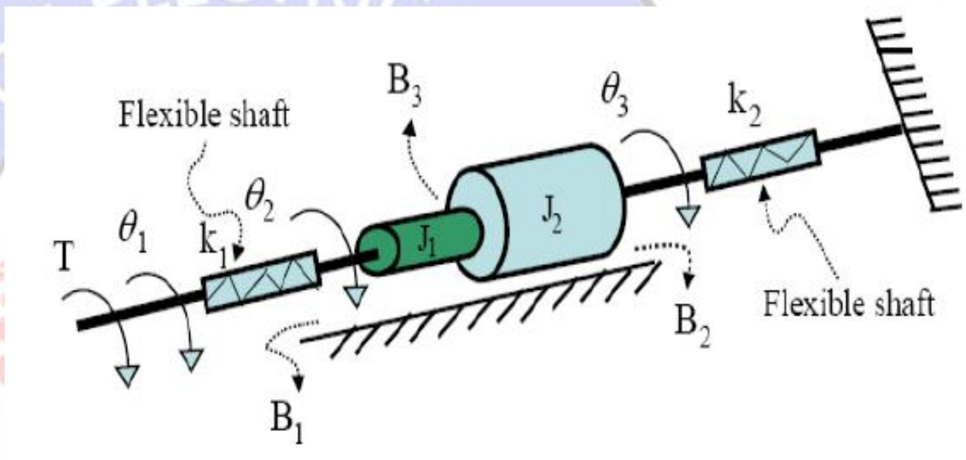
Example (12): Write the mathematical model of the following system

$$T = JD^2\theta_1 + K(\theta_1 - \theta_2)$$

$$K(\theta_1 - \theta_2) = BD\theta_2$$



Example (13): Write the mathematical model of the following system



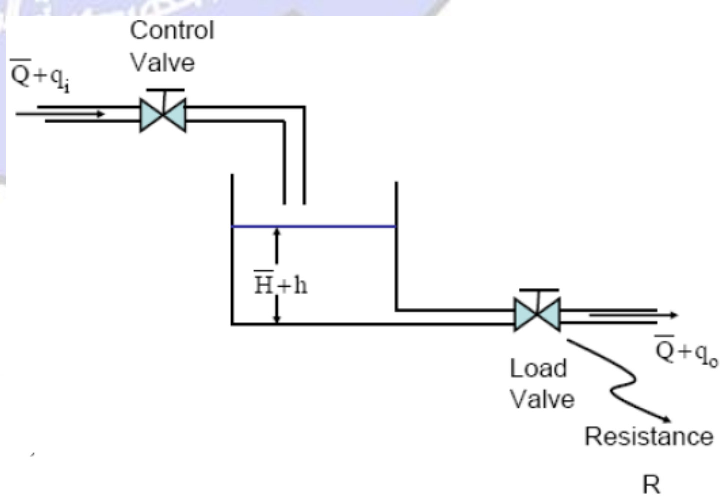
$$T = K_1(\theta_1 - \theta_2)$$

$$T = K_1(\theta_1 - \theta_2) = J_1D^2\theta + B_3D(\theta_2 - \theta_3) + B_1D\theta_2$$

$$B_3D(\theta_2 - \theta_3) = J_2D^2\theta_3 + B_2D\theta_3 + K_2\theta_3$$

2.13. Liquid level systems

Example1.Write the mathematical model of the following system



$$q_i - q_0 = c\frac{dh}{dt}$$

Where

For laminar flow

$$q_0 = \frac{h}{R}$$

For turbulent flow

$$q_0 = K\sqrt{R}$$

$$q_i - \frac{h}{R} = c\frac{dh}{dt}$$

The differential equation will be

$$Rc\frac{dh}{dt} + h = Rq_i$$

Taking the Laplace transform

$$(Rcs+1)H(s) = RQ_i(s)$$

So that the final transfer function will be

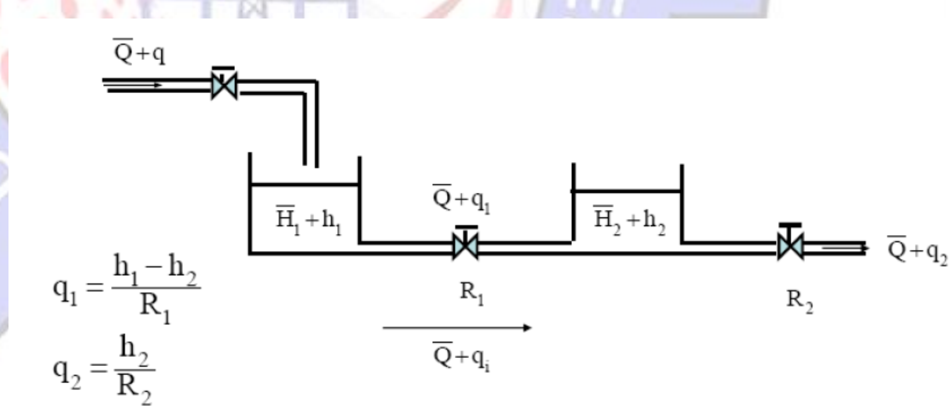
$$\frac{H_0(s)}{Q_i(s)} = \frac{R}{(Rcs+1)}$$

According to that we can find

$$\frac{Q_0(s)}{Q_i(s)} = \frac{1}{(Rcs+1)}$$

Example (14): Find the TF of the interaction liquid level system

$$q_1 = \frac{h_1 - h_2}{R_1}$$



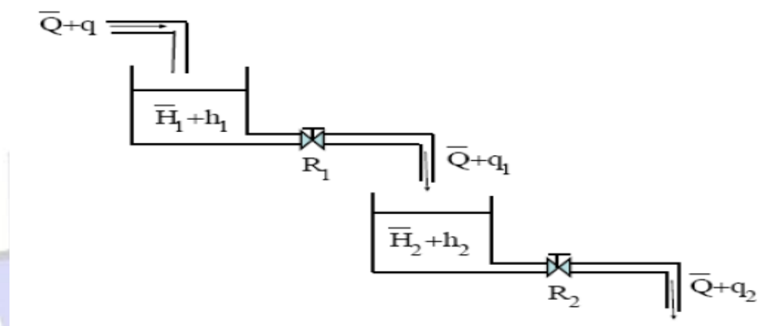
$$q_2 = \frac{h_2}{R_2}$$

$$C_1 = \frac{dh_1}{dt} = q - q_1$$

$$C_2 = \frac{dh_2}{dt} = q_1 - q_2 \quad ; \quad \frac{Q_2(s)}{Q(s)} = \frac{1}{R_1c_1R_2c_2s^2 + s(R_1c_1 + R_2c_2 + R_2c_1) + 1}$$

Home work : Find TFs. $\frac{Q_1(s)}{Q(s)}, \frac{H_1(s)}{Q(s)}, \frac{H_2(s)}{Q(s)}$

Example (15). Find the TF of the non-interaction liquid level system



$$q_1 = \frac{h_1}{R_1}$$

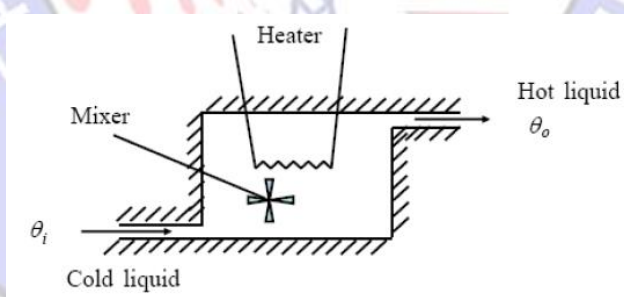
$$q_2 = \frac{h_2}{R_2}$$

$$C_1 = \frac{dh_1}{dt} = q - q_1$$

$$C_2 = \frac{dh_2}{dt} = q_1 - q_2$$

Home work : Find TFs. $\frac{Q_2(s)}{Q(s)}$

2.14. Thermal Systems:



Consider that heat input rate changes from $\overline{H}$ to $\overline{H}+h_i$ then heat outflow will change from $\overline{H}$ to $\overline{H}+h_o$ also the temperature of the out following liquid will change from $\overline{\theta}_o$ to $\overline{\theta}_o+\theta_o$. Considering change only:

$$h_i - h_o = Q\frac{d\theta}{dt}$$

$$\theta = h.R$$

Or

$$RQ\frac{d\theta}{dt} + \theta = Rh_i$$

Note

$$h_o = Gc\theta$$

$$Gc = \frac{1}{R}$$

$$Q = Mc$$

By taking Laplace transform

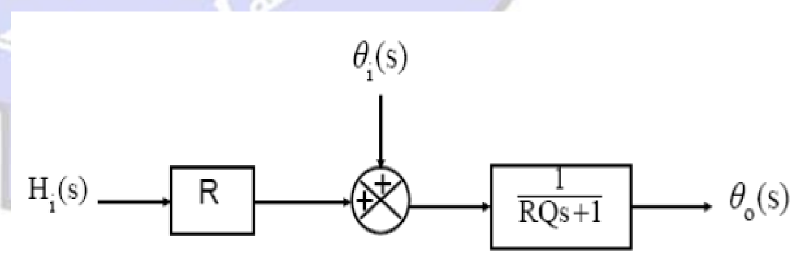
$$\frac{\theta(s)}{H_i(s)} = \frac{R}{RQs+1}$$

$$Q\frac{d\theta}{dt} = Gc\theta_i - h_o$$

$$Q\frac{d\theta}{dt} = \frac{1}{R}\theta_i - \frac{\theta}{R}$$

$$RQ\frac{d\theta}{dt} + \theta = \theta_i$$

$$\frac{\theta(s)}{\theta_i(s)} = \frac{1}{RQs+1}$$



In case of changes in both θ_i & h_i then we have :

$$Rc\frac{d\theta}{dt} + \theta = \theta_i + Rh_i$$

Where

θ_i : Steady state temperature of inflowing liquid, F^o.

θ_o : Steady state temperature of outflowing liquid, F^o.

G : steady state liquid flow rate lb/sec.

M: mass of liquid in tank,lb.

c: specific heat of liquid tu/lb.F^o.

R: thermal resistance, Fo sec/B tu.

Q: thermal capacitance, B tu/F^o.

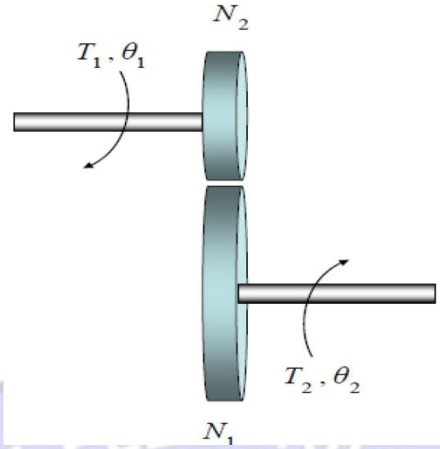
$\overline{H}$: Steady state heat i/p rate ,B tu/sec.

2.15. Extra systems

2.15.1. Gear trains

A gear train is a mechanical device that transmit energy from one part of a system to another in such a way that force, torque ,speed and displacement are altered. Two gears are shown coupled together in following figure. The inertial and friction of the gears are neglected in the ideal case considered.

The relationships between the torque T_1, T_2 and angular displacements θ_1, θ_2 and the teeth numbers N_1, N_2 of the gear train are derived from the following facts.



1. The number of teeth on the surface of the gear is proportional to the radius r_1, r_2 of the gears ,that is.

   $r_1 N_2 = r_2 N_1$

2. The distance traveled along the surface of each gear is same. Therefore

   $r_1 \theta_1 = r_2 \theta_2$

3. The work done by one gear is equal to that of the other since there is assume to be no loss, thus

   $T_1 \theta_1 = T_2 \theta_2$

   If the angular velocities of the two gears are ω_1 and ω_2

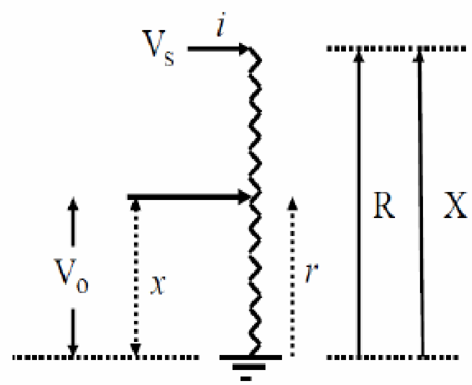
   $$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$

2.15.2. Potentiometer (transducer):

Consider linear resistance

$$x = kr \quad ; \quad X = kR$$

$$i = \frac{V_s}{R} \quad ; \quad V_o = iR$$



From 1 and 2

$$V_o = \frac{V_s}{R} . r$$

$$V_o = \frac{V_s}{\frac{X}{k}} . \frac{x}{k} = \frac{V_s}{X} x$$

If $K_p = \frac{V_s}{X}$; $V_o = K_p x$

2.15.3. Error Detector

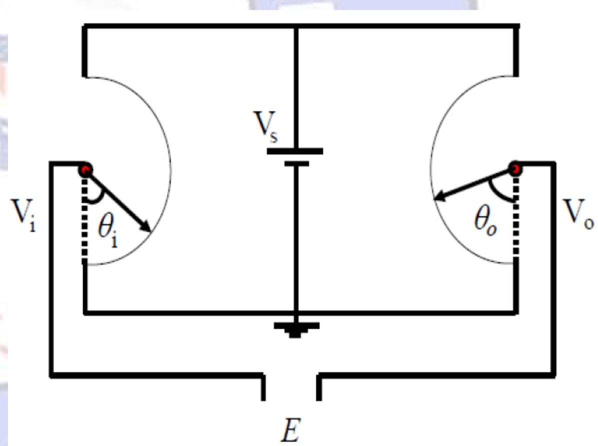
$$V_i = k_1 \theta_1$$

$$V_o = k_2 \theta_2$$

$$E = k_1 \theta_1 - k_2 \theta_2$$

If $k_1 = k_2 = k$

$$E = k \ (\theta_1 - \theta_2)$$



2.15.4. First-Order Op-Amp:

In addition to adding and subtracting signals, op-amps can be used to implement transfer functions of continuous-data systems. While many alternatives are available, we will explore only those that use the inverting op-amp configuration shown in beside Fig.

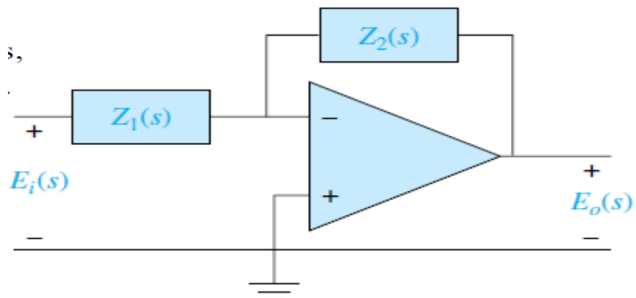


Fig.2.17. Inverting op-amp configuration

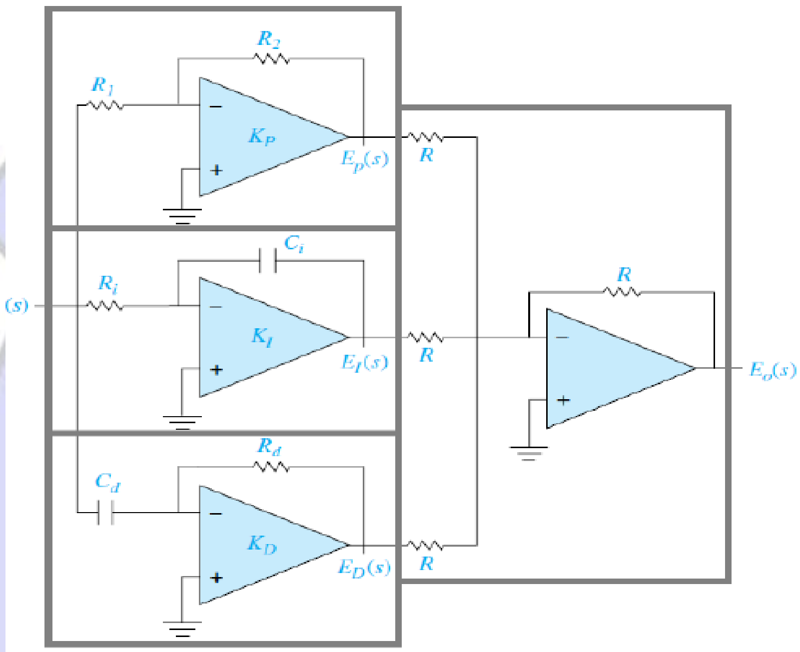
In the figure, Z1(s) and Z2(s) are impedances commonly composed of resistors and capacitors. Inductors are not commonly used because they tend to be bulkier and more expensive. Using ideal op-amp properties, the input-output relationship, or transfer function, of the circuit shown in Fig. can be written in a number of ways, such as

$$G(s)=\frac{E_o(s)}{E_i(s)}=-\frac{Z_2(s)}{Z_1(s)}=-\frac{1}{Z_1(s)Y_2(s)}=-Z_2(s)Y_1(s)=-\frac{Y_1(s)}{Y_2(s)}$$

Where $Y1(s)$ = $1/Z1(s)$ and $Y2(s)$=$1/Z(s)$ are the admittances associated with the circuit impedances.

Example (16): For the Fig. below find The transfer function that Described the relation between Eo(s) and E(s) (i.e. Find **G(s)= Eo(s)/ E(s)?**

Solution:



$$\frac{E_p(s)}{E(s)} = -\frac{R_2(s)}{R_1(s)}$$

$$\frac{E_1(s)}{E(s)} = -\frac{1}{R_i c_i s}$$

$$\frac{E_D(s)}{E(s)} = -R_D c_D s$$

The output voltage is

$$E_o(s) = -[E_p(s) + E_1(s) + E_D(s)]$$

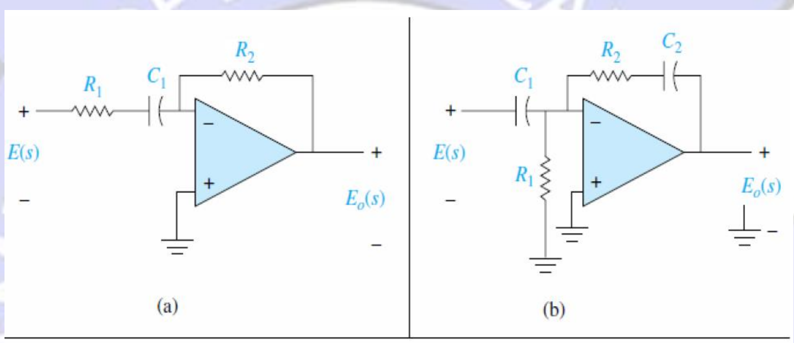
Thus the transfer function of PID operation amplifier is

$$G(s)=\frac{E_0(s)}{E(s)}=\frac{R_2}{R_1}+\frac{1}{R_i c_i s}+R_D c_D s$$

$$G(s)=\frac{R_1 R_i c_i R_d c_d s^2+R_1 R_i c_i s+R_1}{R_1 R_i c_i s}$$

This is transfer function of (proportional , integral , derivative)(PID) controller that will study in details in next time.

H.W. Find the transfer functions Eo(s)/E(s) for each the circuits shown in (a, b) of the below Fig.



2.16. Simulation diagram:

The simulation diagram is a term can be defined as the connection diagram by using analogue tools to describe the differential equation of the mathematical model for any system. The elements are used in simulation diagram can be shown by:

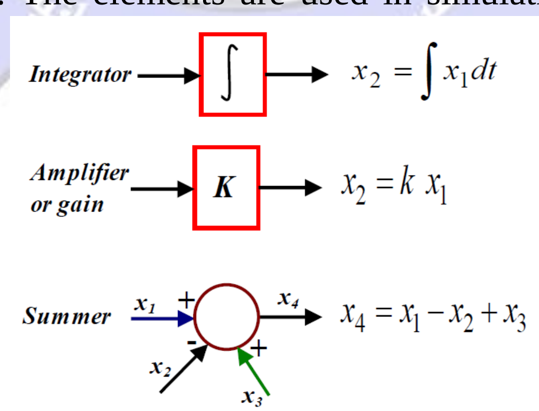


Fig.2.18. Elements used in a simulation diagram.

One of the methods used to obtain a simulation diagram includes the following steps:

1) Start with differential equation.

2) On the left side of the equation put the highest-order derivative of the dependent variable. A first-order or higher-order derivative of the input may appear in the equation. In this case the highest-order derivative of the input is also placed on the left side of the equation. All other terms are put on the right side.

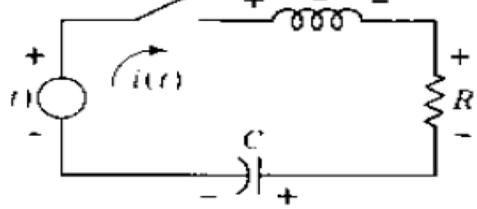
3) Start the diagram by assuming that the signal, represented by the terms on the left side of the equation, is available. Then integrate it as many times as needed to obtain all the lower-order derivatives. It may be necessary to add a summer in the simulation diagram to obtain the dependent variable explicitly.

4) Complete the diagram by feeding back the approximate outputs of the integrators to a summer to generate the original signal of step 2. Include the input function if it is required.

Example (17): Draw the simulation diagram for the series RLC circuit of Fig. below in which the output is the voltage across the capacitor.

Solution:

For the series RLC shown above ,the applied voltage equal to the sum of the voltage drops when the switch is closed.



$$V_l + V_c + V_R = e$$

$$l\frac{di}{dt} + \frac{1}{c}\int i dt + iR = e$$

Step1. When y=V_c and u=e we get

$$LC\ddot{Y} + RC\dot{Y} + Y = u$$

Step 2. Rearrange the terms to the form

$$\ddot{Y} = \frac{1}{LC}u - \frac{R}{L}\dot{Y} - \frac{1}{LC}Y$$

$$\ddot{Y} = bu - a\dot{Y} - bY$$

Where

a= R/L and b=1/LC

step3.the signal $\ddot{Y}$ is integrated twice as shown in simulink implementation (Fig.a)

step4. The complete block diagram can be illustrated as in Fig.b.

the state variables are often selected as the output of the integrators in the simulation diagram.

In this case they are :

$$y = x_1,$$

$$\dot{y} = x_2 = \dot{x}_1$$

$$\ddot{y} = \dot{x}_2$$

The state space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u = AX + Bu$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u = cX + Du$$

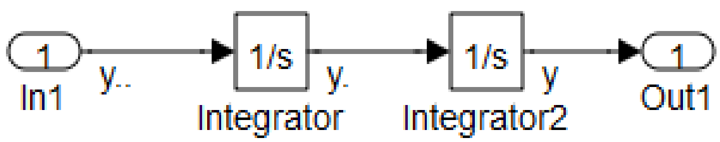


Fig.2.19. Simulink implementation

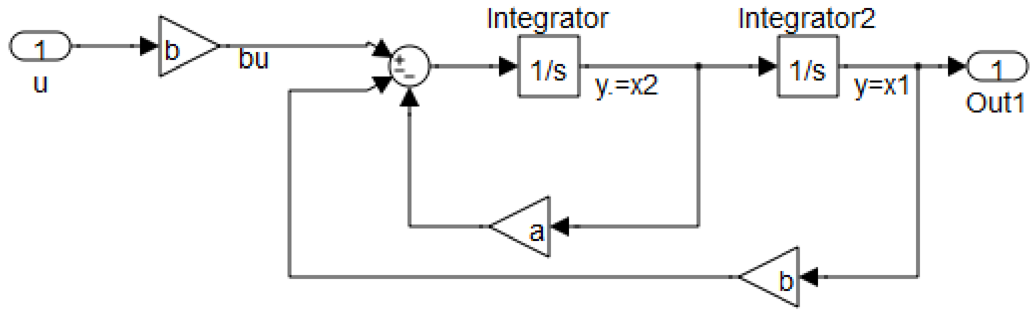


Fig.2.20. Simulink implementation of system state space

Example (18): Draw the simulation diagram that explain the differential equation below:

$$y''' + 2y'' - 5y' - 7y = 5\sin(u)$$

Solution

$$y = x_1 \Rightarrow y' = x'_1 = x_2$$
$$x_2 = y' \Rightarrow x'_2 = y'' = x_3$$
$$y'' = x_3 \Rightarrow x'_3 = y''' = 5\sin(u) - 2y'' + 5y' + 7y$$

From the above eqn's can be draw the following Simulink diagram can be obtained:

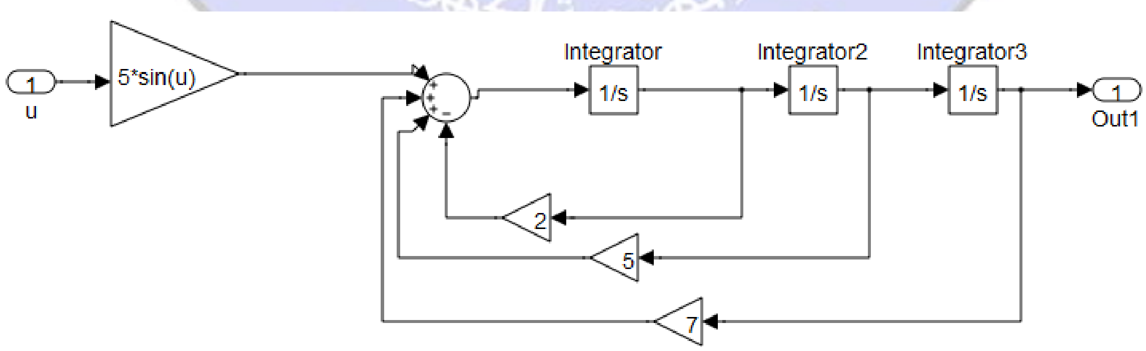


Fig.2.21. Simulink block diagram of the given example

H.W.(1):

For the positional servomechanism obtain the closed-loop T.F. for the positional servomechanism shown Fig.2.22. Assume that the in-put and output of the system are:

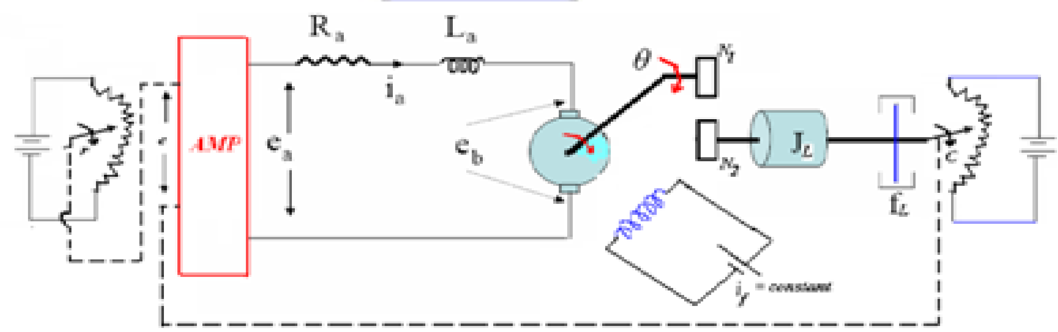


Fig.2.22. Circuit Diagram of the positional servomechanism.

input shaft position and the output shaft

r = reference input shaft, radian

c = output shaft, radian

θ = motor shaft, radian

k_1 = gain of potentiometer error detector = 24/π volt/rad

k_p = amplifier gain = 10

k_b = back emf const.= 5.5*10-2 volts-sec/rad

K = motor torque constant = 6*10-5 Ib-ft-sec2

R_a = 0.2 Ω

L_a = negligible

J_m = 1*10-3 Ib-ft-sec2

f_m = negligble

J_l = 4.4*10-3 Ib-ft-sec2

f_L = 4*10-2 Ib-ft/rad/sec

n = gear ratio N1/N2=1/10

Hint $J = J_m + n_2 J_l$, $f = f_m + n_2 f_L$

Answer $\frac{\theta(s)}{E_a(s)} = \frac{0.72}{s(0.13s+1)}$

H.W.(2): Find the circuits diagram for every transfer function below:

a.

$$V_i \longrightarrow \boxed{\frac{1}{\mathrm{RcD}+1}} \longrightarrow V_o$$

b.

$$V_i \longrightarrow \boxed{\frac{\mathrm{RcD}}{\mathrm{LcD}^2+\mathrm{RcD}+1}} \longrightarrow V_r$$

Lecture No. Three

Block Diagrams Processing

This lecture discusses the following topics :

3.1. Introduction.

3.2. Symbols used in block diagrams (B.D).

3.3. Block Diagram reduction rules.

3.4. Variables in the Block diagram.

3.5. The Block Diagram Components.

3.6. Solved Problems

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Yousif AL Mashhadany
U. O. Anbar

3.1. Introduction

The representation of physical components by blocks is shown in Lecture two for each block the transfer function provides the dynamical mathematical relationship between the input and output quantities. Also, lecture one describe the concept of feedback, which is used to achieve a better response of a control system to a command input. Now, the control systems represents by block diagrams. The blocks represent the functions performed rather than the components of the system.

3.2. Symbols used in block diagrams(B.D):

a. *Block:* the transfer function(TF) of the system element is placed in the block symbolized by,

Fig.3.1. Block diagram of a TF

b. *Summing points:* The operation of addition or subtraction is performed by this system element and symbolized by.

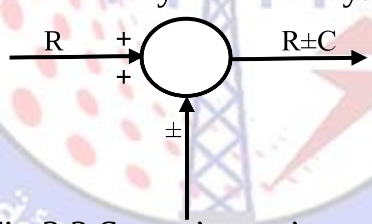


Fig.3.2.Summing point notation

c. *Take off point:* This operation is used to provide a dual input (i/p) or output(o/p) to a system element and it is represented by,

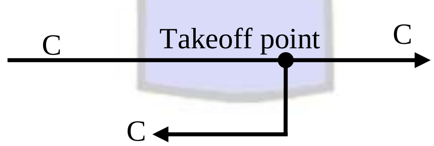
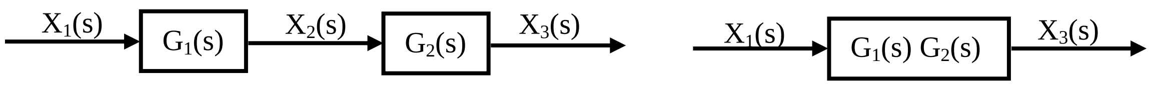


Fig.3.3.Take off notation

d. *Direction arrows:* this symbol defines a unidirectional flow of the signal

Fig.3.4.Arrow notation

e. *Cascaded rule:* If two (or more) no load blocks in the same direction the output as follow:



$$G_1 = \frac{X_2(s)}{X_1(s)} \quad and \quad G_{12} = \frac{X_3(s)}{X_2(s)}; \rightarrow G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)}$$
$$= G_1(s)G_2(s)$$

Fig.3.5. Cascaded rule

3.3. Variables in the Block diagram:

For the block diagram shown in Fig. below can be define the following term, that is represents the standard control system term in the representation of physical system in block diagram form:

Command (v): is the input that is established by some means external to, and independent of, the feedback control system.

Reference input (r): is derived from the command and is the actual signal input to the system.

Controlled variable (c): is the quantity that is directly measured and controlled. It is the output of the controlled system.

Primary feedback (b): is a signal that is a function of the controlled variable and that is compared with the reference input to obtain the actuating signal.

Actuating signal (e): is obtained from a comparison measuring device and is the reference input minus the primary feedback. This signal, usually at a low energy level, is the input to the control elements that produce the manipulated variable.

Manipulated variable(m): is the quantity obtained from the control elements that is applied to the controlled system. The manipulated variable is generally at a higher energy level than the actuating signal and may also be modified in form.

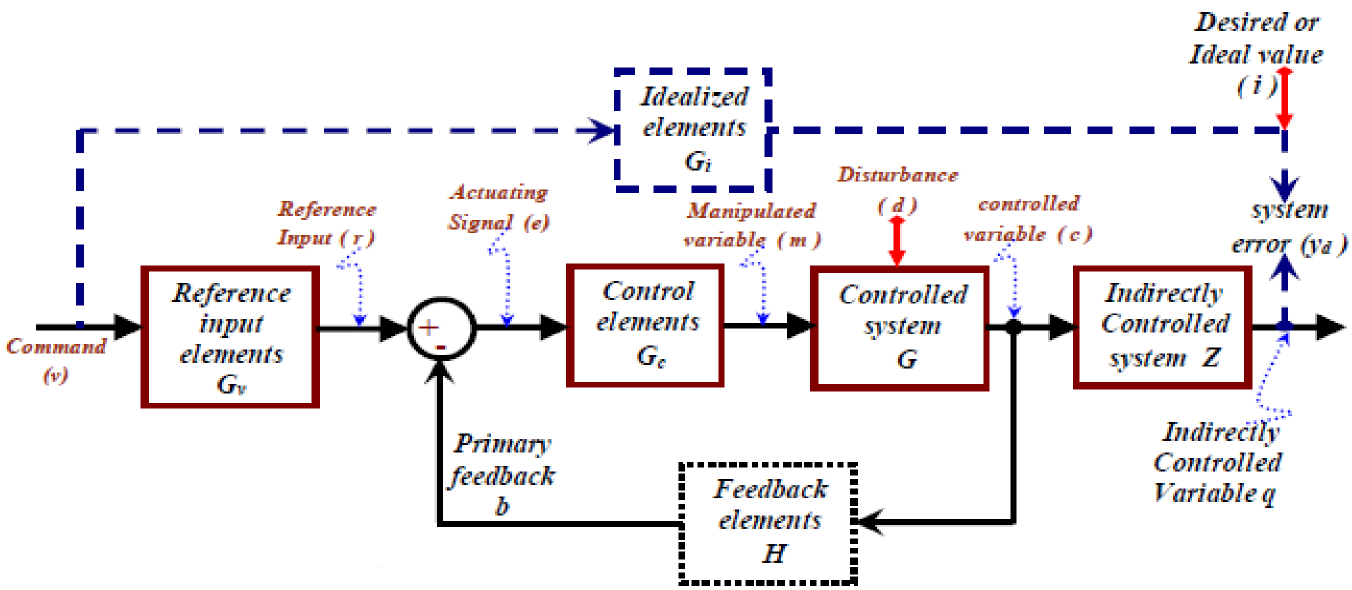


Fig.3.6. closed loop control system with all signals

Indirectly controlled variable (q): is the output quantity that is related through the indirectly controlled system to the controlled variable. It is outside the closed loop and is not directly measured for control.

Ultimately controlled variable: is a general term that refers to the indirectly controlled variable. In the absence of the indirectly controlled variable, it refers to the controlled variable.

Ideal value i: is the value of the ultimately controlled variable that would result from an idealized system operating with the same command as the actual system.

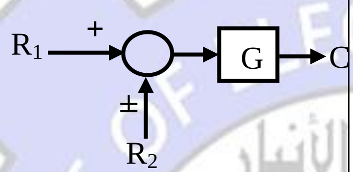
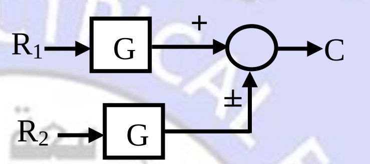
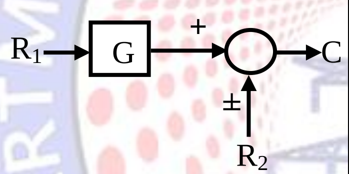
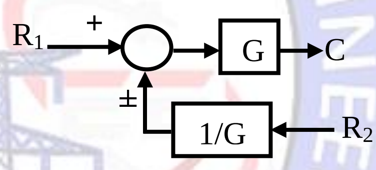
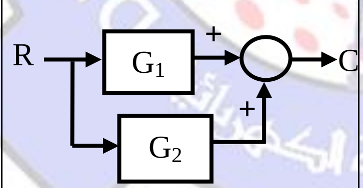
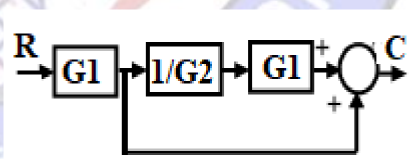
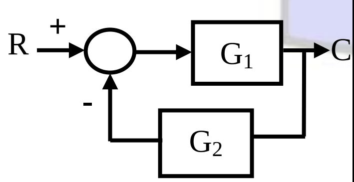
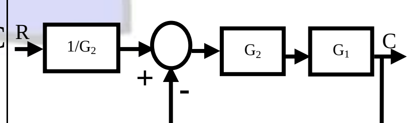
System error (ye): is the ideal value minus the value of the ultimately controlled variable.

Disturbance (d): is the unwanted signal that tends to affect the controlled variable. The disturbance may be introduced into the system at many places.

3.4. Block Diagram reduction rules:

The rules are using in the block diagram reduction to reduce the complex block diagram as single block diagram between input and output.

Table 3.1. Rules of Block Diagram Reduction

<table>
<tr><th>Transformation</th><th>B.D</th><th>Equivalent B.D</th><th>Equation(T.F)</th></tr>
<tr><td>Moving summing point beyond a block</td><td></td><td></td><td>$\frac{C}{R_1 \pm R_2} = G$</td></tr>
<tr><td>Moving summing point a head of a block</td><td></td><td></td><td>$C = R_1 G \pm R_2$</td></tr>
<tr><td>Moving block to the forward path</td><td></td><td></td><td>$C = R_1(G_1 \pm G_2)$</td></tr>
<tr><td>Moving F/B to the forward path</td><td></td><td></td><td>$\frac{C}{R} = \frac{G_1}{1 + G_1 G_2}$</td></tr>
</table>

3.5. The Block Diagram Components:

Reference input elements Gv: produce a signal **(r)** proportional to the command.

Control elements Gc: produce the manipulated variable m from the actuating signal.

Controlled system G: is the device that is to be controlled. This is frequently a high-power element.

Feedback element (H): produces the primary feedback b from the controlled variable. This is generally a proportionality device but may also modify the characteristics of the controlled variable.

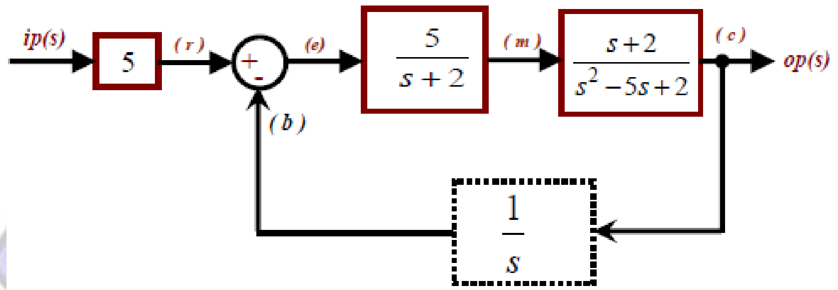
Indirectly controlled system Z: relates the indirectly controlled variable **(q)** to the controlled quantity **(c).** This component is outside the feedback loop.

Idealized system Gi: is one whose performance is agreed upon to define the relationship between the ideal value and the command. This is often called the model or desired system.

Disturbance element N: denotes the functional relationship between the variable representing the disturbance and its effect on the control system.

3.6. Solved Problems

Prob.1. For the block diagram shown below find the overall transfer function ip/op?



Solution:

By Appling the rules of B.D. reduction can be get:

by cascaded no load blocks can be reduce the forward direction:

$$G_1(s) = \frac{5}{s+2}$$

$$G_2(s) = \frac{s+2}{s^2+5s+2}$$

The feed forward TF is:

$$G_1(s) * G_2(s) = \frac{5s+10}{s^3-3s^2-8s+4}$$

By using negative feedback rule we get

$$G(s) = \frac{G_1(s)*G_2(s)}{1+G_1(s)*G_2(s)H(s)}, H(s) = 1/s$$

or

$$G(s) = \frac{G_f(s)}{1+G_f(s)H(s)}, G_f(s) = G_1(s)G_2(s)$$

The final TF will be:

$$G(s) = \frac{25s^2+50s}{s^4-3s^3-8s^2+9s+10}$$

Prob.2. **(Position Control System):**

Fig.3.7. below shows a simplified block diagram of an angular position control system. The reference selector and the sensor, which produce the reference input $R=\theta_R$ and the controlled output position $c=\theta_o$, respectively, consist of rotational potentiometers. The combination of these units represents a rotational comparison unit that generates the actuating signal **E** for the position control system, as shown in Fig.3.8 ,where k_θ, in volts per radian, is the potentiometer sensitivity constant. The symbolic comparator for this systemis shown in Fig.3.9. The transfer function of the motor-generator control is obtained by writing the equations for the schematic diagram shown in Fig.3.8. This figure shows a dc motor that has a constant field excitation and drives an inertia and friction load. The armature voltage for the motor is furnished by the generator, which is driven at constant speed by a prime mover. The generator voltage e_g is determined by the voltage ef applied to the generator field. The generator is acting as a power amplifier for the signal voltage e_f .

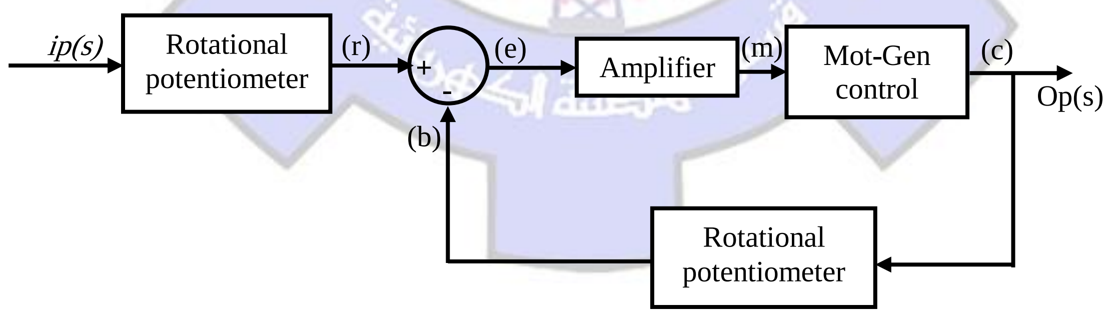


Fig.3.7. Position control system

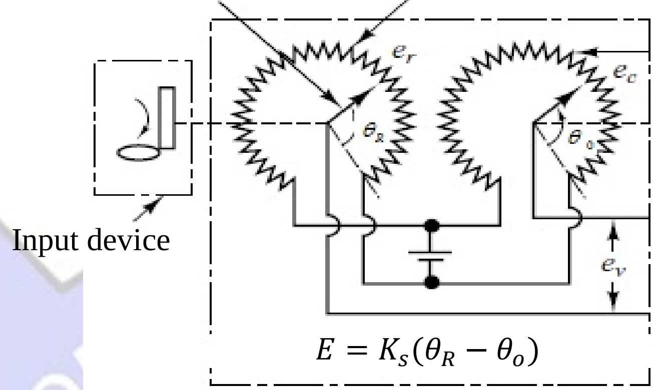


Fig.3.8. Rotational position comparison

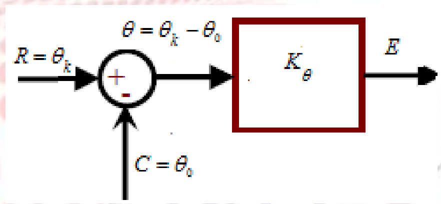


Fig.3.9. its block diagram representation

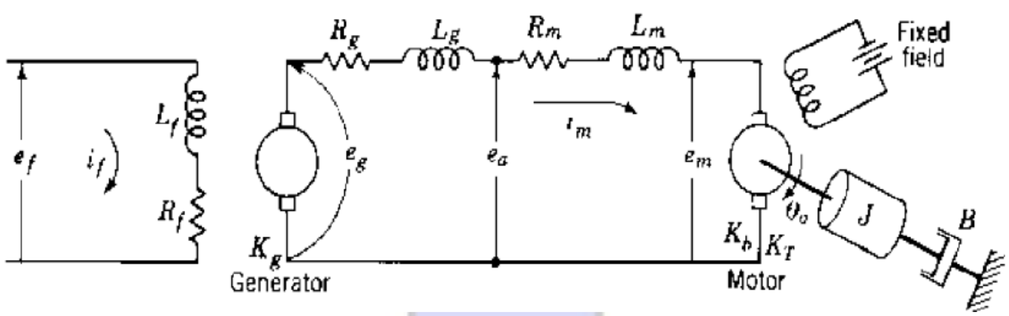


Fig.3.10. Motor-generator control

The equations that described the system in Fig.3.10. can be written as:

$$e_f = L_f D i_f + R_f i_f \tag{3.1}$$

$$e_g = K_g i_f \tag{3.2}$$

$$e_g - e_m = (L_g + L_m)Di_m + (R_g + R_m)i_m \quad (3.3)$$

$$e_m = K_b D\theta_0 \quad (3.4)$$

$$T = K_T i_m = JD^2\theta_0 + BD\theta_0 \quad (3.5)$$

According to the eq's(3.1-3.5) can be draw the block diagram of the system in Fig.3.10. same as the Fig.3.11;

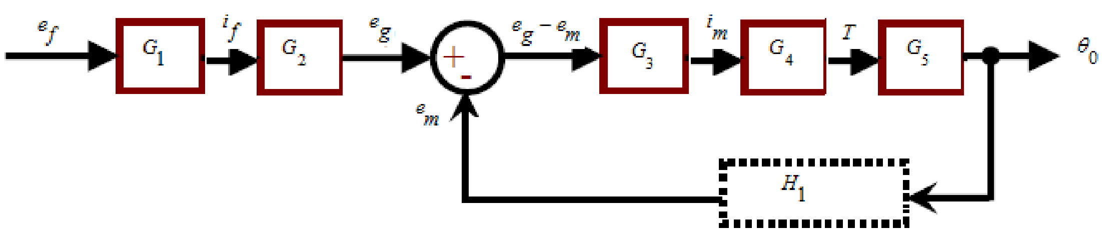


Fig.3.11. Block Diagram of the Motor-generator control system

The transfer functions of each block, as determined in terms of the pertinent Laplace transforms, are as follows:

$$G_1(s) = \frac{I_f(s)}{E_f(s)} = \frac{1/R_f}{1+(L_f/R_f)S} = \frac{1/R_f}{1+T_f S} \quad (3.6)$$

$$G_2(s) = \frac{E_g(s)}{I_f(s)} = K_g \quad (3.7)$$

$$G_3(s) = \frac{I_m(s)}{E_g(s)-E_m(s)} = \frac{1/(R_g+R_m)}{1+[(L_g+L_m)/(R_g+R_m)]S} = \frac{1/R_{gm}}{1+(L_{gm}/R_{gm})S} \quad (3.8)$$

$$G_4(s) = \frac{T(s)}{I_m(s)} = K_T \quad (3.9)$$

$$G_5(s) = \frac{\Theta_0(s)}{T(s)} = \frac{1/B}{s[(1+(J/B)s]} = \frac{1/B}{s(1+T_n s)} \quad (3.10)$$

$$H_1(s) = \frac{E_m(s)}{\Theta_0(s)} = K_b s$$

(3.11)

The block diagram can be simplified, as shown in Fig.3.12, by combining the blocks in cascade. The block diagram is further simplified, to get the final transfer finction of vthe system as follow:

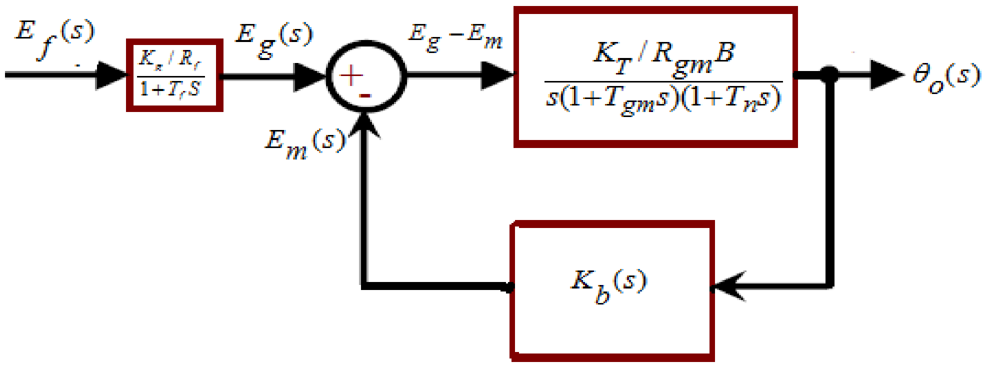
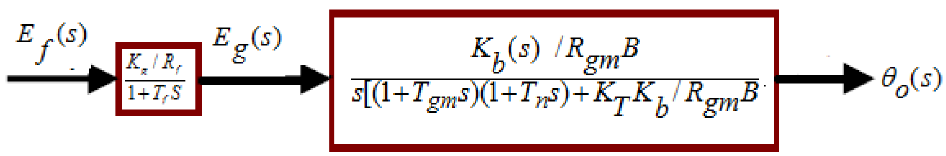


Fig.3.12. simplified block diagram

The next step is to apply the negative rule to get:



$$G_x(s)=\frac{\Theta_o(s)}{E_f(s)}=\frac{K_T K_g / R_f R_{gm} B}{s(1+T_f s)[(1+K_T K_b / BR_{gm})+(T_{gm}+T_n)s+T_{gm}T_n s^2]}$$

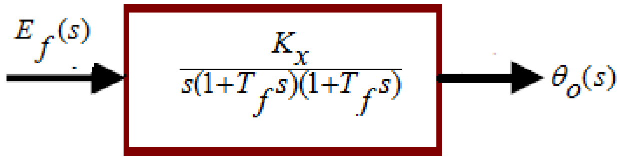
$$G_x(s)=\frac{\Theta_o(s)}{E_f(s)}\approx\frac{K_x}{s(1+T_f s)(1+T_f s)}$$

Where

$$K_x=\frac{K_T K_g}{R_f(BR_{gm}+K_T K_b)};T_f=\frac{L_f}{R_f}\ \text{ and }\ T_m=\frac{JR_{gm}}{BR_{gm}+K_T K_b}$$

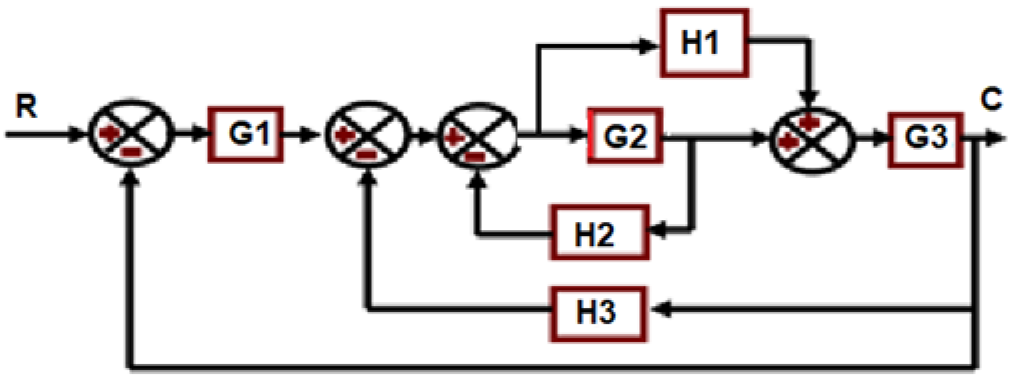
$$K_x=\frac{K_g}{R_f K_b}$$

The final block diagram of the this example will be as follows:



H.W. Find the Simulation Diagram for the transfer function $G_x(s)$?

Prob.3. Simplified the block diagram for the system shown below .

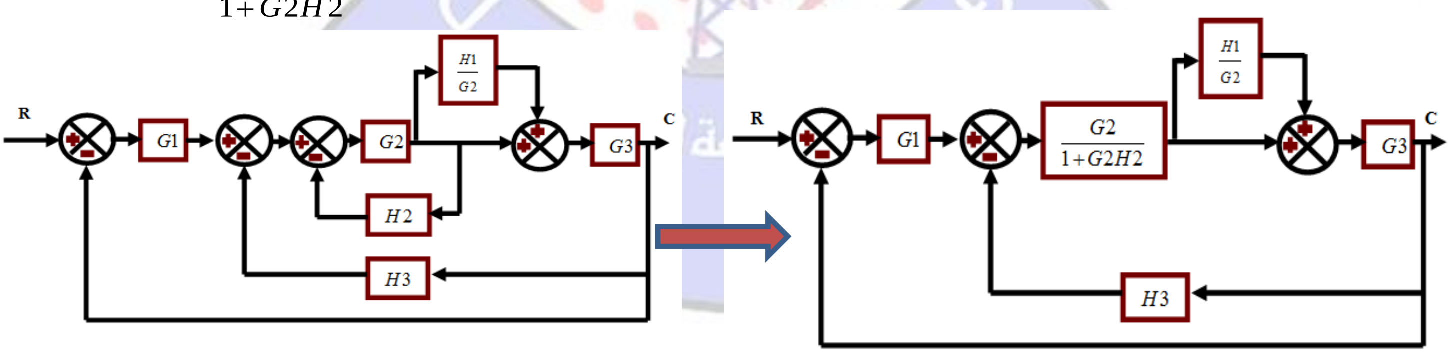


Solution:

We can notice the following

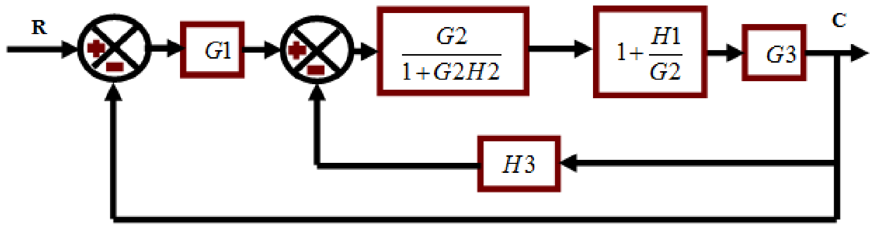
1.G2 and H2 is standard negative feedback system and the transfer function will be as:

$$Gt1 = \frac{G2}{1+G2H2}$$



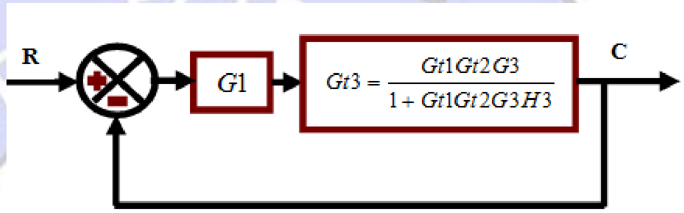
2. The transfer function of the parallel combiation of H1/G2 and unity forward path will be as :

$$Gt2 = 1 + \frac{H1}{G2}$$

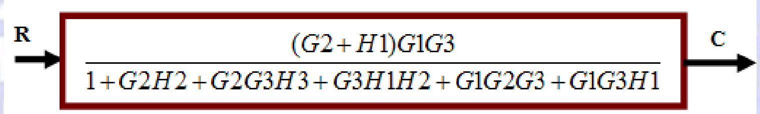


3.Gt1 ,Gt2 ,G3 and H3 combination is a standard negative feedback control system and its transfer function will be:

$$Gt3 = \frac{Gt1Gt2G3}{1 + Gt1Gt2G3H3}$$



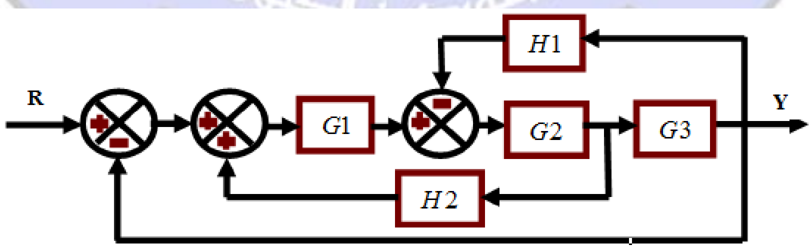
4.Gt3,G1 and unity feedback control system is a standard negative feedback its transfer function will as in the follwing block diagram



The final transfer function of the system is:

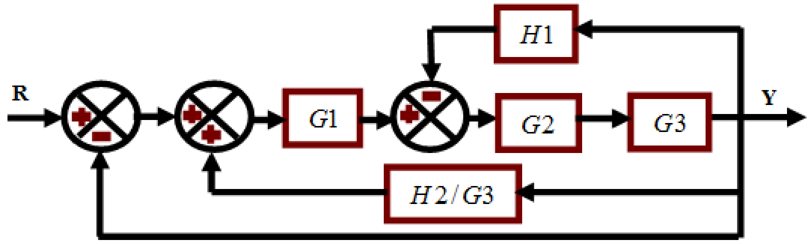
$$\frac{C}{R} = \frac{G1G3(G2 + H1)}{1 + G2H2 + G2G3H3 + G3H1H2 + G1G2G3 + G1G3H1}$$

Example 5. Simplified the following block diagram shown in Fig.3.13.a

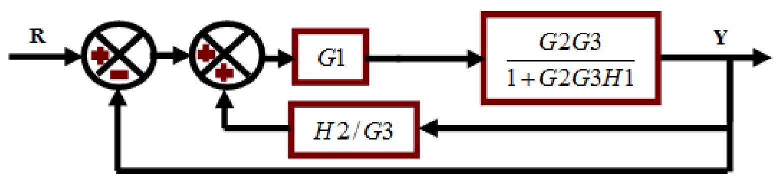


(a)

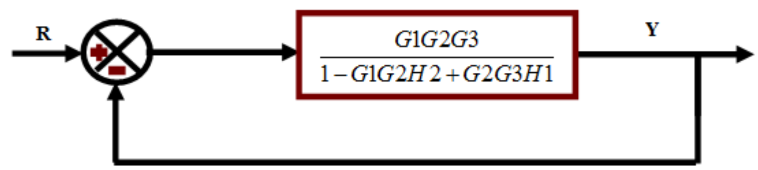
By using the standard rules for reduction can be get the overall all T.F by the steps (b-e), That is shown below:



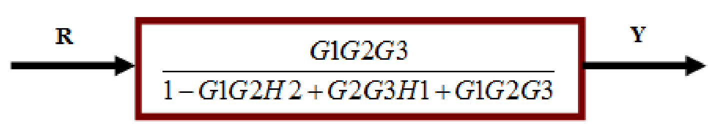
(b)



(c)



(d)



(e)

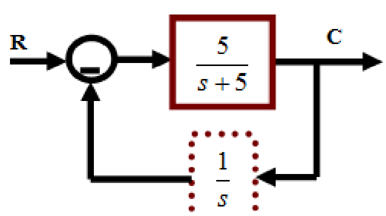
Fig.3.13(b-e): Steps for the solution of reduction the B.D. in Fig.3.13.a.

The final transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G1G2G3}{1 - G1G2H2 + G2G3H1 + G1G2G3}$$

Prob.4. For Matlab :Find the overall transfer function(C(s)/R(s)) of the following control system using Matlab.

Solution:

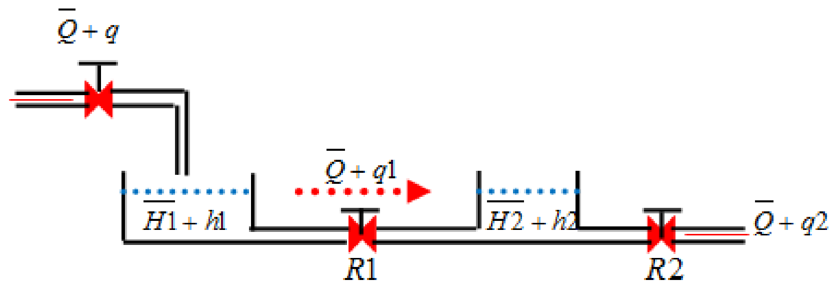


MATLAB provides the ‘‘feedback’’ function

to calculate the overall (closed -loop) transfer function as shown by the following m-file.

```
% The feedback command calculates the closed-loop transfer
% function for a given forward transfer function G(s)
% and feedback transfer function H(s)
% Type "help feedback" for more information
% Example 1:
% Define a forward transfer function
% using the tf command
% System = tf (numerator, denominator) Transfer function: 5/(s +5)
G = tf([5],[1 5])
% Define a feedback transfer function H(s)= 1/s
% using the tf command
H =tf([1],[1 0], % System = tf (numerator, denominator) Transfer function: 1/s
% For a non-unity, negative feedback system the
% closed loop transfer function is
cltf = feedback (G,H,-1)
%Transfer function:5s / (s^2 +5s+ 5)
% For positive feedback use "feedback (G,H,1)"
5s / (s^2 +5s -5)
% The forward transfer function is calculated by multiplying
% G(s) and H(s) ; G *H : Transfer function: 5 / ( s^2+ 5s); The program:
G = tf([5],[1 5])
H =tf([1],[1 0]
cltf = feedback (G,H,-1)
```

Prob.5. Construct the block diagram for the system shown below(q,q1,q2,h1,h2) are changes from steady state.



$$q-q1=c1\frac{dh1}{dt}$$

$$q1=\frac{h1-h2}{R1}$$

$$q1-q2=c2\frac{dh2}{dt}$$

$$q2=\frac{h2}{R2}$$

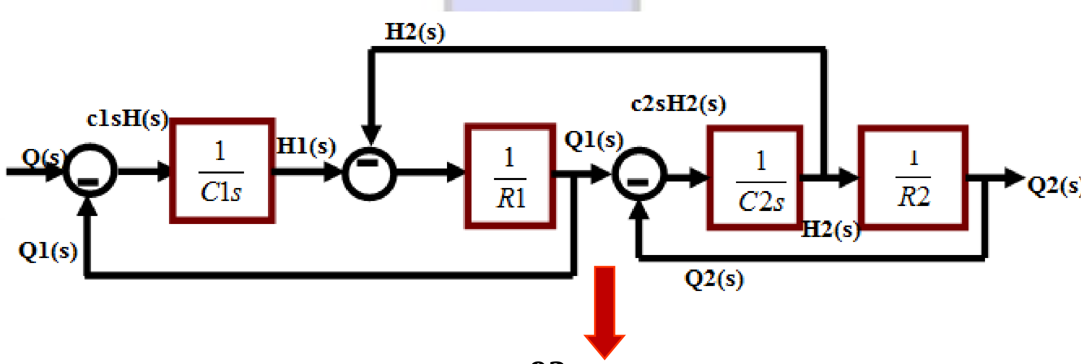
In Laplace transform

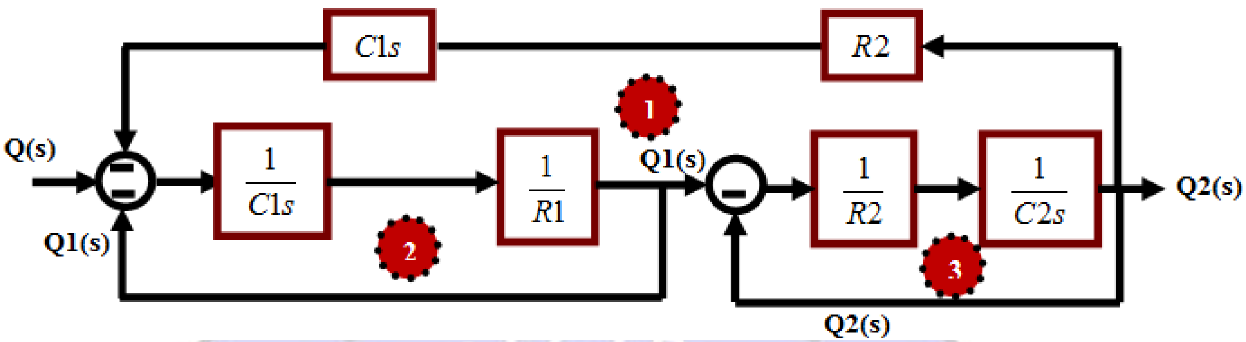
$$Q(s)-Q1(s)=c1sH1(s)$$

$$Q1(s)=\frac{H1(s)-H2(s)}{R1}$$

$$Q1(s)-Q2(s)=c2sH2(s)$$

$$Q2(s)=\frac{H2(s)}{R2}$$



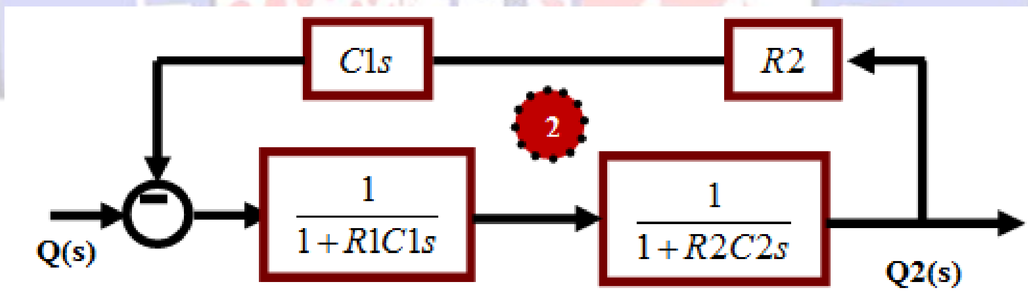


Loop 3:

$$\frac{Q2(s)}{Q1(s)} = \frac{(1/R2)(1/C2s)}{1+(1/R2)(1/C2s)} = \frac{1}{1+C2R2s}$$

Loop 2

$$\frac{Q(s)}{Q1(s)} = \frac{(1/R1)(1/C1s)}{1+(1/R1)(1/C1s)} = \frac{1}{1+C1R1s}$$



Loop 1.

$$\frac{Q2(s)}{Q1(s)} = \frac{(1/R1C1s)(1/R2C2s)}{1+(1/R1C1s)(1/R2C2s)C1R2s}$$

$$\frac{Q2(s)}{Q1(s)} = \frac{1}{C1C2R1R2s^2+(C1R1+C2R2+R2C1)s+1}$$

Lecture No. Four

Signal Flow Graphs

This lecture will discusses the following topics

4.1. Introduction.

4.2. Flow-Graph Definitions.

4.3. Rules of Signal flow graph:

4.4. Signal flow graph for control system.

4.5. State Transition Signal Flow Graph

4.6. Simplification for the system of dual inputs

4.7. Matlab program for signal flow graph

Prof. Dr.
Yousif AL Mashhadany
U. O. Anbar

4.1. Introduction:

The block diagram is a useful tool for simplifying the representation of a system. The block diagram has only one feedback loop and may be categorized as simple block diagrams. When it has two, three, etc, feedback loops; thus it is not a simple system. When intercoupling exists between feedback loops, and when a system has more than one input and one output, the control system and block diagram are more complex. Having the block diagram simplifies the analysis of complex system. Such an analysis can be even further simplified by using a signal flow graph (SFG), which looks like a simplified block diagram.

An SFG is a diagram that represents a set of simultaneous equations .It consists of a graph in which nodes are connected by directed branches. The nodes represent each of the system variables. A branch connected between two nodes acts as a one-way signal multiplier: the direction of signal flow is indicated by an arrow placed on the branch, and the multiplication of general matrix block diagram representing the state and output equations. factor (transmittance or transfer function) is indicated by a letter placed near the arrow. Thus, in Fig.4.1, the branch transmits the signal x_1 from left to right and multiplies it by the quantity a in the process. The quantity a is the transmittance, or transfer function. It may also be indicated by $a=t_{12}$, where the subscripts show that the signal flow is fromnode1to node 2.

Fig.4.1. Signal flow graph for x2=ax1

4.2. Flow Graph Definitions.

The analysis of flow graph is achieved by mathematic process of anode that it performs two functions:

1. Addition of the signals on all incoming branches
2. Transmission of the total node signal (the sum of all incoming signals) to all outgoing branches these functions are illustrated in the graph of Fig.4.2, which represents the equations

$$w = au + bv, \quad x = cw, y = dw \tag{4.1}$$

Three types of nodes are of particular interest:

Source nodes (independent nodes).These represent independent variables and have only outgoing branches. In Fig.4.2, nodes u and v are source nodes.

Sink nodes (dependent nodes). These represent dependent variables and have only incoming branches. In Fig.4.2, nodes x and y are sink nodes.

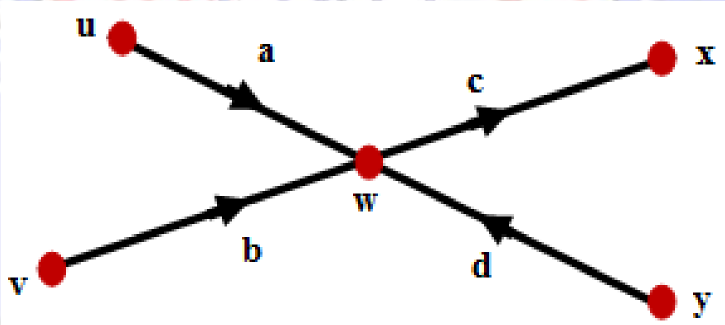


Fig.4.2. Signal flow graph for equation(4.1)

Mixed nodes (general nodes). These have both incoming and outgoing branches. In Fig.4.2, node w is a mixed node. A mixed node may be treated as a sink node by adding an outgoing branch of unity transmittance, as shown in Fig.4.3, for the equation

$$w = au + bv, \ \& \ x = cw = cau + cbv \tag{4.2}$$

A path is any connected sequence of branches whose arrows are in the same direction. A forward path between two nodes is one that follows the arrows of successive branches and in which a node appears only once. In Fig.4.2, the path uwx is a forward path between the nodes u and x.

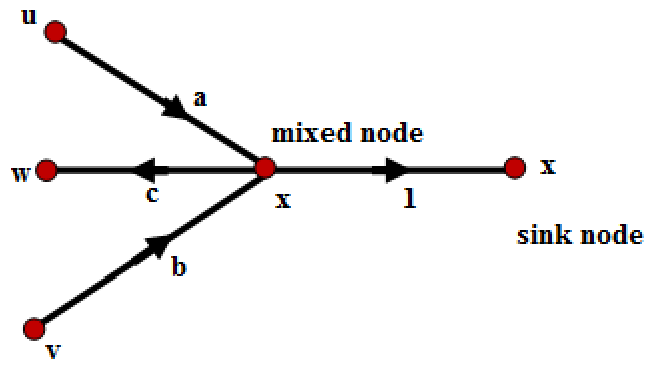


Fig.4.3. Mixed and sink nodes for a variable

Input Node (Source): An input node is a node that has only outgoing branches.

Output Node (Sink): An output node is a node that has only incoming branches. However, this condition is not always readily met by an output node.

Path: A path is any collection of a continuous succession of branches traversed in the same direction. The definition of a path is entirely general, since it does not prevent any node from being traversed more than once.

Forward Path: A forward path is a path that starts at an input node and ends at an output node and along which no node is traversed more than once.

Loop: A loop is a path that originates and terminates on the same node and along which no other node is encountered more than once.

Forward-Path Gain: The forward-path gain is the path gain of a forward path.

Loop Gain: The loop gain is the path gain of a loop.

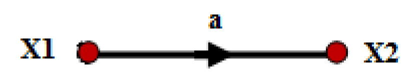
Nontouching Loops: Two parts of an SFG are nontouching if they do not share a common node.

4.3. Rules of Signal flow graph

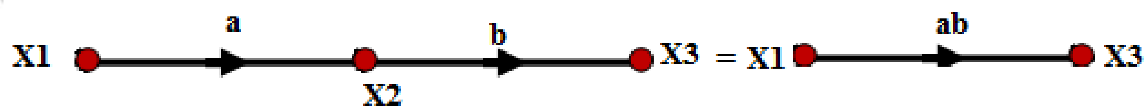
When constructing an SFG, junction points, or nodes, are used to represent variables. The nodes are connected by line segments called branches, according to the cause-and-effect equations. The branches have associated branch gains and directions. A signal can transmit through a branch only in the direction of the arrow.

1. The value of a node with one incoming branch, as shown below is

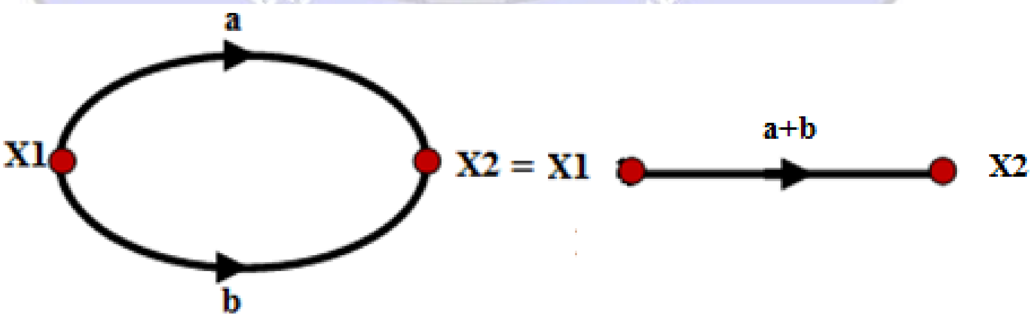
$$X_2 = aX_1 \tag{4.3}$$



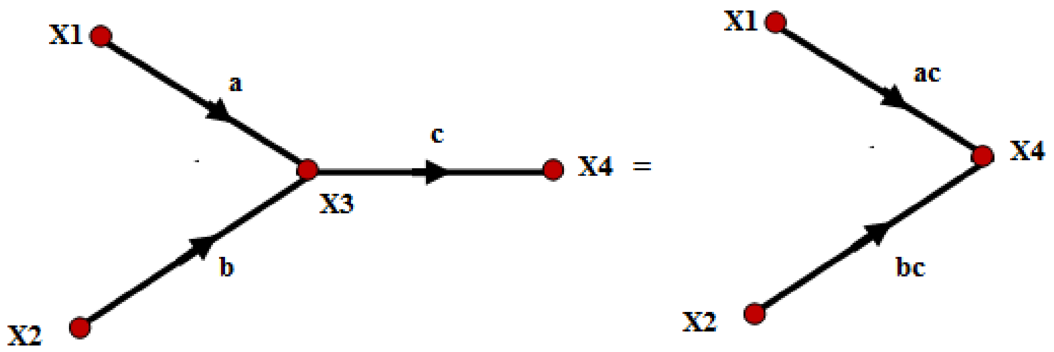
2. The total transmittance of cascaded branches is equal to the product of all branch transmittances. Cascaded branches can be combined into a single branch by multiplying the transmittances ,as shown below.



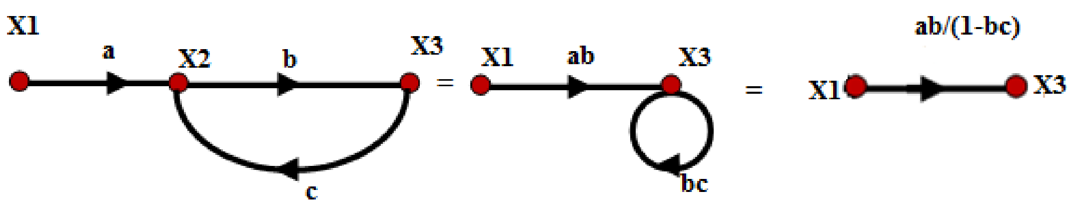
3. parallel branches may be combined by the transmittances, as shown below.



4. A mixed mode may be eliminated as shown below



5. A loop may be eliminated as shown below



The derivation of last relationship can be explained as in the following equations.

$$\left.\begin{aligned} X3 &= bX2 \\ X2 &= aX1 + cX3 \\ X3 = abX1 + bcX3 &\Rightarrow X3 - bcX3 = abX1 \\ X3(1-bc) = abX1 &\Rightarrow X3 = \frac{ab}{1-bc}X1 \end{aligned}\right\} \qquad (4.4)$$

6. Signal flow graph (SFG) applies only to linear systems.

7. The equations for which an SFG is drawn must be algebraic equations in the form of cause and effect.

8. Nodes are used to represent variables. Normally, the nodes are arranged from left to right, from the input to the output, following a succession of cause-and-effect relations through the system.

4.4. Signal flow graph for control system

Some signal flow graphs of simple control system are shown in Fig.4.4. For such simple graphs, the closed loop transfer function C(s)/R(s) can be obtained easily by inspection. For more complicated signal flow graphs, Mason's gain formula is quite useful.

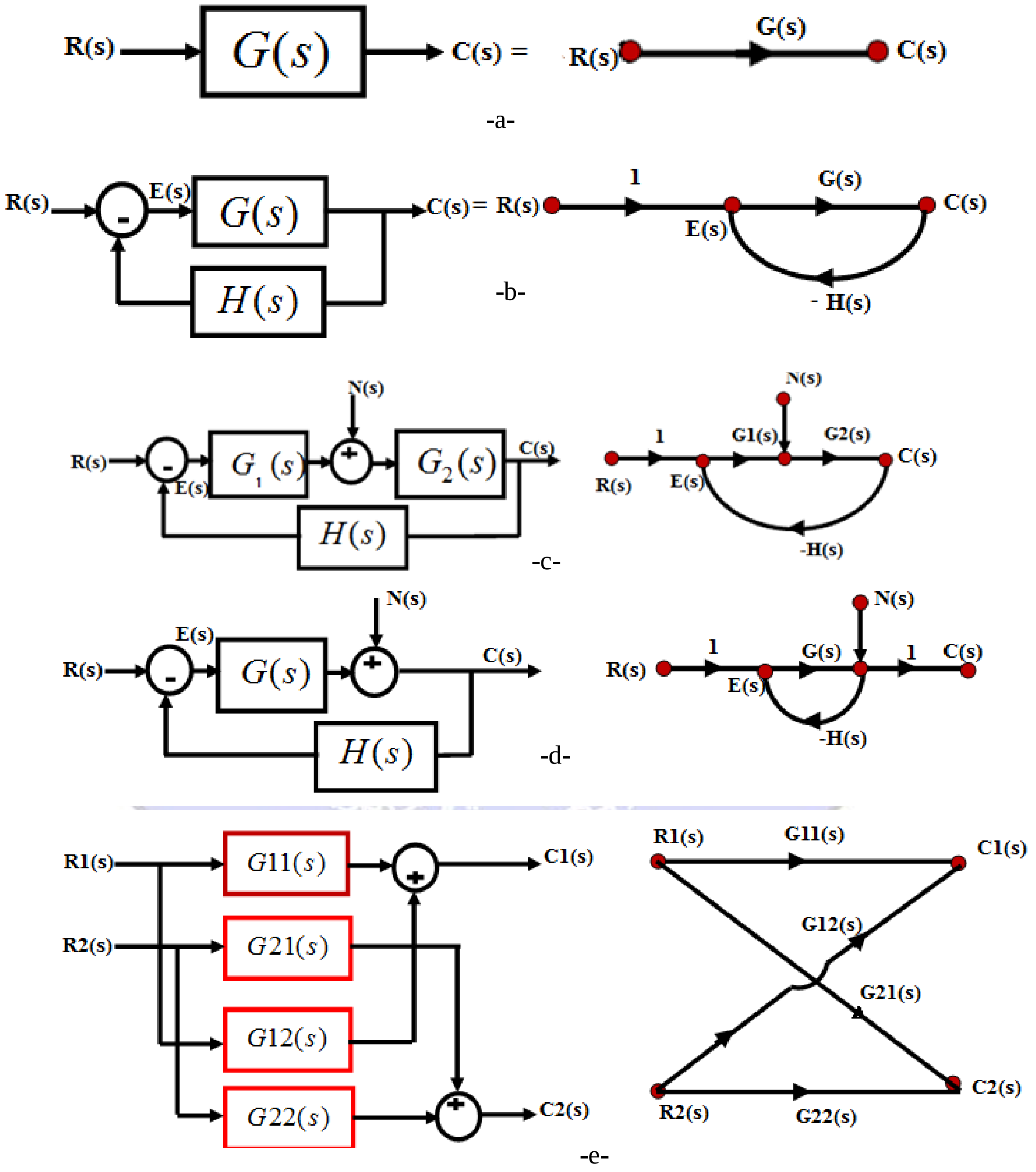


Fig.4.4.(a-e). Signal flow graph forms for simple control system.

In many practical cases e wish to determine the relationship between an input variable and an output variable of the signal flow graph. The transmittance between an input node and an output node is the overall gain, or overall transmittance, between these two nodes. Mason's gain formula, which is applicable to the overall gain, is given by

$$P = \frac{1}{\Delta}\sum_{p} P_k \Delta_k \tag{4.5}$$

Where

P_k=path gain or transmittance of the Kth forward path

Δ_k= cofactor of the *Kth* forward path determinant for the graph with the loops touching the *Kth* forward path removed.

Δ=determinant of the graph.

Δ=1-(sum of all different loop gains)+(sum of gain products of all possible combinations of two nontouching loops)- (sum of gain products of all possible combinations of three nontouching loops)+…….

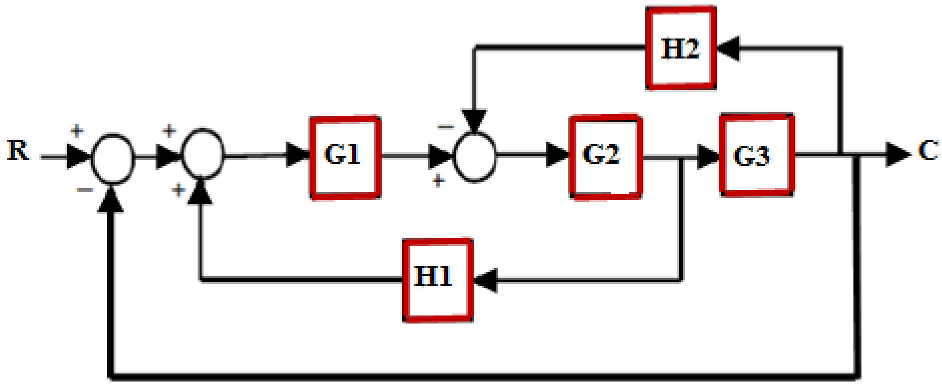
$$\Delta = 1 - \sum_{a} L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + ...$$

$\sum_{a} L_a$ =sum of all different loop gains

$\sum_{b,c} L_b L_c$ =sum of gain products of all possible combinations of two nonteaching loops

$\sum_{d,e,f} L_d L_e L_f$ =sum of gain products of all possible combinations of three nonteaching loops.

Example(1): Find the transfer function C(s)/R(s) for the system block diagram shown below by using Mason' gain formula?



Solution:

In the system there is only one forward path between the input R(s) and the output C(s). The forward path gain is,

$p_1 = G_1G_2G_3$

From the signal flow graph, we see that there are three individual loops.

The gains of these loops are;

$L_1 = G_1G_2H_1$

$L_2 = -G_3G_2H_2$

$L_3 = -G_1G_2G_3$`

Note that since all three loops have a common branch, there are no non-touching loops; hence, the determinant is Δ given by

Δ=1-($L_1+L_2+L_3$)=1-$G_1G_2H_1+G_3G_2H_2+G_1G_2G_3$

The factor Δ_1 of the determinant along the forward path connecting the input node and output node is obtained by removing the loops that touch this path.

Since path P_1 touch all loops, we have

Δ_1=1

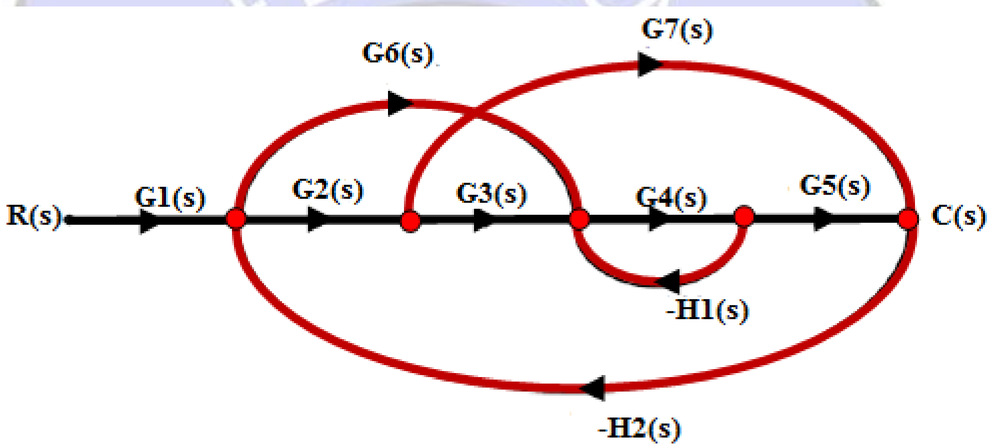
Therefore the overall transfer function of the closed loop system is given by

$$\frac{C(s)}{R(s)} = \frac{P_1\Delta_1}{\Delta}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

HW. Try to obtain the transfer function by block diagram reduction and compare the results.

Example (2): Consider the SFG of a system shown in the following figure. Obtain the closed-loop transfer function C(s)/R(s) by use of Mason's gain formula?



Solution:

In this system there are three forward paths between the input and the output.

$$p_1 = G_1G_2G_3G_4G_5$$

$$p_2 = G_1G_6G_4G_5$$

$$p_3 = G_1G_6G_7$$

There are four individual loops in this system

$$L_1 = -G_4H_1$$

$$L_2 = -G_2G_7H_2$$

$$L_3 = -G_6G_4G_5H_2$$

$$L_4 = -G_2G_3G_4G_5H_2$$

Loop L_1 does not touch Loop L_2 and there are no nontouching loops in this system just L_1 and L_2 so that the determinant of the system Δ will be

$\Delta=1-(L_1+L_2+L_3+L_4)+L_1L_2$

The factor Δ_1 is obtain from Δ by removing the loops that touch p_1 ,therefore by removing L_1,L_2 , L_3, L_4 and $L_1 L_2$,the factor $\Delta_1=1$.

Similarly by eliminating all loops in Δ that touch p_2.

$\Delta_2=1$

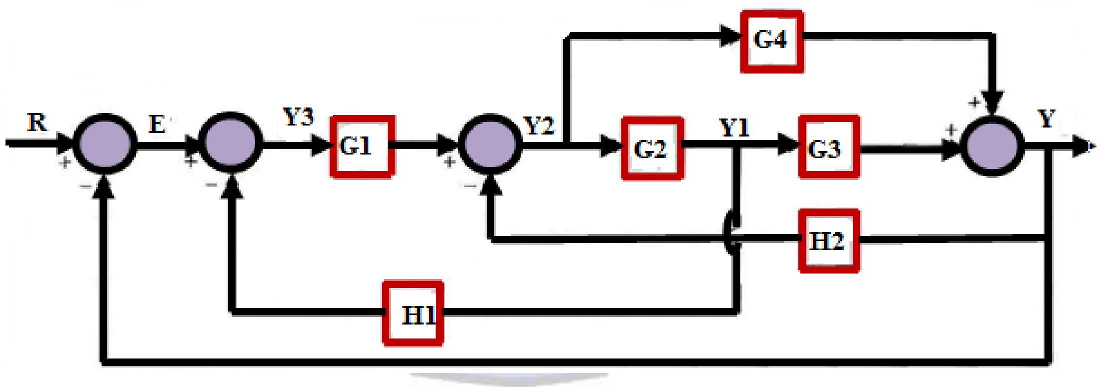
Δ_3 can be obtained by removing ,L_2 , L_3, L_4 and $L_1 L_2$ from the Δ that touch p_3

$\Delta_3=1-L_1$; The transfer function of the closed loop system C(s)/R(s)

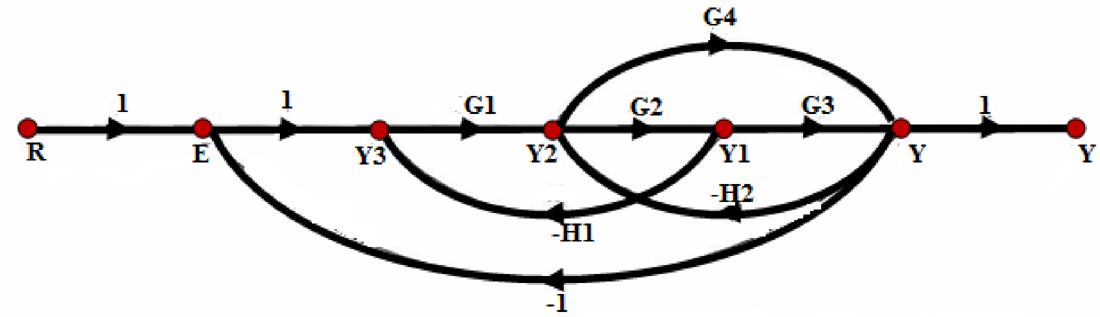
$$\frac{C(s)}{R(s)}=\frac{1}{\Delta}(p_1\Delta_1+p_2\Delta_2+p_3\Delta_3)$$

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4G_5+G_1G_6G_4G_5+G_1G_2G_7(1+G_4H_1)}{1+G_4H_1+G_2G_7H_2+G_6G_4G_5H_2+G_4H_1G_2G_7H_2+G_2G_3G_4G_5H_2}$$

To illustrate how an equivalent SFG of a block diagram is constructed and how the gain formula is applied to a block diagram, consider the block diagram shown in Fig.4.5.(a). The equivalent SFG of the system is shown in Fig. 4.5.(b).



(a)



(b)

Fig. 4. 5. (a) Block diagram of a control system, (b) Equivalent signal flow graph.

Notice that since a node on the SFG is interpreted as the summing point of all incoming signals to the node, the negative feedbacks on the block diagram are represented by assigning negative gains to the feedback paths on the SFG. First we can identify the forward paths and loops in the system and their corresponding gains. That is: forward path gain

$p_1 = G_1G_2G_3$

$p_2 = G_1G_4$

Loop gains

$L_1 = -G_1G_2H_1$

$L_2 = -G_2G_3H_2$

$L_3 = -G_1G_2G_3$

$L_4 = -G_4H_2$

$L_5 = -G_1G_4$

The closed loop transfer function of the system is obtained by applying Mason' gain formula to the SFG or using the block diagram reduction.

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{\Delta}$$

$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4$$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4}$$

4.5. State Transition Signal Flow Graph:

The state transition SFG or, more simply, the state diagram, is a simulation diagram for a system of equations and includes the initial conditions of the states. Since the state diagram in the Laplace domain satisfies the rules of Mason's SFG, it can be used to obtain the transfer function of the system and the transition equation. The basic elements used in a simulation diagram are a gain, a summer, and an integrator. The signal-flow representation in the Laplace domain for an integrator is obtained as follows:

$$\dot{x}_1(t) = x_2(t); \Rightarrow sx_1(t) = x_2(t) \tag{4.6}$$

$$X_1(t) = \frac{x_2(t)}{s} + \frac{x_1(t_0)}{s} \tag{4.7}$$

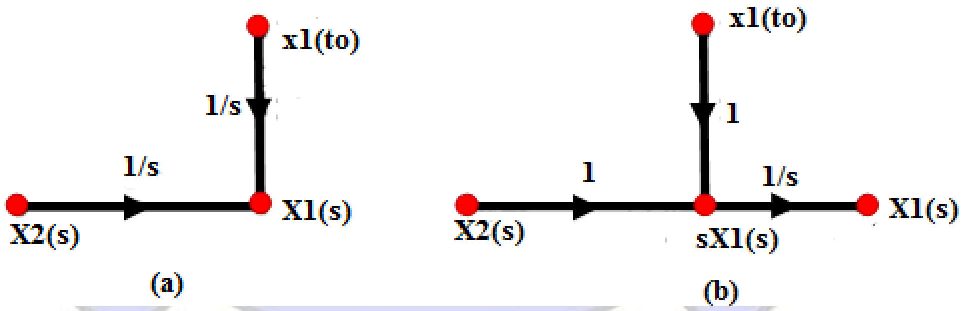
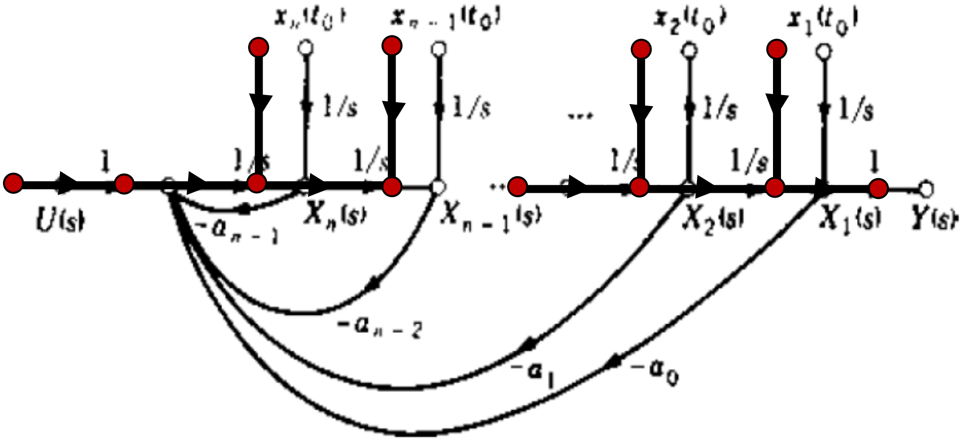


Fig.4.6. Representations of an integrator in the Laplace domain in a signal flow graph

The above equation may be represented either by Fig.4.6.a or Fig.4.6.b. A differential equation that contains no derivatives of the input, as given by:

$$D^n y + a_{n-1}D^{n-1}y + a_1D\ y + a_0y = u \tag{4.8}$$



Example 4: For the following equation find:

$$y^{\cdot\cdot} + \frac{R}{L} y^{\cdot} + \frac{1}{LC} y = \frac{1}{LC} u$$

(a) Draw the state diagram. (b) Determine the state transition equation?

Solution:

(a) The state Diagram, Fig. below, includes two integrators, because this is a second-order equation. The state variables are selected as the phase variables that are the outputs of the integrators, that is, x_1=y and x_2= x_1.

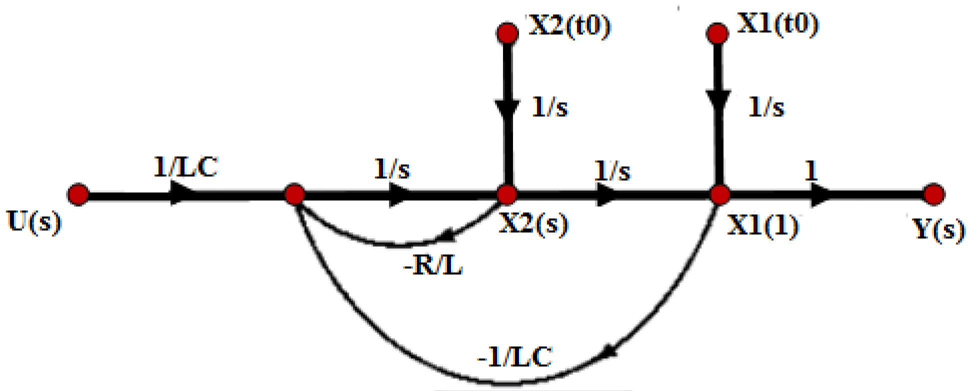
(b) The state transition equations are obtained by applying the Mason gain formula with the three inputs u, $x_1(t_0)$, and $x_2(t_0)$:

$$X_1(s) = \frac{s^{-1}(1+s^{-1}R/L}{\Delta(s)} x_1(t_0) + \frac{s^{-2}}{\Delta(s)} x_2(t_0) + \frac{s^{-2}/LC}{\Delta(s)} U(s)$$

$$X_2(s) = \frac{s^{-2}/LC}{\Delta(s)} x_1(t_0) + \frac{s^{-1}}{\Delta(s)} x_2(t_0) + \frac{s^{-1}/LC}{\Delta(s)} U(s)$$

$$\Delta(s) = 1 + \frac{s^{-1}R}{L} + \frac{s^{-2}}{LC}$$

The signal flow graph for this system is



After simplification these equations become

$$X(s)=\begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix}$$

$$X(s)=\frac{1}{s^2+(R/L)s+1/LC}\left\{\begin{bmatrix} s+R/L & 1 \\ -1/LC & s \end{bmatrix}\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}+\begin{bmatrix} 1/LC \\ s/LC \end{bmatrix}U(s)\right\}$$

4.6. Simplification for the system of dual inputs:

According to the principle of superposition theory, we must find the output by considering one input at a time and cancelled another courses. For the system is shown in Fig.4.7, we find C1(s)/R(s), and then $C_2(s)/D_1(s)$, and $C_3(s)/D_2(s)$, the final output of system is achieved by summation of these three inputs;

$C= C_1+ C_2+ C_3$

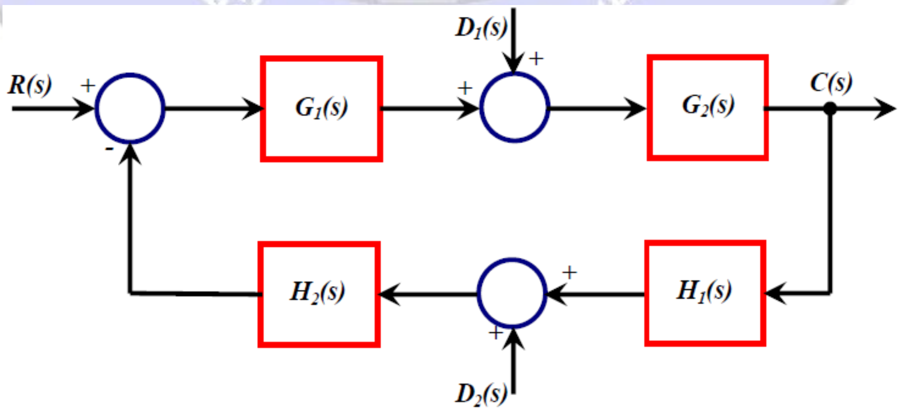
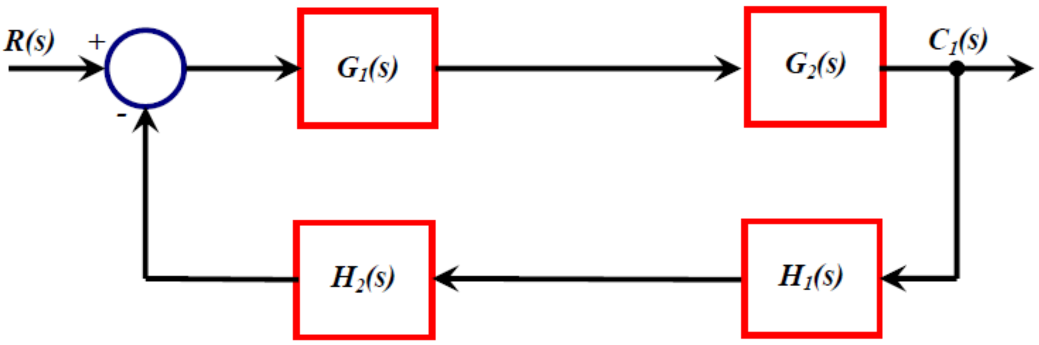


Fig.4.7. Block diagram for dual inputs control system.

Output due to input R(s):

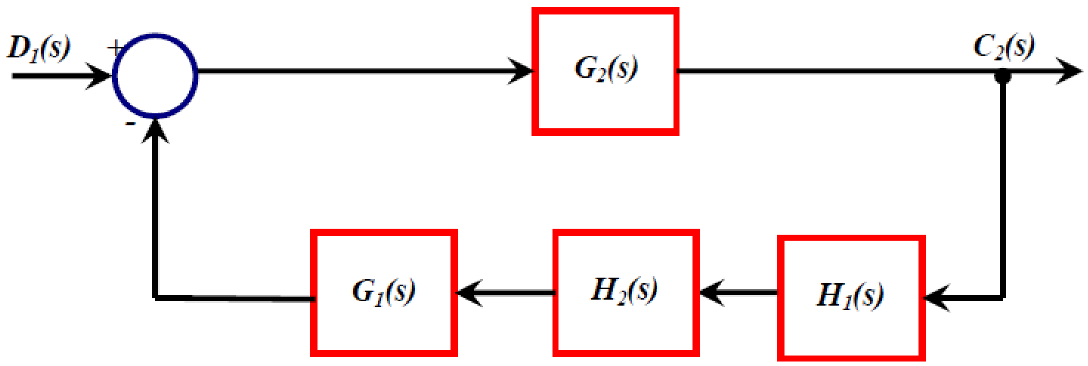
Let D1(s)=0 and D2(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_1(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \qquad (4.9)$$

Output due to input D1(s):

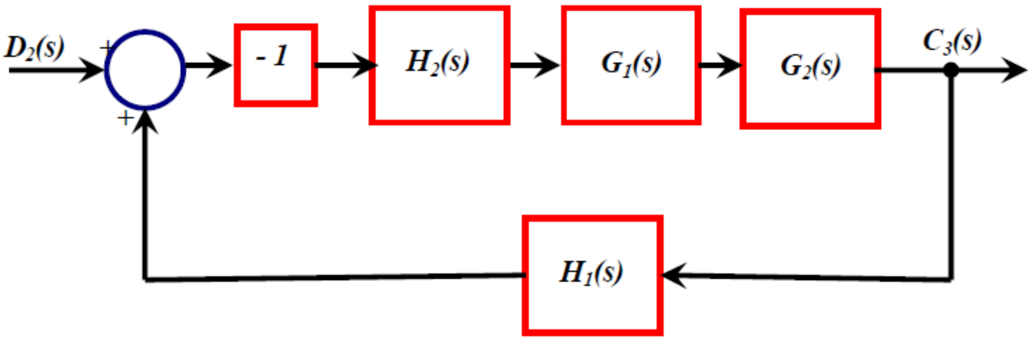
Let R(s)=0 and D2(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_2(s)}{D_1(s)} = \frac{G_2(s)}{1+G_1(s)H_1(s)H_2(s)} \qquad (4.10)$$

Output due to input D2 (s):

Let R1(s)=0 and D1(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_3(s)}{D_2(s)}=\frac{-G_2(s)G_1(s)H_2(s)}{1+G_1(s)G_2H_1(s)H_2(s)} \tag{4.11}$$

The total output of the system is the summation of outputs with respect to a corresponding input:

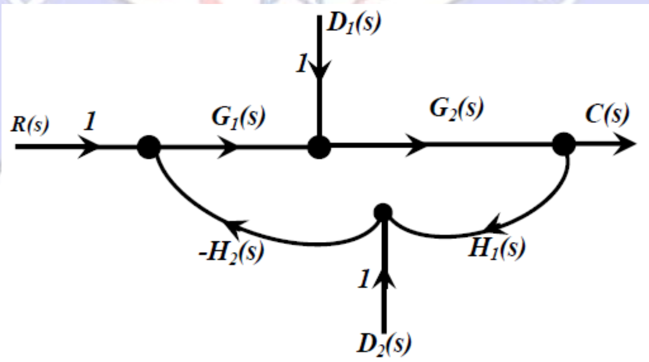
$C= C_1+ C_2+ C_3$

$$C=\frac{RG_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)}+\frac{D_1G_2(s)}{1+G_1(s)H_1(s)H_2(s)}+\frac{-D_2G_2(s)G_1(s)H_2(s)}{1+G_1(s)G_2H_1(s)H_2(s)}$$

$$C=\frac{G_2(s)[RG_1(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)}+\frac{D_1}{1+G_1(s)H_1(s)H_2(s)}+\frac{-D_2G_1(s)H_2(s)]}{1+G_1(s)G_2H_1(s)H_2(s)}$$

The solution by using signal flow graph:

Firstly draw the signal flow graph diagram for original B.D;

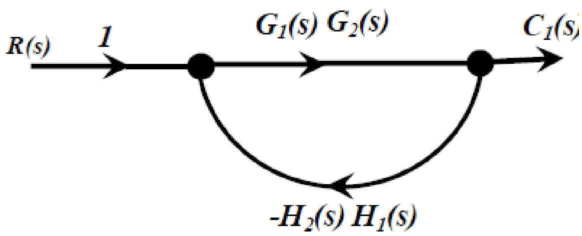


$$P=\frac{1}{\Delta}\sum_{p}P_k\Delta_k$$

When apply the superposition theory can be get;

Output due to input R(s):

Let D1(s)=0 and D2(s)=0, the signal flow graph becomes;



By apply the Mason's formula**:**

$p_1 = G_1(s)G_2(s)$

$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$

$L_2 = 0$

$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$

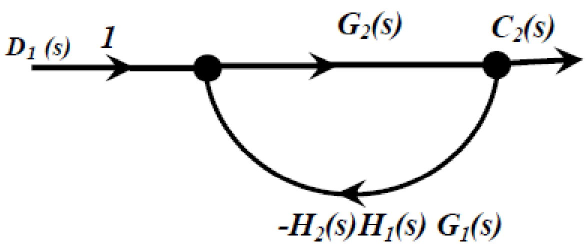
$\Delta_1 = 1$

$$P_R = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_R = \frac{C_1(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

Output due to input D1(s):

Let R(s)=0 and D2(s)=0, the signal flow graph becomes;



$p_1 = G_2(s)$

$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$

$$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$$

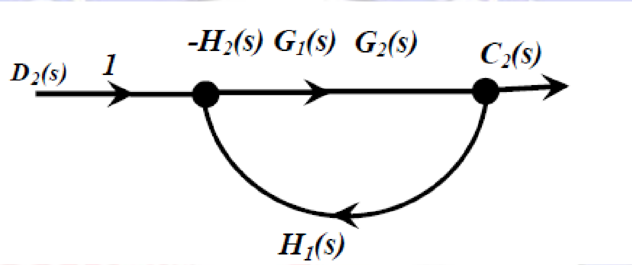
$$\Delta_1 = 1$$

$$P_{D_1} = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_R = \frac{C_2(s)}{D_1(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

Output due to input D2(s):

Let R(s)=0 and D1(s)=0, the signal flow graph becomes;



$$p_1 = -G_1(s)G_2(s)H_2(s)$$

$$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$$

$$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$$

$$\Delta_1 = 1$$

$$P_{D2} = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_{D2} = \frac{C_2(s)}{D_2(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

$$P_{total} = P_R + P_{D1} + P_{D2}$$

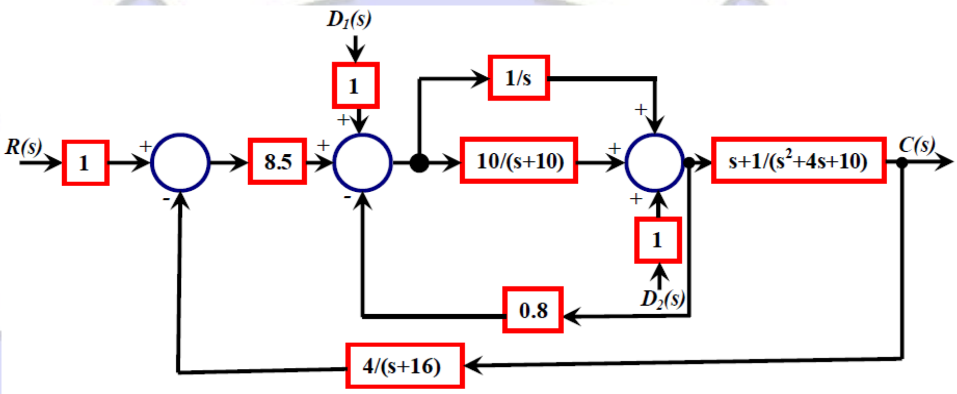
$$P_{total} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}R + \frac{G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}D_1$$

$$+ \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}D_2$$

$$C = \frac{G_2(s)[G_1(s)R + D_1 - -G_1(s)H_2(s)D_2]}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

Note: Where the denominator represents the polynomial of the system therefore; it has the same form in all inputs affect and in all form where the polynomial refer to clc's of the system (i.e. A matrix in state space form).

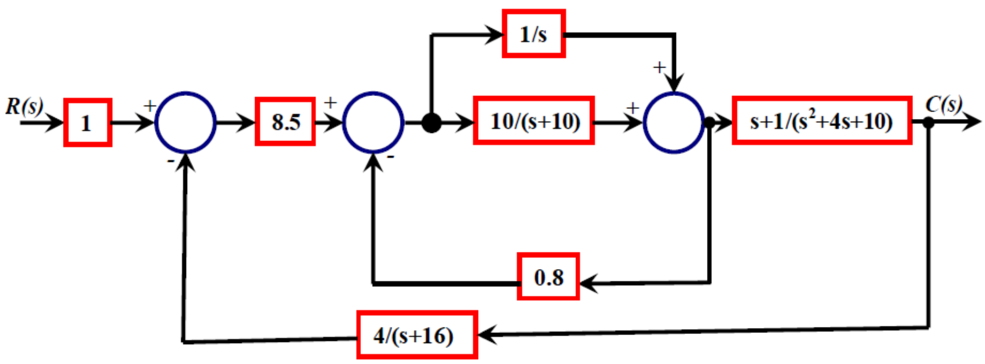
Example 5: Find the T.F for the block diagram shown below?



Solution:

By using superposition theory:

1. C/R(s) by set D1 and D2=0; the block becomes:



And can be continue with solution by using B.D.R and By using S.F.G. and this is a home work. From all the above examples, we can see that all loops and forward paths are touching in this case.

As a general rule, if there are no nontouching loops and forward paths in the block diagram or SFG of the system, then the Mason' gain formula can be putted in a simplified formula, as shown next.

$$\frac{C}{R} = \sum \frac{forward\ \ path\,gains}{1 - loop\ gains}$$

4.7. Matlab program for signal flow graph

Also all the complex solution can be minimize by using MATLAB with the following program, for the control system that is shown in Example 5, we can find the solution by using Matlab program. The following program for MISO system to get the final transfer function which can be get analytically by block diagram reduction or signal flow graph.

Program in MATLAB to find T.F. of MISO system:

```
n1=[1];d1=[1];
n2=[8.5];d2=[1];
n3=[1];d3=[1 0];
n4=[10];d4=[1 10];
n5=[.8];d5=[1];
n6=[4];d6=[1 16];
n7=[1 1];d7=[1 4 10];
n8=[1];d8=[1];
n9=[1];d9=[1];
nblocks=9;
blkbuild;
q=[ 1 0 0 0
2 1 -6 0
3 2 -5 8
```

```
4 2 -5 8
5 3 4 9
6 7 0 0
7 3 4 9
8 0 0 0
9 0 0 0 ];
iu=9;
iy=7;
[ac,bc,cc,dc]=connect(a,b,c,d,q,iu,iy);
[num,den]=ss2tf(ac,bc,cc,dc,1)
printsys (num,den)
%i/p R & O/P C
% 4.2633e-014 s^4 + 93.5 s^3 + 1674.5 s^2 + 2941 s + 1360
%T.F. R/C= ------------------------------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
%i/p D1 & O/P C
% 2.8422e-014 s^4 + 11 s^3 + 197 s^2 + 346 s + 160
% T.F C/D1= ----------------------------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
%i/p D1 & O/P C
% s^4 + 27 s^3 + 186 s^2 + 160 s - 1.728e-011
% T.F. D2/C -----------------------------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
```

Lecture No. Five

Transient Response Analysis

This lecture will discusses the following topics

5.1. Analysis of typical control system.

5.2. Samples of systems Response.

5.3. Test inputs.

5.4. Second –order systems and T.R. specifications.

5.5. Parameters of transient-response.

5.6. Solved problem.

Prof. Dr.
Yousif AL Mashhadany
U. O. Anbar

5.1. Analysis of a typical control system

Consider a first differential equation $\frac{dx(t)}{dt}+ax(t)=u(t)$

This may be the equation of a physical system with input u(t) and output x(t) i)function) is that part of t response which occurs near t =0 and then decays .this part of t response is due to the characteristics of the system only .

ii) Steady state part (s.s) (particular integral) is that part of the response which is present throughout the period t=0 to t= .but at t this the complete solution because the transient part is absent.

The nature of steady state response depends of external input only .complete solution=Tr part +S.S part

i) Auxiliary equation (characteristic equation): $m+a=0 \Rightarrow m=-a$

Transient part Ae^{-at}

ii) Steady state part

let u(t)=U(constant)

$\frac{dx(t)}{dt}=0$ at steady state: $aX_{ss}=U \Rightarrow X_{ss}=\frac{U}{a}$

$$X(t)=Ae^{-at}+\frac{U}{a}$$

If we know x(t)=0 at t=0: $X(t)=\frac{U}{a}(1-e^{-at})$

At t=0 , x(t)=0

at $t=\infty$, $x(t)=U/a$

at $t=1/a$, $x(t)=0.63\ U/a$

$T=1/a$ =time constant (see Fig.5.1)

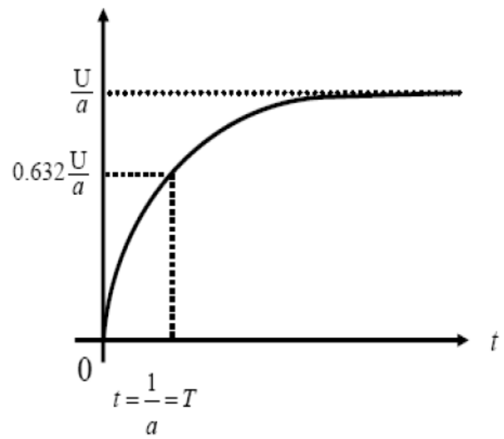


Fig. 5.1. Response of first order system

5.2. Samples of systems Response.

i) R-C , R-L circuit with constant voltage input.

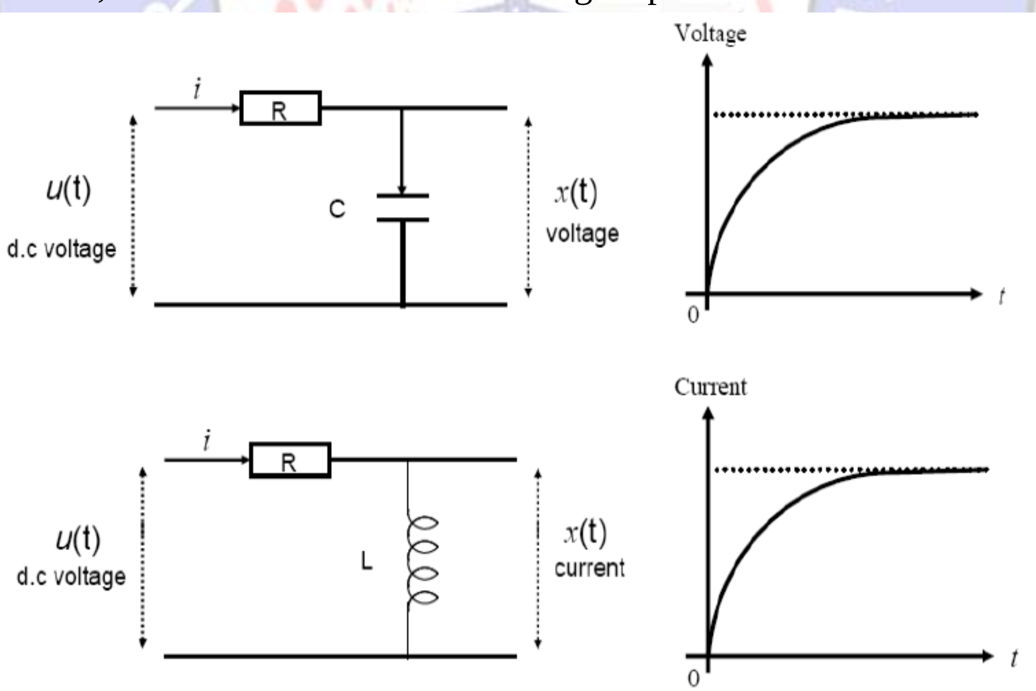
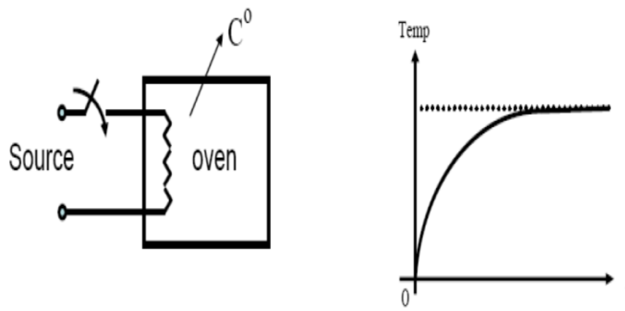
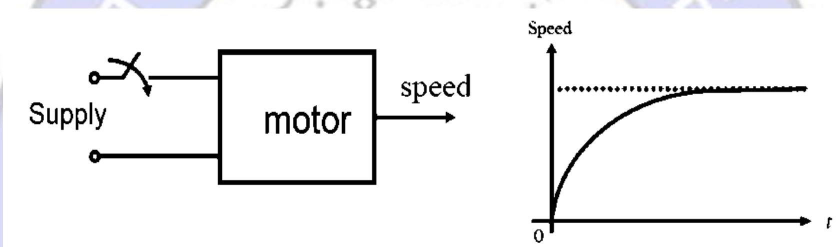


Fig.5.2. Example of RLC circuit with constant voltage input.

ii) A sudden voltage applied an electric oven.



iii) Constant supply voltage switched on to motor.



All these system may be represented by differential equation of first order.

SUCH SYSTEMS ARE CALLED FIRST ORDER SYSTEMS

Consider a D.C. motor operating with a constant field current i_f.If the input to the motor is taken as e_1(armature voltage) and the output is taken as speed ω .The differential equation of the motor may be written as :-$\frac{d\omega(t)}{d(t)} + a\omega(t) = e_1(t)$

Using 'D' operator $D\omega(t) + a\omega(t) = e_1(t)$

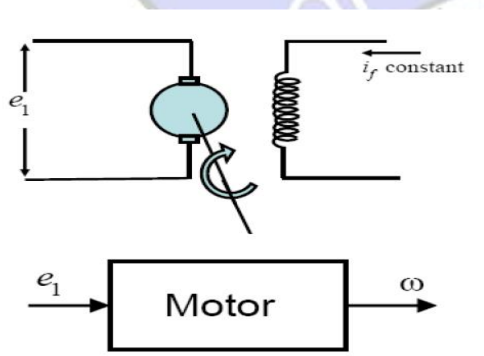
$$\frac{\omega(t)}{e_1(t)} = \frac{1}{D+a}$$

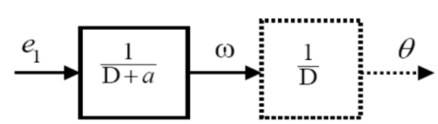
If we define θ as the output variable, $\omega = \frac{d\theta}{dt}$

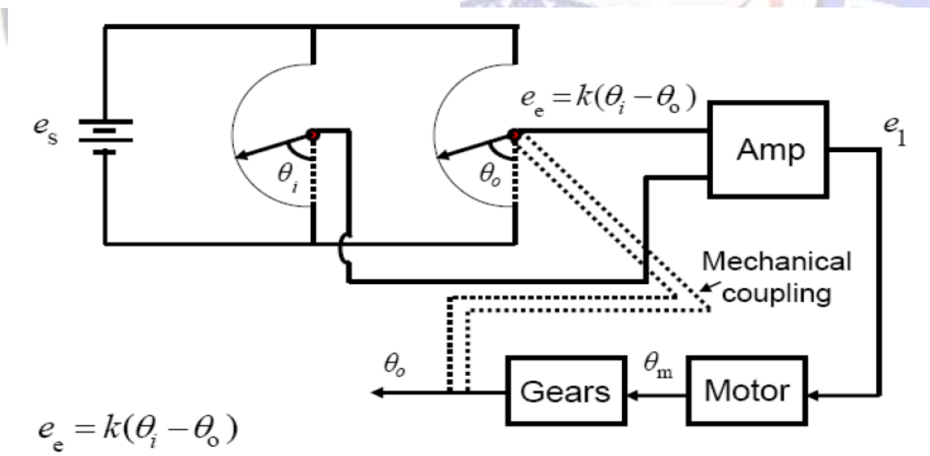
$$\frac{d^2\theta(t)}{dt^2} + a\frac{d\theta(t)}{dt} = e_1(t)$$

$$D^2\theta + aD\theta = e_1(t)$$

$$\frac{\theta(t)}{e_1(t)} = \frac{1}{D(D+a)}$$





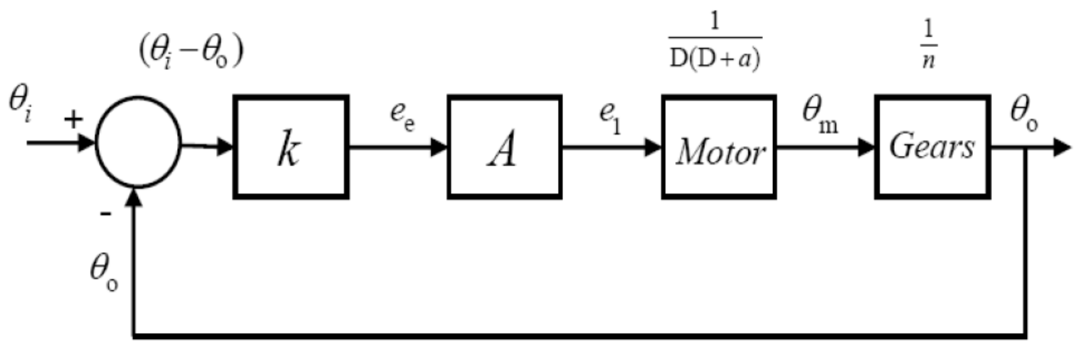


Analysis of a position control system:

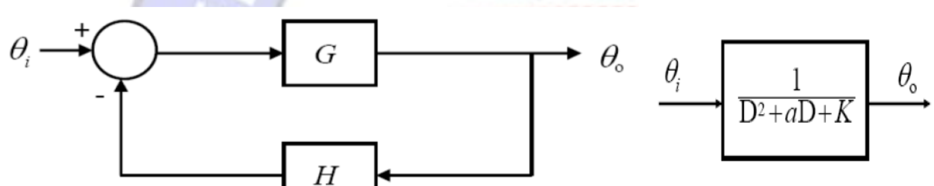
$$k = \frac{e_e}{\theta_{i} - \theta_o}$$

$$k = \frac{e_s}{\theta_{max}}$$

Block diagram:



Simplified block diagram:



$$G = \left(k.A\frac{1}{D(D+a)}\frac{1}{n}\right) = \frac{K}{D(D+a)}$$

Where K = system gain

$$\frac{\theta_o}{\theta_i} = \frac{G}{1+GH} = \frac{\frac{K}{D(D+a)}}{1+\frac{K}{D(D+a)}} = \frac{K}{D^2+aD+K}$$

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = k\theta_i(t)$$

a is parameter of the motor:

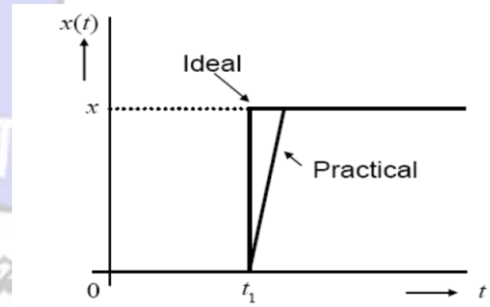
$$K = k.A\frac{1}{n}$$

Where A is amplifier gain and $\frac{1}{n}$ is gear ratio.

5.3. Test inputs

i) Step function: a step is a sudden change in the value of the physical quantity $x(t)$ from one level (usually zero) to another level, in zero time.

$$x(t) = \begin{cases} x & t > t_1 \\ 0 & t \leq t_1 \end{cases}$$



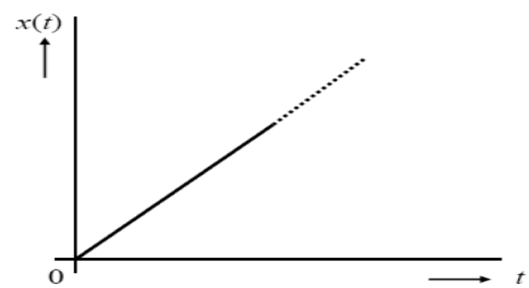
Step function at t_1

Unit step:

$$u(t) = \begin{cases} 1 & t > t_1 \\ 0 & t \leq t_1 \end{cases}$$

ii) Ramp Function: ramp is a signal which starts from a zero level and increase linearly with espect to time.

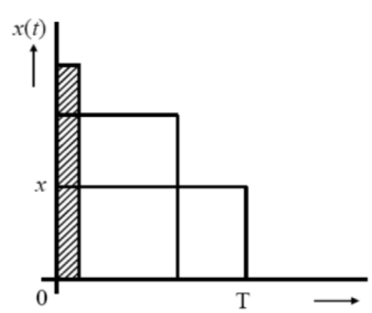
$$x(t) = \begin{cases} kt & t > 0 \\ 0 & t \leq 0 \end{cases}$$



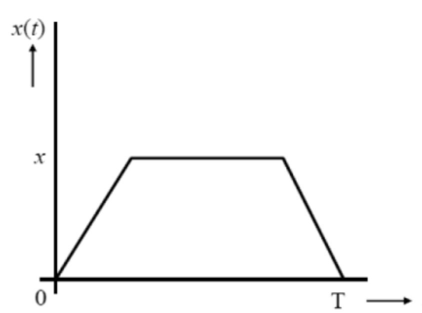
Ramp function at $t = 0$

iii) Pulse Function: a pulse may be considered as a step function which is present for limited period.

$$x(t) = \begin{cases} x & 0 < t \leq T \\ 0 & elsewhere \end{cases}$$

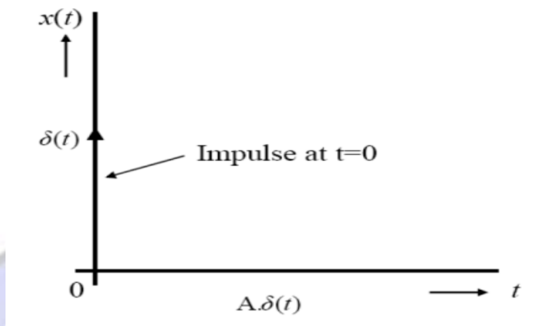


Ideal



Practical

iv) Impulse Function: if in the pulse, the width is decreased and the height is increased such that.



$\lim_{\substack{T\to 0\\ x\to\infty}} x.T = A$, the resulting function is impulse $A\delta(t)$.

5.4. Second –order systems and T.R. specifications.

Differential Equation of the C.L position control system:

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = k\theta_i(t)$$

For step input , $\theta_i(t) = R$, $t > 0$

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = kR$$

Solve the differential equation.

i) S.S solution ($\dot{\theta}_o(t) = \ddot{\theta}_o(t) = 0$)

$(\theta_o)_{s.s} = R$

ii) Transient solution

Auxiliary equation: $r^2 + ar + k = 0$

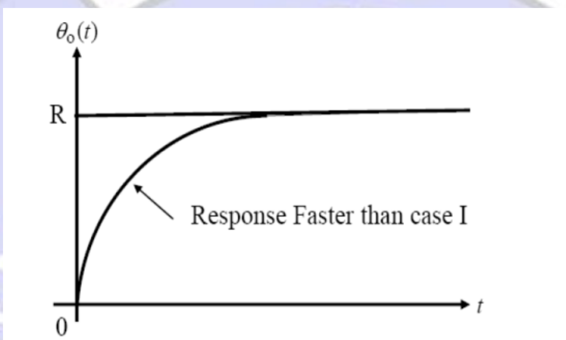
(characteristic equation) $r_1, r_2 = \frac{-a\pm\sqrt{a^2-4k}}{2}$

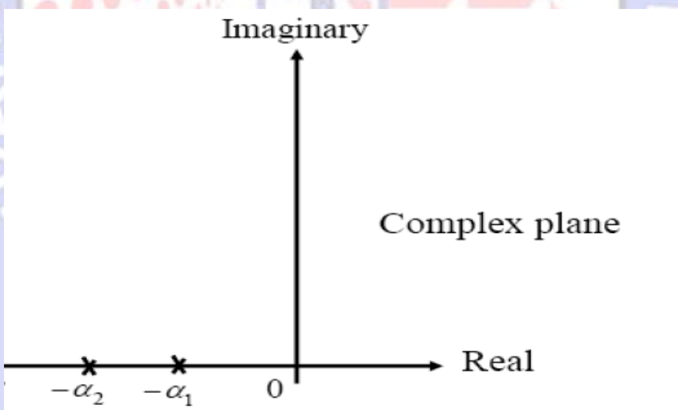
Case I: The two roots are distinct.

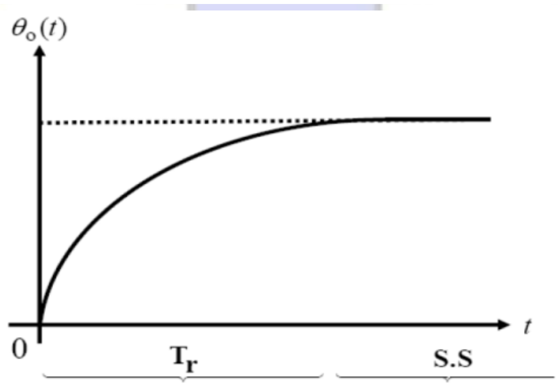
$$r_1, r_2 = -\alpha_1, -\alpha_2 \quad ; \; a^2 > 4k$$

$$(\theta_o)_{Tr} = C_o e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}$$

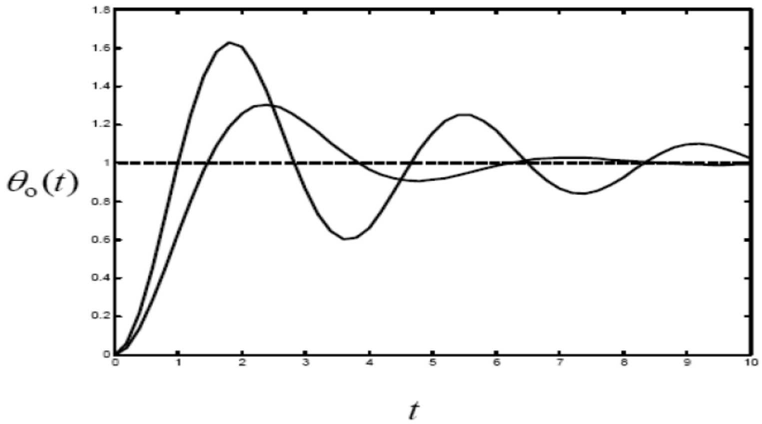
$\theta_o(t) = R + C_o e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}$



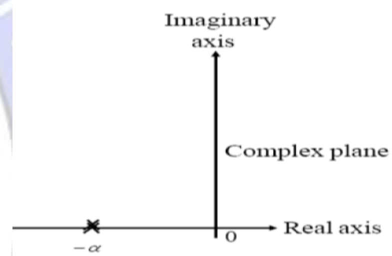




<u>Case II:</u> Repeated Roots.



$r_1, r_2 = -\alpha, -\alpha \quad ; \ a^2 = 4k$



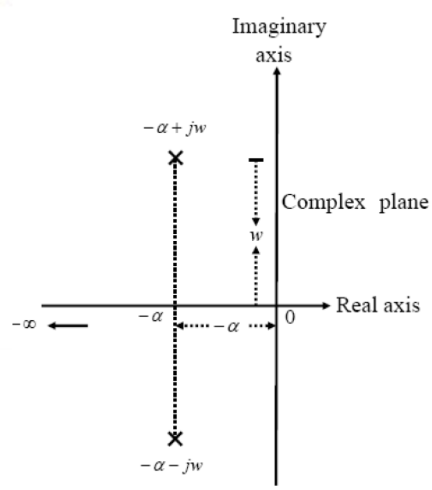
$(\theta_o)_{Tr} = (C_o + C_1 t)e^{-\alpha t}$

$$\theta_o(t) = R + (C_o + C_1 t)e^{-\alpha t}$$

<u>Case III:</u> Complex Conjugate Roots.

$$r_1, r_2 = -\alpha \pm jw \quad ; \ a^2 < 4k$$

$(\theta_o)_{Tr} = e^{-\alpha t}(C_o \cos wt + C_1 \sin wt)$



$(\theta_o)_{Tr} = C_2 e^{-\alpha t} \sin(wt + C_3)$

$$\theta_o(t) = R + C_2 e^{-\alpha t} \sin(wt + C_3)$$

Response of Position Control System:

i) Distinct Roots. $\theta_o(t) = R + C_o e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}$

ii) Repeated Roots. $\theta_o(t) = R + (C_o + C_1 t)e^{-\alpha t}$

iii) Complex Conjugate Roots. $\theta_o(t) = R + C_2 e^{-\alpha t}\sin(wt + C_3)$

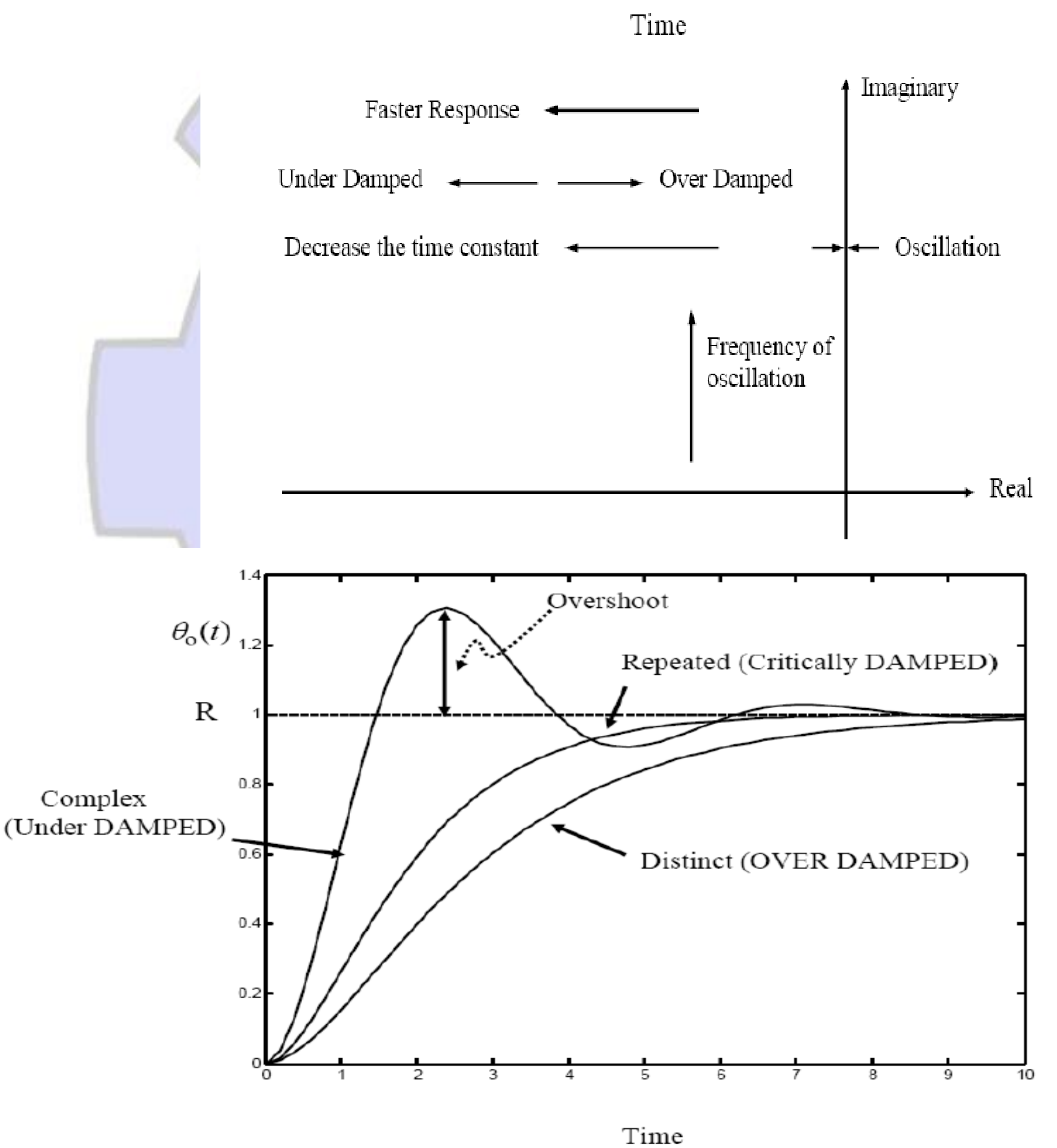


Fig. 5.3. Cases of Response for Control System.

1. Response becomes faster and faster as the roots moved along the –ve real axis. The time constant $\frac{1}{\alpha}$ also decreases progressively.
2. Damping increases as the roots moves away in the –ve real dirction.
3. Frequency of oscillation increases as the roots move away from the real axis (along the imaginary axis direction).

All control system design methods attempt to shift the roots of the characteristic equation from an undesirable location to a desirable location.

5.5. Parameters of transient-response.

In many practical cases , the desired performance characteristics of control systems are specified in terms of time domain quantities .Systems with energy storage cannot respond instantaneously and will exhibit transient response whenever they are subjected to inputs or disturbances. Frequently , the performance characteristics of a control system are specified in terms of the transient response to a unit-step input since it is easy to generate and is sufficiently drastic.(If the response to a step input is known , it is mathematically possible to compute the response to any input).

The transient response of a system to a unit-step depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition that the system is at rest initially with output and all time derivatives thereof zero. Then the response characteristics can be easily compared.

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input , it is common to specifying the following:

1. Delay time , t_d
2. Rise time , t_r
3. Peak time , t_p
4. Maximum overshoot , M_p
5. Settling time , t_s

These specifications are defined in what follows and are shown graphically in fig.

1. Delay time, t_d : the delay time is the time required for the response to reach half the final value the very first time.

2. Rise time , t_r :the rise time is the time required for the response to rise from 10% to 90% ,5% to 95% , or 0% to 100% of its final value. For under-damped second –order systems, the 0% to 100% rise time is normally used. For over-damped systems, the 10% to 90% rise time is commonly used.

3. Peak time, t_p: the peak time is the time required for the response to reach the first peak of the overshoot.

4. Maximum (percent) overshoot, M_p : the maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent over-shoot. It is defined by

$$Maximum\ percent\ overshoot = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

6. Settling time , t_s:the settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system .Which percentage error criterion to use may be determined from the objectives of the system design in question. The time-domain specifications just given are quite important since most control systems are time-domain systems; that is , they must exhibit acceptable time responses. (This means that the control system must be modified until the transient response is satisfactory). Note that if we specify the values of $t_d, t_r, t_p, t_s \text{ and } M_p$, then the shape of the response curve is virtually determined. This may be seen clearly from Fig.5.3. Second-order systems and transient-response specifications. In the following , we shall obtain the rise time, peak time , maximum overshoot, and settling time of the second-order system given by Equation below. These values will be obtained in terms of ζ and ω_n.The system is assumed to be under-damped.

Rise time t_r:Referring to Equation, we obtain the rise time t_r by letting $c(t_r) = 1$ or

$$c(t_r) = 1 = 1 - e^{-\zeta w_n t_r}(\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r)$$

Since $e^{-\zeta w_n t_r} \neq 0$, we obtain from Equation the following equation:

$$\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r = 0$$

Or $\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$

Thus ,the rise time t_r is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}$$

Where β is defined in fig. clearly for a small value of t_r , ω_d must be large.

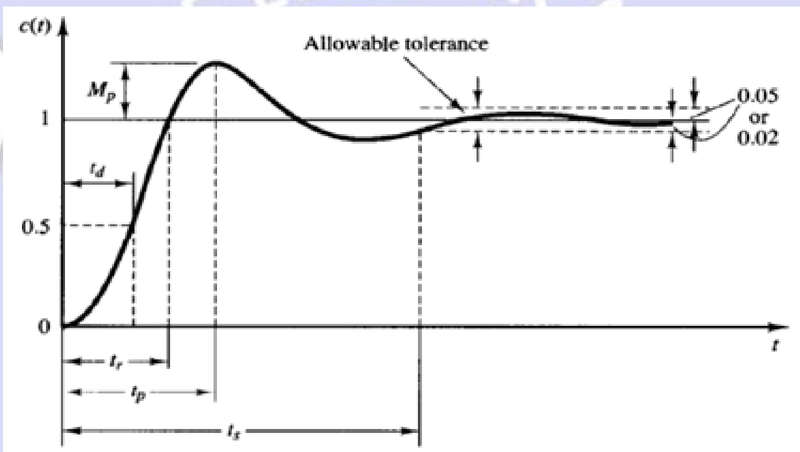


Fig. 5.4. Transient Response Parameters

Peak time , t_p: Referring to Equation , we may obtain the peak time by differentiating c(t)

with respect to time and letting this derivative equal can be simplified to

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}(cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t)$$

$$+e^{-\zeta\omega_n t}(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t)$$

and the cosine terms in this last equation cancel each other , dc/dt , evaluated at $t = t_p$

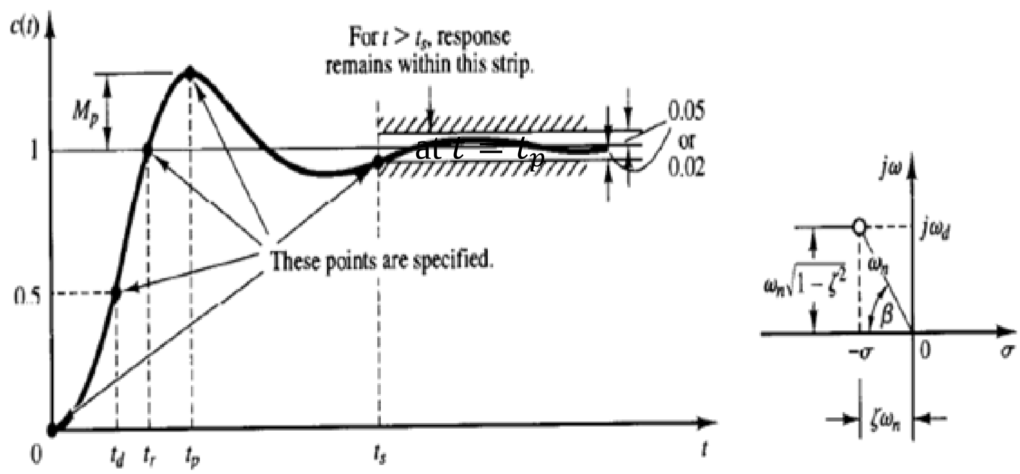


Fig. 5.5. Transient – response specifications and definition of the angle β

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin \omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation :

$$\sin \omega_d t_p = 0$$

$$\text{Or} \quad \omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot,

$\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d}$$

The peak time t_p corresponds to one-half cycle of the frequency of damped oscillation.

Maximum overshoot M_p: The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$ .Thus , M_p is obtained as

$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}(cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi$$

$$= -e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

Settling time t_s: For an under-damped second-order system, the transient response is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta^2}) \qquad , for\ t \geq 0$$

The curves $1 \pm (e^{-\zeta\omega_n t\sqrt{1-\zeta^2}/})$ are the envelope curves of the transient response for a unit-step input. The response curve c(t) always remains within a pair of the envelope curves , as shown in Fig. below. The time constant of these envelope curves is $1/\zeta\omega_n$. .

The speed of decay of the transient response depends on the value of the time constant $1/\zeta\omega_n$. For a given ω_n , the setting time t_s is a function of the damping ratio ζ.

5.6. Solved problem

Prob.1. A field controlled d.c motor is characterized by the following differential equation.

$$0.5\frac{dw(t)}{dt} + w(t) = 1.57 i_f(t)$$

Where, w(t) is the angular velocity of the motor in radians/second and if is the field current in mA.

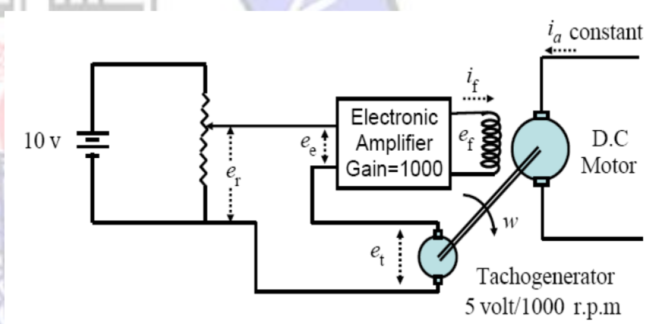
a)if the motor is supplied with a step input of 100mA what is the steady state speed in r.p.m.

at S.S $\Rightarrow$ $\dot{w}=0$

$$w_{ss} = 1.57 * 100 = 157\frac{rad}{second}$$

$$= 157\frac{60}{2\pi}\, r.p.m$$

$$= 1499.23\, r.p.m$$



b)in (a) how much time would be taken by the motor to reach i)25% , ii)50% and iii)75% of the steady state speed?

Characteristic equation

(0.5m+1)=0

$m = -2$

$w_{Tr} = Ae^{-2t}$

At t=0 , w(0)=0

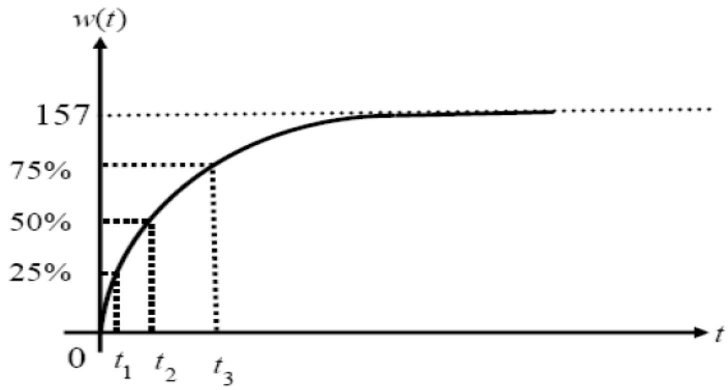
0=157+A

A=−157

$w(t) = 157 * (1 - e^{-2t})$

i) $w(t_1) = 25\%$ of the S.S speed (157 rad/second)

$\frac{25}{100} * 157 = 157 * (1 - e^{-2t_1}) \Rightarrow t_1 = 0.1438\ sec$



$\frac{50}{100} * 157 = 157 * (1 - e^{-2t_2}) \Rightarrow \quad t_2 = 0.3466\ sec$

$\frac{75}{100} * 157 = 157 * (1 - e^{-2t_3}) \Rightarrow \quad t_3 = 0.6931\ sec$

c)The above motor is used in a speed control scheme as shown in figure bolow.

Draw the block diagram of the system and write down the differential equation of the closed loop system. Given that field resistance =100Ω, inductance 20H.

i) M

$$e_f = R_f i_f + L_f \frac{di_f}{dt}$$

$$e_f = 100 * i_f + 20Di_f$$

$$\frac{i_f}{e_f} = \frac{1}{100 + 20D} = \frac{1000}{100 + 20D} mA$$

from system equation.

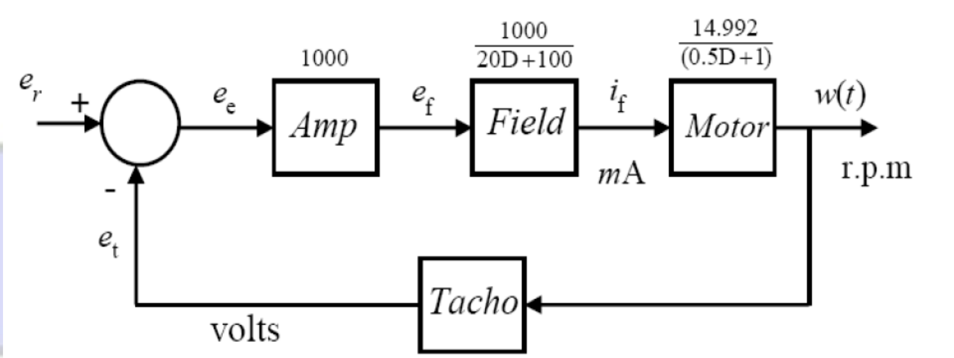
$(0.5\,D + 1)w = 1.57 i_f$

$\frac{w}{i_f} = \frac{1.57}{0.5D+1}$ in rad/second $= \frac{14.992}{0.5\,D+1}$ in r.p.m

d) calculate the setting of the potentiometer to get a steady state speed of

i) 900 r.p.m

ii) 1100 r.p.m



$$G = 1000 * \frac{1000}{20D + 100} * \frac{14.992}{0.5D + 1} = \frac{1499200}{(D+2)(D+5)}$$

H=0.005 volt/r.p.m

$$\frac{w(t)}{e_r(t)} = \frac{G}{1 + GH} = \frac{1499200}{D^2 + 7D + 7506}$$

$$\frac{d^2 w(t)}{e_r(t)} + 7\frac{dw(t)}{dt} + 7506\,w(t) = 1499200\,e_r(t)$$

Differential Equation of the C.L system

i) For $w(t)|_{t=\infty} = 900\, r.p.m$

$D = D^2 = 0$ at steady state

$$w(t)_{S.S} = 900 = e_r \frac{1499200}{7506}$$

$e_r = 4.506$ volts

Potentiometer factor =0.4506

ii) For $w(t)_{S.S} = 1100$ r.p.m $e_r = 5.507$ volts

Potentiometer factor =0.5507

if the amplifier gain suddenly decreases by 25% what would be the range in the motor speed if it was earlier running at 900 r.p.m. when the motor is running at 900 r.p.m

$e_r = 4.506$ volts

Amplifier gain=750

$$\frac{w(t)}{e_r(t)} = \frac{1124400}{D^2 + 7D + 5632}$$

At S.S $w(t)|_{t=\infty} = \frac{4.506*1124400}{5632} = 899.6$ r.p.m

Prob.2. A small electric oven is known to have a first order differential equation as its describing equation. when the rated input of 20 volt is applied to the oven at 25°C, the steady state temperature is found to be 1225°C and a temperature of 625°C is reached in 30 seconds.

a) Write down the differential equation of the oven.

General first order differential equation.

$$\frac{dT(t)}{dt} + aT(t) = be_1(t)$$

Oven Temperature   Oven Voltage

t=0 , T=25°

t=30 , T=625°

t=∞ , T=1225°

$$T_{ss} = \frac{b}{a}e_1 \quad ; \; T_{tr} = Ae^{-at}$$

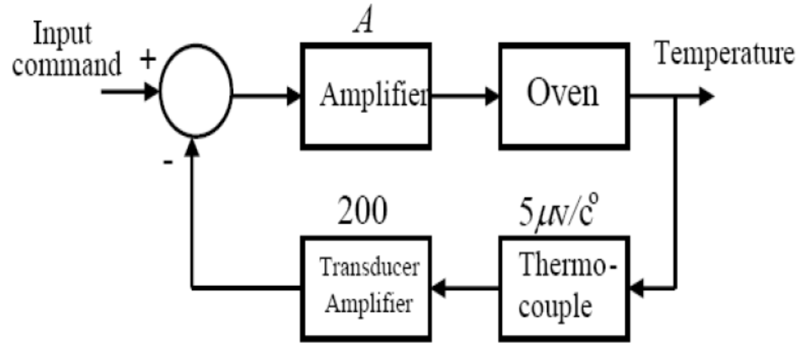
$$T_{total} = Ae^{-at} + \frac{b}{a}e_1$$

Initial condition at t=0 , T(t)=25

$$25 = A + \frac{b}{a}e_1 \,, \qquad A = 25 - \frac{b}{a}e_1$$

$$T(t) = \left(25 - \frac{b}{a}e_1\right) * e^{-at} + \frac{b}{a}e_1$$

At t=∞ (steady state);



T(t)=1225°C

$$1225 = \frac{b}{a}20$$

At t=30 , T(t)=625

$$625 = \left(25 - \frac{b}{a}20\right) * e^{-30a} + \frac{b}{a}20$$

$$625 = (25 - 1225) * e^{-30a} + 1225$$

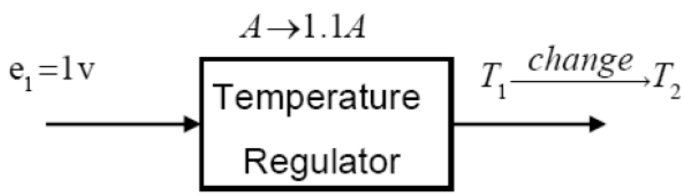
a=0.0231049

b=1.4151755

Oven equation is

$$\frac{dT(t)}{dt} + 0.023T(t) = 1.415e_1(t)$$

b) it is now required to control the temperature of the oven by a close loop feedback system as shown in figure below. Obtain the differential equation of the overall system.



$$G = A\frac{1.415}{D+0.023} \quad ; H = 5*10^{-6}*200 = 10^{-3}$$

$$\frac{T(t)}{e_1} = \frac{G}{1+GH} = \frac{1.415A}{D+0.023+A*1.415*10^{-3}}$$

c)calculate the value of 'A' such that if 'A' increases by 10% the steady state change in the oven temperature does not exceed 0.5°C for $e_1 = 1$ volts

$$T_1 - \mathrm{T}_2 = 0.5$$

$$\frac{1.415*1.1*A}{0.023+1.415*1.1*10^{-3}*A} - \frac{1.415*A}{0.023*1.415*10^{-3}*A} = 0.5$$

$$1.10122375*A^2 + 34.172251*A + 264.5 = 3254.5*A$$

$$1.10122375*A^2 - 3220.32775*A + 264.5 = 0$$

$$A = \frac{3220.3277 \pm \sqrt{(3220.3277)^2 - 4*1.10122375*264.5}}{2*1.10122375}$$

A=2924.158

d) Calculate the time constant of the closed loop system for the value of 'A' calculated in part (c).

$$\frac{T(t)}{e_1} = \frac{1.415A}{D+0.023+A*1.415*10^{-3}} = \frac{K}{(D+a)}$$

$a = 0.023 + 2924.153 * 1.415 * 10^{-3} = 4.160676$

Time constant= $T = \frac{1}{a} = 0.240345\ sec$

e)what is the range of input command in volts required for controlling the temperature from 100°C to 1000°C.

At $\quad S.S \quad T = \frac{1.415*2924.153}{0.023+2924.153*1.415*10^{-3}} e_1 = \frac{4137.676496}{4.1606764} e_1$

$T = 994.472 * e_1$

at T=100

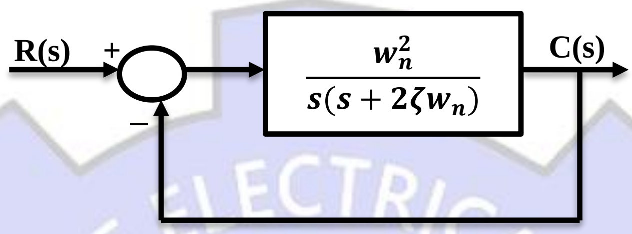
100=994.472*e_1 , $e_1 = 0.100555$ volt

at T=1000

$1000 = 994.472 * e_1$, $e_1 = 1.00555$ volt

The range of input command is $0.100555 \leq e_1 \leq 1.00555$

Prob. 3: **.**For the system shown in Fig. below where $\zeta = 0.6$ and $\omega_n = 5\, rad/sec$. Let us obtain the rise time t_r,peak time t_p,maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step unit.



Solution:

From the given values of ζ and ω_n , we obtain

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

<u>Rise time t_r:</u> The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta - \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93\, rad$$

The rise time t_r is thus :

$$t_r = \frac{3.14 - 0.93}{4} = 0.55\, sec$$

<u>peak time t_p:</u> The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785\, sec$$

<u>Maximum over shoot M_p:</u> The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.025$$

The maximum percent overshoot is thus 9.5%

<u>Settling time t_s:</u> For the 2% criterion , the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33\, sec$$

For the 5% criterion

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1\, sec$$

Lecture No. Six

Steady-State Error

This lecture will discusses the following topics

6.1. Introduction.

6.2. Steady-State Step Error Coefficient.

6.3. Comparison between steady state error in open loop & closed loop system.

6.4. Solved problems

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6.1. Introduction.

The simple closed-loop feedback system, with unity feedback, shown in Fig. 6.1, may be called a tracker since the output c(t) is expected to track or follow the input r(t). The open-loop transfer function for this system is (G(s)=C(s)/E(s)),which is determined by the components of the actual control system. Generally G(s) has one of the following mathematical forms:

$$\left.\begin{aligned} G(s) &= \frac{K_o(1+T_1 s)(1+T_2 s)......}{(1+T_a s)(1+T_b s).....} \\ G(s) &= \frac{K_1(1+T_1 s)(1+T_2 s)......}{s(1+T_a s)(1+T_b s).....} \\ G(s) &= \frac{K_2(1+T_1 s)(1+T_2 s)......}{s^2(1+T_a s)(1+T_b s).....} \end{aligned}\right\} \qquad (6.1)$$

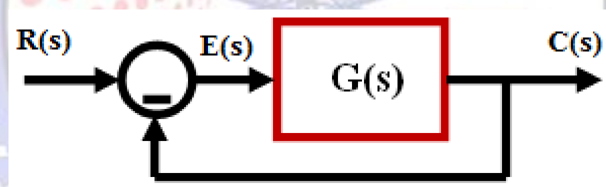


Fig. 6.1. Unity–feedback control system

Note that the constant term in each factor is equal to unity. The preceding equations are expressed in a more generalized manner by defining the standard form of the transfer function as:

$$G(s) = \frac{K_m(1+b_1 s+b_2 s^2+b_w s^w)......}{s^m(1+a_1 s+a_2 s^2+a_u s^u).....} = K_m G'(s) \qquad (6.2)$$

Where

$a_1, a_2, \ldots$=constant coefficients

$b_1, b_2, \ldots$=constant coefficients

K_m = gain constant of the transfer function G(s)

m = 0,1,2,…. Denotes the transfer function type

G'(s) = forward transfer function with unity gain

The degree of the denominator is n=m u. For a unity-feedback system, E and C have the same units. Therefore, K_o is non-dimensional, K_1 has the units of seconds-1, K_2 has the units of seconds-2.

In order to analyze each control system, a ‘‘type’’ designation is introduced. The designation is based upon the value of the exponent m of s in Equation(6.1). Thus, when m=0, the system represented by this equation is called a Type 0 system; when m=1, it is called a Type 1 system; when m=2, it is called a Type 2 system; etc. Once a physical system has been expressed mathematically, the analysis is independent of the nature of the physical system. It is immaterial whether the system is electrical, mechanical, hydraulic, thermal, or a combination of these. The most common feedback control systems have Type 0, 1, or 2 open-loop transfer functions. It is important to analyze each type thoroughly and to relate it as closely as possible to its transient and steady-state solution. The various types exhibit the following steady-state properties:

Type 0: A constant actuating signal results in a constant value for the controlled variable.

Type 1: A constant actuating signal results in a constant rate of change (constant velocity) of the controlled variable.

Type 2: A constant actuating signal results in a constant second derivative (constant acceleration) of the controlled variable.

Type 3: A constant actuating signal results in a constant rate of change of acceleration of the controlled variable.

These classifications lend themselves to definition in terms of the differential equations of the system and to identification in terms of the forward transfer function. For all classifications the degree of the denominator of G(s)H(s) usually is equal to or greater than the degree of the numerator because of the physical nature of feedback control systems. That is, in every physical system there are energy-storage and dissipative elements such that there can be no instantaneous transfer of energy from the input to the output. However, exceptions do occur.

6.2. Steady-State Step Error Coefficient.

The error coefficients are independent of the system type. They apply to any system type and are defined for specific forms of the input, i.e., for a step, ramp, or parabolic input. These error coefficients are applicable only for stable unity feedback systems. The results are summarized in Table 6.1.

Table 6.1. Definitions of Steady-State Error Coefficients for Stable Unity-Feedback Systems

Error coefficient	Definition of error coefficient	Value of error coefficient	Form of input signal r(t)
Step (K_p)	$\frac{c(t)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} G(s)$	$R_0 u_{-1}(t)$
Ramp (K_v)	$\frac{(Dc)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} sG(s)$	$R_1 t u_{-1}(t)$
Parabolic(K_a)	$\frac{(D^2c)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} s^2G(s)$	$\frac{R_2 t^2 u_{-1}(t)}{2}$

The step error coefficient is defined as:

$step\ \ error\ coefficient = \dfrac{steadystate\ value\ of\ output c(t)_{ss}}{steadystate\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_p$ and implies only for a step input, $r(t)=R_0u_{-1}(t)$.the steady state value of the output is obtained by apply final value theorem.

$$C(t)_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0}[\frac{sG(s)}{1+G(s)}\frac{R_0}{s}] == \lim_{s\to 0}[\frac{G(s)}{1+G(s)}R_0]$$

Similarly for $e(t)_{ss}$

$$e(t)_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0}[s\frac{1}{1+G(s)}\frac{R_0}{s}] == \lim_{s\to 0}[\frac{1}{1+G(s)}R_0]$$

Substitute the above two equation to get step error coefficient

$$step\ \ error\ coefficient = \frac{\lim_{s\to 0}[\frac{G(s)}{1+G(s)}R_0]}{\lim_{s\to 0}[\frac{1}{1+G(s)}R_0]}$$

Since both numerator and denominator of the above equation in the limit can't be zero or infinity simultaneously, where $K_m \neq 0$,the indeterminate $0/0$ or ∞/∞ never occur. Thus this equation reduces to K_p. Therefore applying (step error coefficient $= \lim_{s\to 0} G(s) = K_p$) to each type system yields

$$K_p = \lim_{s\to 0}\frac{K_o(1+T_1s)(1+T_2s)......}{(1+T_as)(1+T_bs).....} = K_0 \text{ for type zero system}$$

$K_p = \infty$ for type one system

$K_p = \infty$ for type two system

The ramp error coefficient is defined as:

$$Ramp\,error\ coefficient = \frac{steadystate\ derivative of\ output Dc(t)_{ss}}{steadystate\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_v$$

Therefore applying (ramp error coefficient $=\lim_{s\to 0} sG(s) = K_v$) to each type system yields

$K_v = \lim_{s\to 0} s \dfrac{K_o(1+T_1 s)(1+T_2 s)......}{(1+T_a s)(1+T_b s).....} = 0$ for type zero system

$K_v = K_1$ for type one system

$K_v = \infty$ for type two system

The parabolic error coefficient is defined as:

$$Parabolic\ error\ coefficient = \frac{steady state\ of\ second\ derivative\ of\ output D^2 c(t)_{ss}}{steady state\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_a$$

Therefore applying (ramp error coefficient $=\lim_{s\to 0} s^2 G(s) = K_a$) to each type system yields

$K_a = \lim_{s\to 0} s^2 \dfrac{K_o(1+T_1 s)(1+T_2 s)......}{(1+T_a s)(1+T_b s).....} = 0$ for type zero system

$K_a = 0$ for type one system

$K_a = K_2$ for type two system

Table 6.2. below gives the values of the error coefficients for the Type 0,1, and 2 systems. These values are determined from Table 6.1. The reader should be able to make ready use of Table 6.2 for evaluating the appropriate error coefficient. The error coefficient is used with the definitions given in Table 6.1 to evaluate the magnitude of the steady-state error.

Table 6.2. Steady-State Error Coefficients for Stable Systems

System type	Step error coefficient K_p	Ramp error coefficient K_v	parabolic error coefficient K_a
0	K_0	0	0
1	∞	K_1	0
2	∞	∞	K_2

6.3. Comparison of steady state errors in open loop and closed loop systems:

Consider the open loop control system and closed loop control system shown in Fig. 6.2(a,b).

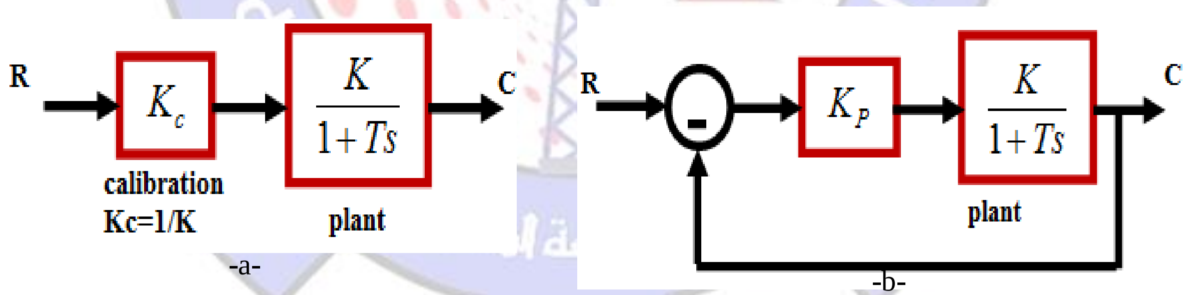


Fig.6.2. (a) block diagram of open loop, (b) closed loop system

In the open loop control system the gain *Kc* is calibrated so that *Kc=1/K*, thus the transfer function of open loop control system is:

$$G_0(s) = \frac{1}{K}\frac{K}{1+sT} = \frac{1}{1+sT}$$

In the closed loop control system the gain K_p of the controller is set so that $K_pK>>1$

Assuming a step input, let us compare the steady state errors for those control systems. For the open loop control system the error signal is:

$e(t) = r(t) - c(t)$; Or

$E(s) = R(s) - C(s)$; $\longrightarrow$ $E(s) = [1 - G_0(s)]R(s)$

The steady state error in a unit step response is

$e_{ss} = \lim_{s\to 0} sE(s)$

$e_{ss} = \lim_{s\to 0} s[1 - G_0(s)]\frac{1}{s}$; $\longrightarrow$ $= 1 - G_0(0)$

Table 6. 3. Steady state error in term of gain K

	Step input r(t)=1	ramp input r(t)=t	parabolic input r(t)=1/2 t^2
Type 0 system	$\frac{1}{1+k}$	∞	∞
Type 1 system	0	$\frac{1}{k}$	∞
Type 2 system	0	0	$\frac{1}{k}$

If the $G_0(0)$,dc gain of the open loop control system is equal to unity, then the steady state error is zero. Due environmental change and aging of the components, however the dc gain $G_0(0)$ will drift from unity as time elapses and steady state error will no longer be equal to zero. Such steady state error will remain until the system is recalibrated see table (6.3).

For the closed loop control system the error signal is:

$$E(s) = R(s) - C(s)$$

$$E(s) = [\frac{1}{1+G(s)}]R(s)$$

Where

$$G(s) = \frac{KK_p}{1+Ts}$$

The steady state error in the unit step response is

$$e_{ss} = \lim_{s\to 0} s[\frac{1}{1+G(s)}]\frac{1}{s}$$

$$= \frac{1}{1+G(0)} \quad = \frac{1}{1+K_pK}$$

In the closed loop system, gain K_p is set at a large value compared to 1/K. Thus the steady state error can be made small but not exactly zero.

Let us assume the following variation in the transfer function of the plant, assuming Kc and K_p constant.

$$\frac{K+\Delta K}{1+Ts}$$

As an example let us assume that K=10,ΔK=1 or ,$\Delta K/K$=0.1. Then the steady state error in the unit step response becomes

$$e_{ss} = 1 - {}^{1}\!/_{K}\,(K+\Delta K) = 1 - 1.1 = -0.1$$

For the closed loop system, if gain K_p is set at 100/K, then the steady state error in the unit step response becomes

$$e_{ss} = \frac{1}{1+G(0)}$$

$$e_{ss} = \frac{1}{1 + \frac{100}{k}(K + \Delta K)}$$

$$e_{ss} = \frac{1}{1 + 110} = 0.009$$

Thus closed loop control system is superior to open loop control system in the presence of environmental changes, aging of components and the like, which definitely affect the steady state performance.

6.4. Solved problems

Prob. 1. A closed loop system as a forward transfer function given by:

$$G(s) = \frac{k}{2s^2 + 16s + 16}, H(s) = \frac{1}{s}$$

Evaluate steady state error for input (r(t)=2+t)); when the gain is equal to (2.2)?

Solution:

Open loop T.F.= G(s)*H(s) $= \frac{k}{s(2s^2 + 16s + 16)}$

$$R(s) = \frac{2}{s} + \frac{1}{s^2}.$$

$$e_{ss} = e_{ss\,position} + e_{ss\,velocity}$$

$$e_{ss\ \ position} = \frac{A}{1 + Kp}, A = constant$$

$$K_p = \lim_{s \to 0} G(s) = \lim \frac{K}{s(2s^2 + 16s + 16)} = \infty$$

So that

$$e_{ss\ \ position} = \frac{2}{1 + \infty} = 0, A = 2$$

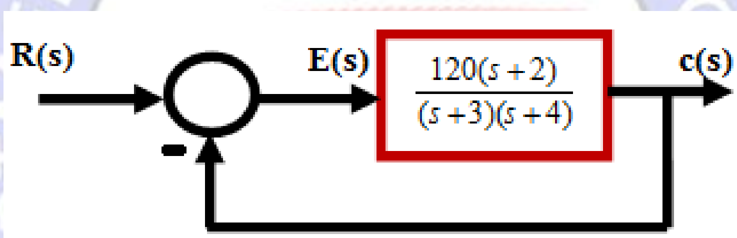
$$K_v = \lim_{s\to 0} G(s) = \lim s \frac{K}{s(2s^2+16s+16)} = 1/4, K = 2$$

$$e_{ss\ velocity} = \frac{B}{Kv} = \frac{1}{1/4} = 4, B = 1$$

$$e_{ss} = e_{ss\ velocity} + e_{ss\ position} = 4+0=4$$

Note; these two values cannot be added where everyone represents system response for certain input.

Prob. 2. Find the steady state errors for the inputs,*5u(t),5tu(t) and 5t²u(t)* to the system shown below, the function *u(t)* is the unit step.



For the input *5u(t)*,the Laplace transform is 5/s, the steady state error will be :

$$e(\infty) = e_{step}(\infty) = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{1+\frac{120*2}{3*4}} = \frac{5}{1+20} = \frac{5}{21}$$

For the input *5tu(t)*,the Laplace transform is $5/s^2$, the steady state error will be :

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

For the input $5t^2$ *u(t)*,the Laplace transform is $10/s^3$, the steady state error will be :

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s\to 0} s^2 G(s)} = \frac{10}{0} = \infty$$

Example 3.A unity feedback system has the following forward transfer function:

$$G(s)=\frac{1000(s+8)}{(s+7)(s+9)}$$

Use Matlab to find $K_{p,}e_{step}(\infty)$ and the closed loop poles to check the stability for the system.

```
numg=1000*[1 8];
deng=poly([-7 -9]);
G=tf (numg,deng);
Kp=dcgain(G)
Estep=1/(1+Kp)
T=feedback(G,1);
poles=pole(T)
```

H.W. Find the value of K to yield a 10% error in the steady state for a unity feedback who has the following forward transfer function. Try to write Matlab code to solve this problem.

$$G(s)=\frac{K(s+12)}{(s+14)(s+18)}$$

Answer: K=189

Lecture No. Seven

Routh's Stability Criterion

7.1. Introduction.

7.2. Routh's Criteria Rules.

7.3. Solved problem for Checking System Stability.

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7.1. Introduction.

The response transform $X_2(s)$ has the general form given by Equation (7.1),which is repeated here in slightly modified form. $X_1(s)$ is the driving transform.

$$X_2(s) = \frac{P(s)}{Q(s)} X_1(s) = \frac{P(s)X_1(s)}{b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s \ + b_o} \qquad (7.1)$$

The stability of the response $X_2(t)$ requires that all zeros of Q(s) have negative real parts. Since it is usually not necessary to find the exact solution when the response is unstable, a simple procedure to determine the existence of zeros with positive real parts is needed. If such zeros of Q(s) with positive real parts are found, the system is unstable and must be modified. Routh's criterion is a simple method of determining the number of zeros with positive real parts without actually solving for the zeros of Q(s). Note that zeros of Q(s) are poles of $X_2(s)$. The characteristic equation is

$$Q(s) = b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s \ + b_o = 0 \qquad (7.2)$$

If the bo term is zero, divide by s to obtain the equation in the form of Equation (7.2). The b's are real coefficients, and all powers of s from s^n to s^0 must be present in the characteristic equation. A necessary but not sufficient condition for stable roots is that all the coefficients in Equation (7.2) must be positive. If any coefficients other than b_0 are zero, or if all the coefficients do not have the same sign, then there are pure imaginary roots or roots with positive real parts and the system is unstable. In that case it is unnecessary to continue if only stability or instability is to be determined. When all the coefficients are present and positive, the system may or may not be stable because there still may be roots on the imaginary axis or in the right-half s plane.

Routh's criterion is mainly used to determine stability. In special situations it may be necessary to determine the actual number of roots in the right half s plane. For these situations the procedure described in this section can be used.

7.2. Routh's Criteria Rules:

The coefficients of the characteristic equation are arranged in the pattern shown in the first two rows of the following Routhian array. These coefficients are then used to evaluate the rest of the constants to complete the array.

S^n	b_n	b_{n-2}	b_{n-4}	b_{n-6}	…….
S^{n-1}	b_{n-1}	b_{n-3}	b_{n-5}	b_{n-7}	…….
S^{n-2}	C1	C2	C3	…….	
S^{n-3}	d1	d2	d3	……	
.	.				
.	.				
S	j1				
S^0	k1				

The constants c1, c2, c3, … etc., in the third row are evaluated as follows:

$$C_1 = \frac{(b_{n-1})(b_{n-1}) - (b_{n-3})(b_n)}{b_{n-1}}$$

$$C_2 = \frac{(b_{n-1})(b_{n-4}) - (b_{n-5})(b_n)}{b_{n-1}}$$

$$C_3 = \frac{(b_{n-1})(b_{n-6}) - (b_{n-7})(b_n)}{b_{n-1}}$$

This pattern is continued until the rest of the c's are all equal to zero. Then the d row is formed by using the sn-1 and sn-2 rows. The constants are:

$$d_1 = \frac{C_1(b_{n-3}) - (b_{n-1})C_2}{C_1}$$

$$d_2 = \frac{C_1(b_{n-5}) - (b_{n-1})C_3}{C_1}$$

$$d_3 = \frac{C_1(b_{n-7}) - (b_{n-1})C_4}{C_1}$$

This process is continued until no more d terms are present. The rest of the rows are formed in this way down to the s0 row. The complete array is triangular, ending with the s0 row. Notice that the s1 and s0 rows contain only one term each. Once the array has been found, Routh's criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes of sign of the coefficients in the first column. Therefore, the system is stable if all terms in the first column have the same sign.

7.3. Solved problem for Checking System Stability.

Prob.1. Check the stability of the control system that it has characteristic equation in the following:

$Q(s)= s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$?

Solution:

The Routhian array is formed by using the procedure described above**:**

$$\begin{array}{llll} S^5 & 1 & 10 & 152 \\ S^4 & 1 & 72 & 240 \\ S^3 & -62 & -88 & \\ S^2 & 70.6 & 240 & \\ S & 122.6 & & \\ S^0 & 240 & & \end{array}$$

In the first column there are two changes of sign, from 1 to -62 and from 62 to 70.6; therefore, Q(s) has two roots in the right-half s plane (RHP).Note that this criterion gives the number of roots with positive real parts but does not tell the values of the roots. If the characteristic equation is factored, the roots are $s_1 = -3$, $s_{2,3} = -1 \pm j\sqrt{3}$, and $s_{4,5} = +2 \pm j\sqrt{4}$. This calculation confirms that there are two roots with positive real parts. The Routh criterion does not distinguish between real and complex roots.

Prob.2. Check the stability of the control system that it has the following characteristic equation(C.E):

C.E.= $s^2 + 3s + 2$?

Solution:

$$\begin{array}{lll} S^2 & 1 & 2 \\ S^1 & 3 & 0 \\ S^0 & \frac{3*2-0*1}{3}=2 & \end{array}$$

Because no change in the first column (pivoted column), there are no poles in the right hand side (RHS) and hence the system is stable.

Prob.3. Check the stability of the control system that it has clc's eqn. in the following:

Q(s)= $s^4 + 3s^3 + s^2 + 3s + 1$?

Solution:

The routh's array:

$$\begin{array}{llll} S^4 & 1 & 1 & 1 \\ S^3 & 3 & 3 & 0 \\ S^2 & 0 & 1 & \\ S & ? & ? & \\ S & ? & & \end{array}$$

This is one of the special cases ,so that when we get zero in routh's array to fill full this theory replace the zero by symbol (δ) and then can be determined the range of stability for this system:

S^4 1 1 1
S^3 3 3 0
S^2 δ 1
S $(3\delta-3)/\delta$ 0
S 0

If we consider δ a very small positive number [it has either a very small positive or a very small negative and this is optional and both of them gives same final result]

A=(3δ – 3)/ δ=3-3/ δ

Lim A= 3-∞ , A= -ve

s→0

This mean, there are two sign changes (from +ve to –ve and from –ve to +ve) . In other words two poles in the right hand side of s-plane, therefore the system is unstable.

Prob.4. The open loop transfer function of a unity negative feedback control system shown below, find the number of poles in the left half ,right half of s-plane and on imaginary axis(jw).

$$G_{open\,loop}(s)=\frac{128}{S(S^7+3S^6+10S^5+24S^4+48S^3+96S^2+128S+192}$$

Solution:

The characteristic equation of system is

$$Q(S) = S^8 + 3S^7 + 10S^6 + 24S^5 + 48S^4 + 96S^3 + 128S^2 + 192S + 128$$

Routh's table can constructed as follows

S^8	1	10	48	128	128
S^7	3	24	96	196	0
S^6	2	16	64	128	
S^5	0	0	0	0	
S^4	?	?	?	?	
S^3	?				
S^2	?				
S^1	?				
S^0	?				

This is the second special case ,when all the row elements are zeros.

To solve this, return to first even polynomial (S^6) and form a new polynomial which is called auxiliary equation as follows:

$$P(s) = 2S^6 + 16S^4 + 64S^2 + 128$$

But the auxiliary equation in the simplest form and this can be done for each row of the Routh's table.

$$P(s) = S^6 + 8S^4 + 32S^2 + 64$$

Next step ,differentiate this polynomial with respect to S to form the coefficients that replace the row of zeros:

$$\frac{dP(s)}{ds} = 6S^5 + 32S^3 + 64S = 0$$

Now the coefficients of S^5 in the main table will be as follows:

S^5 6 32 64

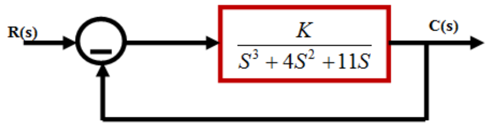
Then complete the table as in the previous examples. If your calculation is correct you find two sign changes from the even polynomial(sixth order). Hence ,the system has two right half plane poles. Because of the symmetry about the origin ,the even polynomial must have an equal number in the left half plane poles. The remaining two will be on J-w axis. There are no sign change from the beginning of the table down to the even polynomial(sixth order). Therefore the rest of the polynomial has no right half plane poles.

The final result will be two poles in the right half, four poles in the left half and two poles on the imaginary axis. Hence the system is unstable.

In the Matlab ,we will come to the closed loop control system and the code will be as follows:

```
numg=128;
deng =[1 3 10 24 . . .
48 96 128  196  0]
G1=tf(numg,deng);
G=feedback(G1,1)
poles=pole(G)
```

Prob.5. For the system shown below, find the minimum possible values of K at which the system is unstable?



Solution:

C.E= 1+O/L.T.F

Routh's array

$$\begin{array}{llll} S^3 & 1 & 11 & 0 \\ S^2 & 4 & K & 0 \\ S^1 & (44-K)/4 & 0 & \\ S^0 & K & & \end{array}$$

O/L.T.F= G(s).H(s); H(s)=1;

C.E. = 1+ K/s^3+4s^2 +11s= s^3+4s^2 +11s+K=0

For stable sys. (44 – K)/ 4 $\geq$ 0

For unstable sys. (44 – K)/ 4 $\leq$ 0

Therefore; the min. value for stable is

K $\geq$ 44.

Lecture No. Eight

Root Locus.

8.1. Introduction.

8.2. General Rules of Root Locus.

8.3. Examples.

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8.1. Introduction.

To facilitate the application of the root-locus method, the following rules are established for K > 0. These rules are based upon the interpretation of the angle condition and an analysis of the characteristic equation. These rules can be extended for the case where K < 0. The rules for both K > 0 and K < 0 are listed in Sec. 7.16 for easy reference. The rules presented aid in obtaining the root locus by expediting the plotting of the locus. The root locus can also be obtained by using the MATLAB program. These rules provide checkpoints to ensure that the computer solution is correct. They also permit rapid sketching of the root locus, which provides a qualitative idea of achievable closed-loop system performance.

8.2. General Rules of Root Locus.

Rule 1: Number of Branches of the Locus:

The characteristic equation C.E.(s)=1+G(s)H(s)=0 is of degree n=mu; therefore, there are n roots. As the open-loop sensitivity K is varied from zero to infinity, each root traces a continuous curve. Since there are n roots, there are the same numbers of curves or branches in the complete root locus. Since the degree of the polynomial C.E.(s) is determined by the poles of the open-loop transfer function, the number of branches of the root locus is equal to the number of poles of the open-loop transfer function.

Rule 2: Real-Axis Locus:

In Fig. 1 are shown a number of open-loop poles and zeros. If the angle condition is applied to any search point such as s1 on the real axis, the

angular contribution of all the poles and zeros on the real axis to the left of this point is zero. The angular contribution of the complex-conjugate poles to this point is 360°. (This is also true for complex-conjugate zeros.) Finally, the poles and zeros on the real axis to the right of this point each contribute 180° (with the appropriate sign included). From Eq.(1) the angle of G(s)H(s) to the point s1 is given by

$$\beta = \sum(\text{angles of denominator terms}) - \sum(\text{angles of numerator terms}) = \begin{cases} (1+2h)180^\circ & \text{for } K > 0 \\ h360^\circ & \text{for } K < 0 \end{cases} \quad \text{- - - - - - (1)}$$

or

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 + [(\phi_4)_{+j} + (\phi_4)_{-j}] - (\psi_1 + \psi_2) = (1+2h)180^\circ$$

$$180^\circ + 0^\circ + 0^\circ + 0^\circ + 360^\circ - 0^\circ - 0^\circ = (1+2h)180^\circ$$

Therefore, s1 is a point on a branch of the locus. Similarly, it can be shown that the point s2 is not a point on the locus. The poles and zeros to the left of a point s on the real axis and the 360° contributed by the complex-conjugate poles or zeros do not affect the odd-multiple-of-180° requirement. Thus, if the total number of real poles and zeros to the right of a search point s on the real axis is odd, this point lies on the locus. In Fig.1 the root locus exists on the real axis from p0 to p1, z1 to p2, and p3 to z2. All points on the real axis between z1 and p2 in Fig.1 satisfy the angle condition and are therefore points on the root locus.

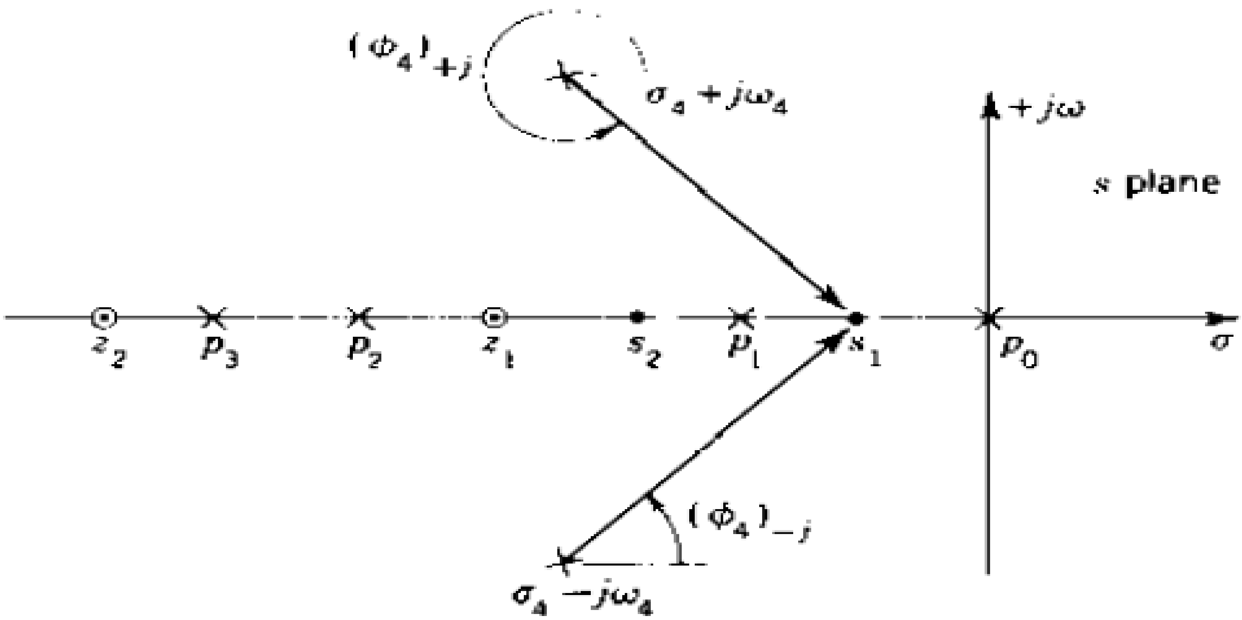


Fig.1. Determination of the real-axis locus.

However, there is no guarantee that this section of the real axis is part of just one branch. Fig.2 a illustrate the situation where part of the real axis between a pole and a zero is divided into three sections that are parts of three different branches.

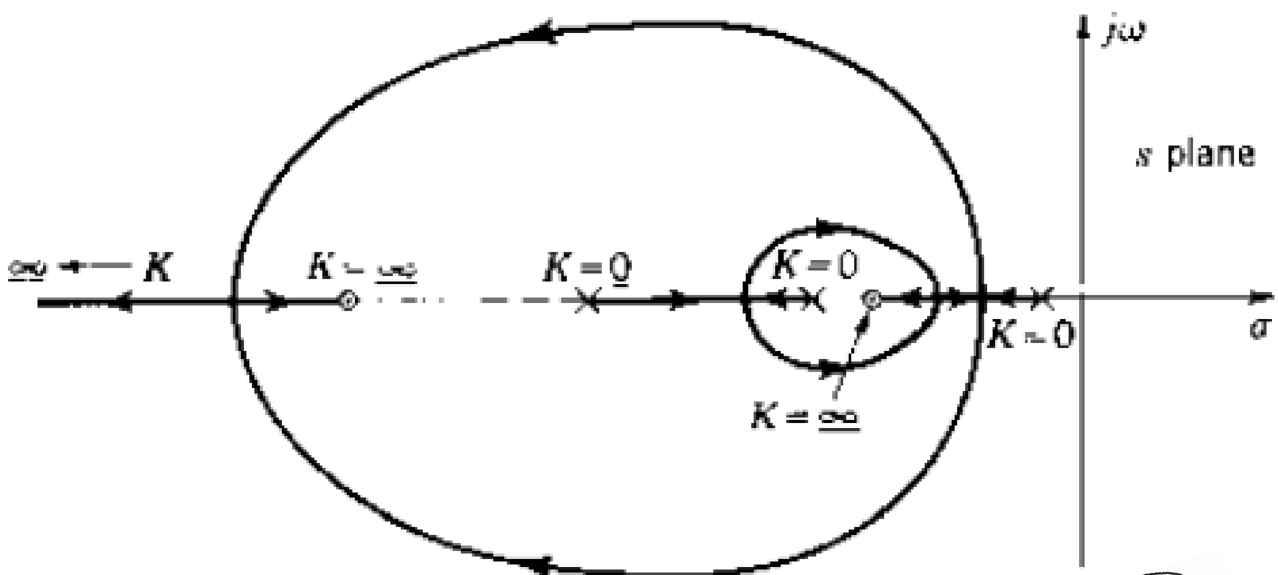


Fig.2.

د. يوسف المشهداني

Rule 3: Locus End Points:

The magnitude of the loop sensitivity that satisfies the magnitude condition is given by Eq.(3) and has the general form in Eq.(4),

$$|K| = \frac{|s^m| \cdot |s-p_1| \cdot |s-p_2| \cdots |s-p_u|}{|s-z_1| \cdots |s-z_w|} = \text{loop sensitivity} \quad \text{- - - - - (3)}$$

$$|W(s)| = K = \frac{\prod_{c=1}^{n} |s-p_c|}{\prod_{h=1}^{w} |s-z_h|} \quad \text{- - - - - (4)}$$

Since the numerator and denominator factors of Eq.(4) locate the poles and zeros, respectively, of the open-loop transfer function, the following conclusions can be drawn:

1) **When s=pc (the open-loop poles), the loop sensitivity K is zero.**
2) **When s=zh (the open-loop zeros), the loop sensitivity K is infinite.**

When the numerator of Eq.(4) is of higher order than the denominator, then s=1 also makes K infinite, thus being equivalent in effect to a zero. Thus, the locus starting points (K=0) are at the open-loop poles and the locus ending points (K=1) are at the open-loop zeros (the point at infinity being considered as an equivalent zero of multiplicity equal to the quantity n - w).

Rule 4: Asymptotes of Locus as s Approaches Infinity.

Plotting of the locus is greatly facilitated by evaluating the asymptotes approached by the various branches as s takes on large values. Taking the limit of G(s)H(s) as s approaches infinity, based on Eqs.(5) and (6), yields

$$G(s)H(s) = \frac{K(s-z_1)\cdots(s-z_w)}{s^m(s-p_1)\cdots(s-p_u)} = \frac{K\prod_{h=1}^{w}(s-z_h)}{s^m\prod_{c=1}^{u}(s-p_c)} \qquad \text{------- (5)}$$

$$G(s)H(s) = \frac{K(s-z_1)\cdots(s-z_w)}{s^m(s-p_1)\cdots(s-p_u)} = -1 \qquad \text{----------(6)}$$

$$\lim_{s\to\infty} G(s)H(s) = \lim_{s\to\infty}\left[K\frac{\prod_{h=1}^{n}(s-z_h)}{\prod_{c=1}^{n}(s-p_c)}\right] = \lim_{s\to\infty}\frac{K}{s^{n-w}} = -1 \qquad \text{-------(7)}$$

Remember that K in Eq.(7) is still a variable in the manner prescribed previously, thus allowing the magnitude condition to be met. Therefore, as s →∞, There are n-w asymptotes of the root locus, and their angles are given by

$$-K = s^{n-w}$$

$$|-K| = |s^{n-w}| \qquad \text{Magnitude condition} \qquad \text{----(8)}$$

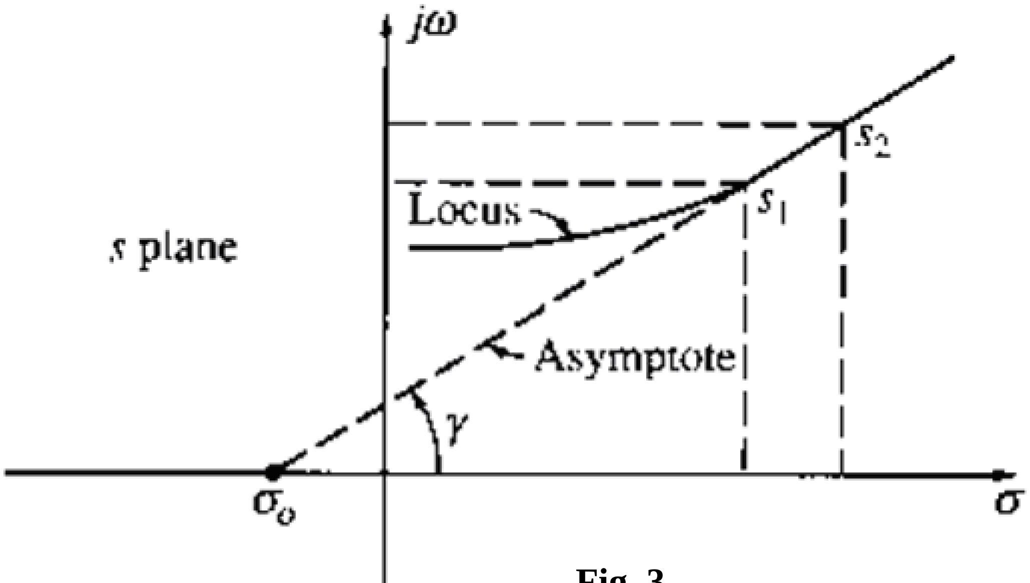
$$\angle -K = \angle s^{n-w} = (1+2h)180^\circ \qquad \text{Angle condition}$$

Rewriting Eq. (9) gives $(n-w)\angle s = (1+2h)180^\circ$ or

$$\gamma = \frac{(1+2h)180^\circ}{n-w} \qquad \text{as } s\to\infty \qquad \text{---------(9)}$$

$$\gamma = \frac{(1+2h)180^\circ}{[\text{number of poles of } G(s)H(s)] - [\text{number of zeros of } G(s)H(s)]} \qquad \text{---(10)}$$

Eq.(10) reveals that, no matter what magnitude s may have, after a sufficiently large value has been reached, the argument (angle) of s on the root locus remains constant. For a search point that has a sufficiently large magnitude, the open-loop poles and zeros appear to it as if they had collapsed into a single point. Therefore, the branches are asymptotic to straight lines whose slopes and directions are given by Eq. (10) (see Fig.3). These asymptotes usually do not go through the origin. The correct real-axis intercept of the asymptotes is obtained from Rule 5.



$s_1=|s_1|\angle 1$

$s_2=|s_2|\angle 2$

.

.

$s_x=|s_x|\angle x$

.

$s_x=\infty\angle \gamma$

Fig. 3

Rule 5: Real-Axis Intercept of the Asymptotes

The real-axis crossing so of the asymptotes can be obtained by applying the theory of equations. The result is

$$\sigma_o = \frac{\sum_{c=1}^{n} \mathrm{Re}(p_c) - \sum_{h=1}^{w} \mathrm{Re}(z_h)}{n-w}$$

The asymptotes are not dividing lines, and a locus may cross its asymptote. It may be valuable to know from which side the root locus approaches its asymptote. The locus lies exactly along the asymptote if the

pole-zero pattern is symmetric about the asymptote line extended through the point so. Rule 6: Breakaway Point

Rule 6: Breakaway Point on the Real Axis

The branches of the root locus start at the open-loop poles where K=0 and end at the finite open-loop zeros or at s=1.When the root locus has branches

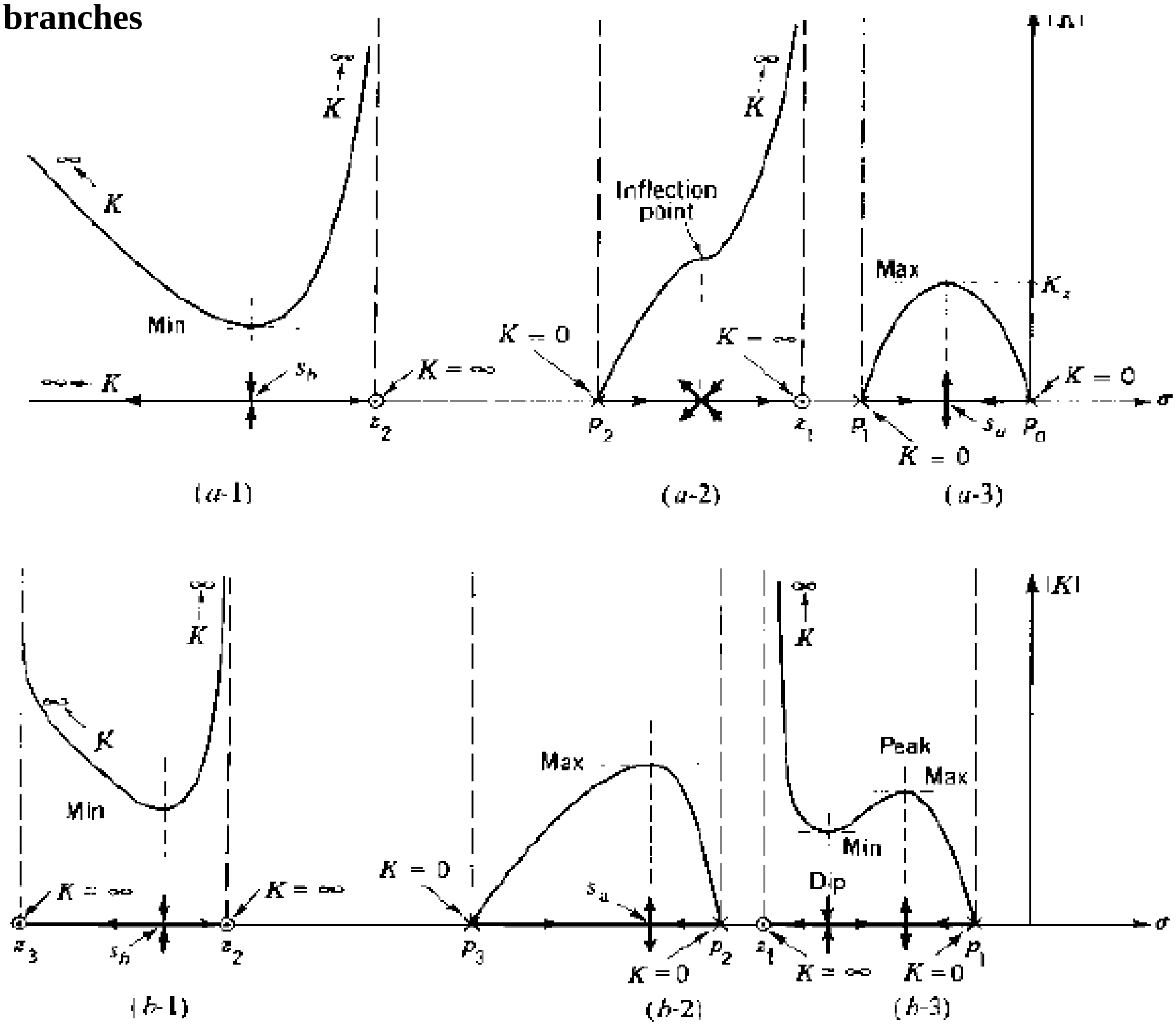


Fig.4.

on the real axis between two poles, there must be a point at which the two branches breakaway from the real axis and enter the complex region of

the s plane in order to approach zeros or the point at infinity. (Examples are shown in Fig. 4.a-3: between p0 and p1, and in Fig. 4.b-2: between p2 and p3.) For two finite zeros (see Fig. 4.b-1) or one finite zero and one at infinity (see Fig. 4.a-1) the branches are coming from the complex region and enter the real axis. In Fig. 4.a-3 between two poles there is a point s a for which the loop sensitivity K z is greater than for points on either side of s a on the real axis.

In other words, since K starts with a value of zero at the poles and increases in value as the locus moves away from the poles, there is a point somewhere in between where the K's for the two branches simultaneously reach a maximum value. This point is called the breakaway point. Plots of K vs. s utilizing Eq.(3) are shown in Fig.4 for the portions of the root locus that exist on the real axis for K > 0. The point sb for which the value of K is a minimum between two zeros is called the break-in point. The breakaway and break-in points can easily be calculated for an open-loop pole-zero combination for which the derivatives of W(s)=K is of the second order. As an example, if

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

then

$$W(s) = s(s+1)(s+2) = -K$$

Multiplying the factors together gives

$$W(s) = s^3 + 3s^2 + 2s = -K \qquad \text{- - - - - - - - - - - - (11)}$$

When s^3+s^2+2s is a minimum, -K is a minimum and K is a maximum. Thus, by taking the derivative of this function and setting it equal to zero, the points can be determined:

د. يوسف المشهداني

$$\frac{dW(s)}{ds} = 3s^2 + 6s + 2 = 0$$

or

$$s_{a,b} = -1 \pm 0.5743 = -0.4257, \; -1.5743$$

Since the breakaway point's s a for K > 0 must lie between s=0 and s=-1, in order to satisfy the angle condition, the value is s_a =-0:4257;

The other point, sb=-1.5743, is the break-in point on the root locus forK <0.

Substituting s_a= 0.4257 into Eq. (11) gives the value of K at the breakaway Point for K > 0 as

$$K = -[(-0.426)^3 + (3)(-0.426)^2 + (2)(-0.426)] = 0.385$$

When the derivative of W(s) is of higher order than 2,a digital-computer program can be used to calculate the roots of the numerator polynomial of dW(s)/ds; these roots locate the breakaway and break-in points. Note that it is possible to have both a breakaway and a break-in point between a pole and zero (finite or infinite) on the real axis, as shown in Figs.4a-1, 7.11a-2,

and 4.b-3. The plot of Kvs. s for a locus between a pole and zero falls into one of the following categories:

1. **The plot clearly indicates a peak and a dip, as illustrated between p1 and z1 in Fig. 4.b-3. The peak represents a 'maximum' value of K that identifies a break-in point.**
2. **The plot contains an inflection point. This occurs when the breakaway and break-in points coincide, as is the case between p2 and z1 in Fig. 4a-2.**

3. The plot does not indicate a dip-and-peak combination or an inflection point. For this situation there are no break-in or breakaway points.

The next geometrical shortcut is the rapid determination of the direction in which the locus leaves a complex pole or enters a complex zero. Although in Fig.5.a complex pole is considered, the results also hold for a complex zero.

In Fig.5.a, an area about p2 is chosen so that l2 is very much smaller than l0, l1, l3, and (l)1. For illustrative purposes, this area has been enlarged many times in Fig.5b. Under these conditions the angular contributions from all the other poles and zeros, except p2, to a search point anywhere in this area are approximately constant. They can be considered to have values determined as if the search point were right at p2.Applying the angle condition to this small area yields

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 - \psi_1 = (1 + 2h)180^\circ$$

or the departure angle is

$$\phi_{2_D} = (1 + 2h)180^\circ - (\phi_0 + \phi_1 + 90^\circ - \psi_1)$$

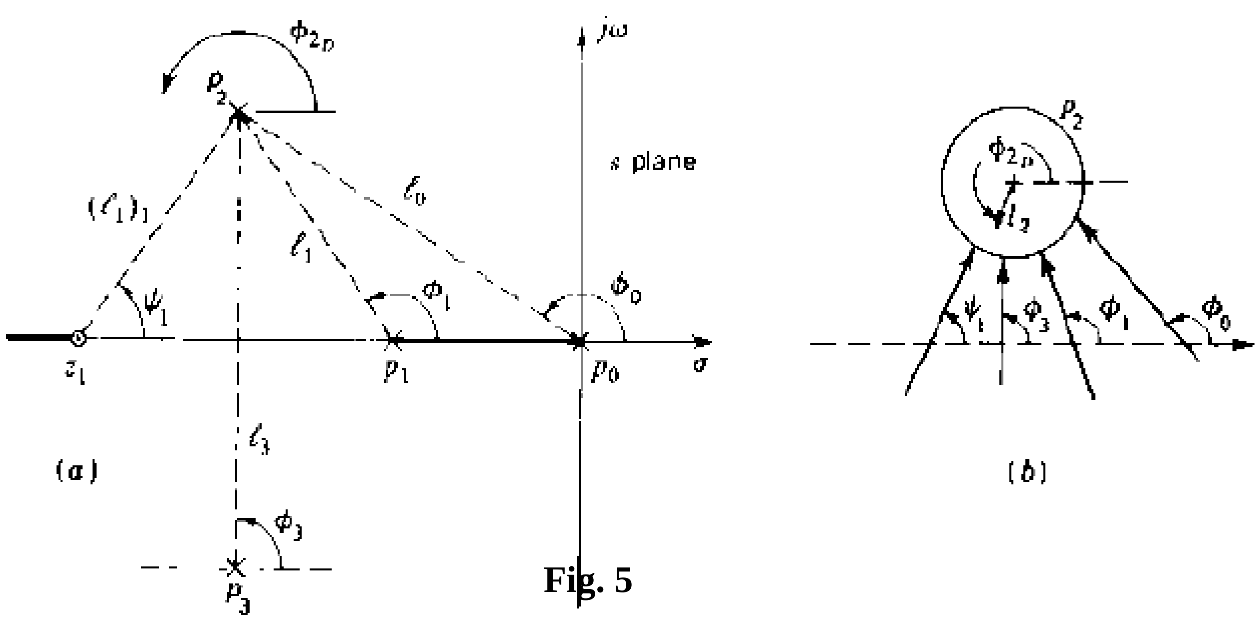


Fig. 5

In a similar manner the approach angle to a complex zero can be determined. For an open-loop transfer function having the pole-zero arrangement shown in Fig.6, the approach angle c1 to the zero z1 is given by

$$\psi_{1_A} = (\phi_0 + \phi_1 + \phi_2 - 90^\circ) - (1 + 2h)180^\circ$$

In other words, the direction of the locus as it leaves a pole or approaches a zero can be determined by adding up, according to the angle condition, all the

angles of all vectors from all the other poles and zeros to the pole or zero in question. Subtracting this sum from (1+2h)180^{o} gives the required direction.

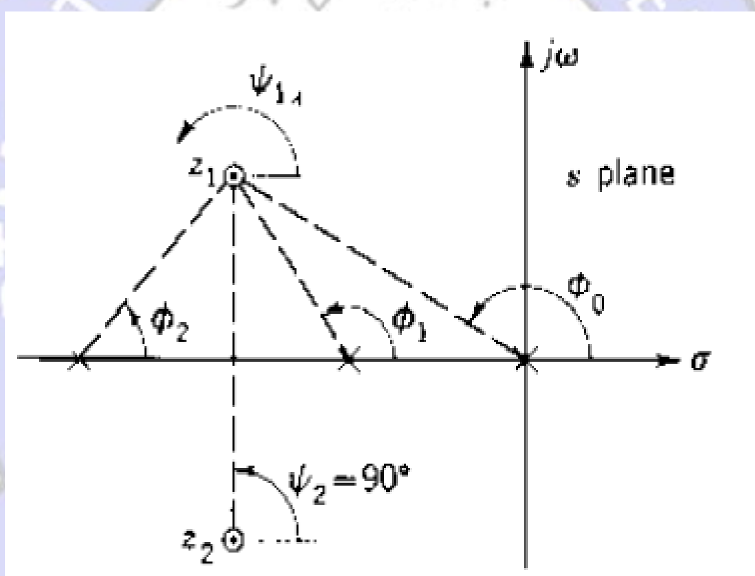


Fig. 6

Rule 7: Complex Pole (or Zero): Angle of Departure:

The next geometrical shortcut is the rapid determination of the direction in which the locus leaves a complex pole or enters a complex zero. Although inFig.7.a complex pole is considered, the results also hold for a complex zero.

In Fig. 7.a, an area about p2 is chosen so that l2 is very much smaller than l0, l1, l3, and (l) 1. For illustrative purposes, this area has been enlarged many times in Fig. 7.b. Under these conditions the angular contributions from all the other poles and zeros, except p2, to a search point anywhere

in this area are approximately constant. They can be considered to have values determined as if the search point were right at p2.Applying the angle condition to this small area yields. In a similar manner the approach angle to a complex zero can be determined. For an open-loop transfer function having the pole-zero

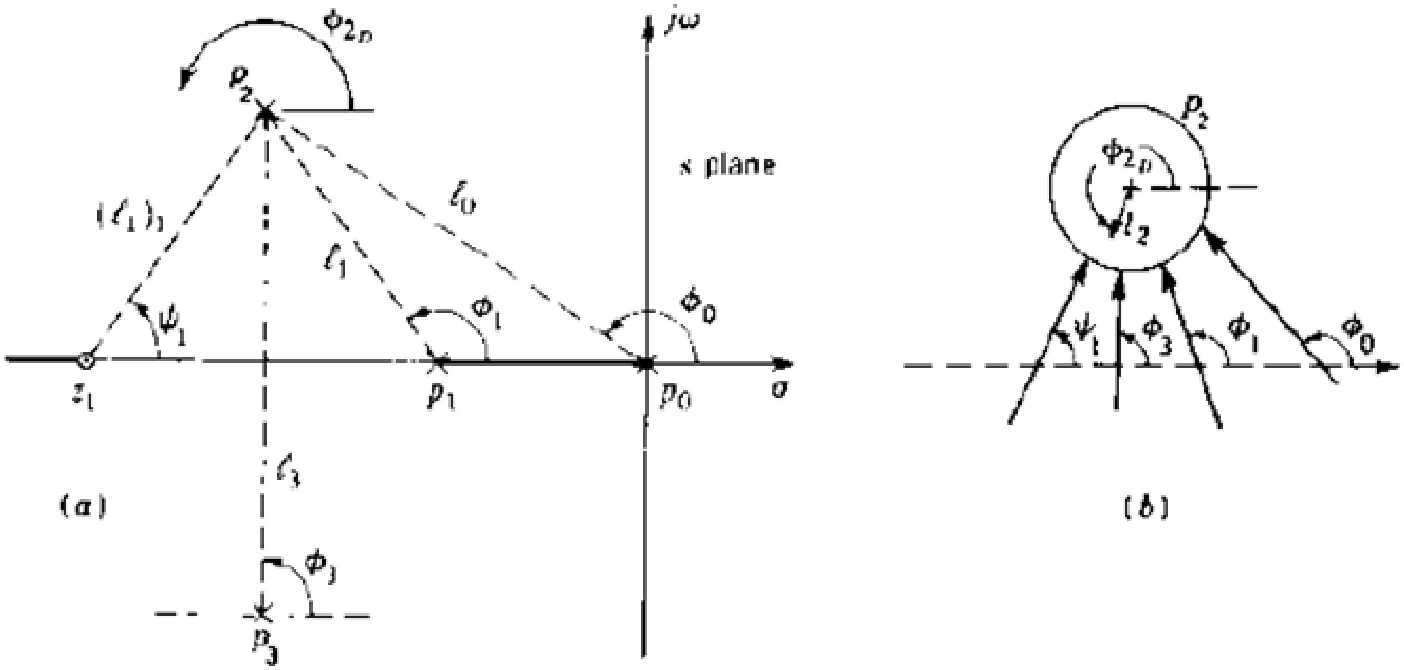


Fig.7. a ,b: Angle condition in the vicinity of a complex pole.

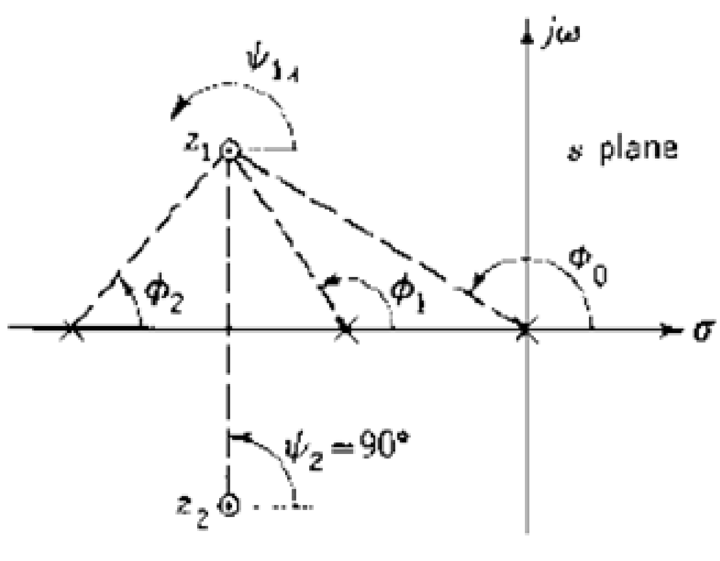


Fig. 8. Angle condition in the vicinity of a complex zero.

Arrangement shown in Fig.8, the approach angle c1 to the zero z1 is given by

$$\psi_{1_A} = (\phi_0 + \phi_1 + \phi_2 - 90^\circ) - (1 + 2h)180^\circ$$

In other words, the direction of the locus as it leaves a pole or approaches a zero can be determined by adding up, according to the angle condition, all the angles of all vectors from all the other poles and zeros to the pole or zero in question. Subtracting this sum from (1+2h)180 gives the required direction.

Rule 8: Imaginary-Axis Crossing Point:

In cases where the locus crosses the imaginary axis into the right-half s plane,
the crossover point can usually be determined by Routh's method or by similar means. For example, if the closed-loop characteristic equation D1D2+ N1N2=0 is of the form.

$$s^3 + bs^2 + cs + Kd = 0$$

the Routhian array is

$$\begin{array}{c|ll} s^3 & 1 & c \\ s^2 & b & Kd \\ s^1 & (bc - Kd)/b & \\ s^0 & Kd & \end{array}$$

An undamped oscillation may exist if the s_1 row in the array equals zero. For this condition the auxiliary equation obtained from the s 2 row is

$$bs^2 + Kd = 0$$

and its roots are

$$s_{1,2} = \pm j\sqrt{\frac{Kd}{b}} = \pm j\omega_n \qquad \text{- - - - - - - - - - - - (12)}$$

The loop sensitivity term K is determined by setting the s1 row to zero:

K =bc/d

For K > 0, Eq. (12) gives the natural frequency of the undamped oscillation. This corresponds to the point on the imaginary axis where the

locus crosses over into the right-half s plane. The imaginary axis divides the s plane into stable and unstable regions. Also, the value of K from Eq. (*K =bc/d*) determines the value of the loop sensitivity at the crossover point. For values of K < 0 the term in the s0 row is negative, thus characterizing an unstable response. The limiting values for a stable response are therefore

$$0 < K < bc/d$$

In like manner, the crossover point can be determined for higher-order characteristic equations. For these higher-order systems care must be exercised in analyzing all terms in the first column that contain the term K in order to obtain the correct range of values of gain for stability.

Rule 9: Intersection or Non-intersection of Root-Locus Branches:

The theory of complex variables yields the following properties:

1. A value of s that satisfies the angle condition of Eq. (1) is a point on the root locus. If dW(s)/ds ≠ 0 at this point, there is one and only one branch of the root locus through the point.

2. If the first y_1 derivatives of W(s) vanish at a given point on the root locus, there are y branches approaching and y branches leaving this point; thus, there are root-locus intersections at this point. The angle between two adjacent approaching branches is given by

$$\lambda_y = \pm\frac{360^\circ}{y}$$

Also, the angle between a branch leaving and an adjacent branch that is approaching the same point is given by

$$\theta_y = \pm\frac{180^\circ}{y}$$

Fig. 9. illustrates these angles at s=-3 ,with $\theta y=45^o$ and $\lambda y=90^o$.

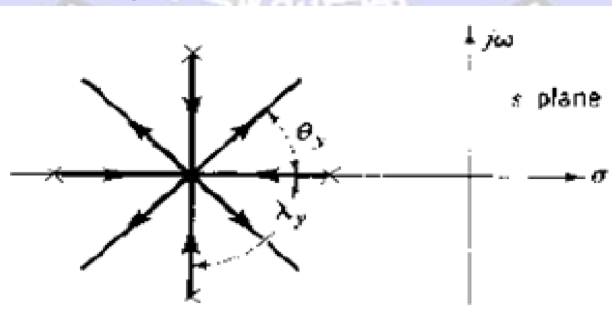


Fig. 9. Root locus for

$$G(s)H(s) = \frac{K}{(s+2)(s+4)(s^2+6s+10)}.$$

Ex.(1).

Find $C(s)/R(s)$ with $\zeta = 0.5$ for the dominant roots (roots closest to the imaginary axis) for the feedback control system represented by

$$G(s) = \frac{K_1}{s(s^2/2600 + s/26 + 1)} \qquad \text{and} \qquad H(s) = \frac{1}{0.04s + 1}$$

Rearranging gives

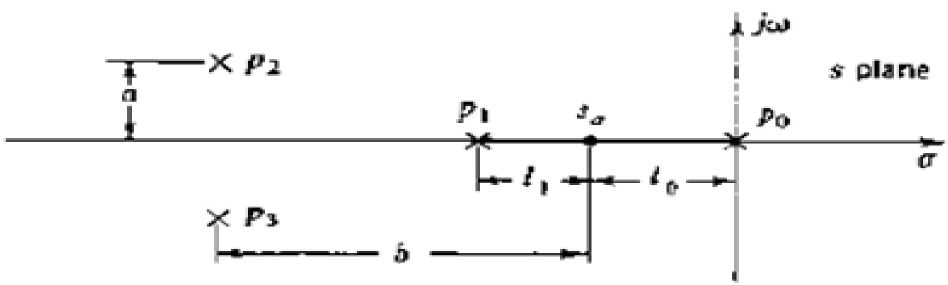
$$G(s) = \frac{2600K_1}{s(s^2 + 100s + 2600)} = \frac{N_1}{D_1} \qquad \text{and} \qquad H(s) = \frac{25}{s+25} = \frac{N_2}{D_2}$$

Thus,

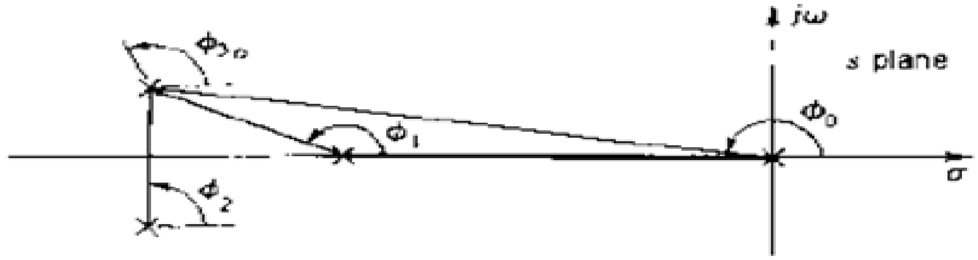
$$G(s)H(s) = \frac{65{,}000K_1}{s(s+25)(s^2+100s+2600)} = \frac{K}{s^4 + 125s^3 + 5100s^2 + 65{,}000s}$$

where $K = 65{,}000K_1$.

1. The poles of $G(s)H(s)$ are plotted on the s plane in Fig. below the values of these poles are $s = 0, -25, -50 + j10, -5\bar{0} - j10$.



Location of the breakaway point.



Determination of the departure angle.

The system is completely unstable for $K < 0$. Therefore, this example is solved only for the condition $K > 0$.

2. There are four branches of the root locus.
3. The locus exists on the real axis between 0 and -25.
4. The angles of the asymptotes are

$$\gamma = \frac{(1+2h)180^\circ}{4} = \pm 45^\circ,\ \pm 135^\circ$$

5. The real-axis intercept of the asymptotes is

$$\sigma_o = \frac{0-25-50-50}{4} = -31.25$$

6. The breakaway point s_a on the real axis between 0 and -25 is found by solving $dW(s)/ds = 0$

$$-K = s^4 + 125s^3 + 5100s^2 + 65{,}000s$$
$$\frac{d(-K)}{ds} = 4s^3 + 375s^2 + 10{,}200s + 65{,}000 = 0$$
$$s_a = -9.15$$

7. The angle of departure ϕ_{3_D} from the pole $-50 + j\,10$ is obtained from

$$\phi_0 + \phi_1 + \phi_2 + \phi_{3_D} = (1+2h)180^\circ$$
$$168.7^\circ + 158.2^\circ + 90^\circ + \phi_{3_D} = (1+2h)180^\circ$$
$$\phi_{3_D} = 123.1^\circ$$

Similarly, the angle of departure from the pole $-50 + j\,10$ is -123.1°.

8. The imaginary-axis intercepts are obtained from

$$\frac{C(s)}{R(s)} = \frac{2600K_1(s+25)}{s^4 + 125s^3 + 5100s^2 + 65{,}000s + 65{,}000K_1}$$

The Routhian array for the denominator of $C(s)/R(s)$, which is the characteristic polynomial, is

$$\begin{array}{c|lll}
s^4 & 1 & 5100 & 65{,}000K_1 \\
s^3 & 1 & 520 \text{ (after division by 125)} & \\
s^2 & 1 & 14.2K_1 \text{ (after division by 4580)} & \\
s^1 & 520 - 14.2K_1 & & \\
s^0 & 14.2K_1 & &
\end{array}$$

Pure imaginary roots exist when the s^1 row is zero. This occurs when $K_1 = 520/14.2 = 36.6$. The auxiliary equation is formed from the s^2 row:

$$s^2 + 14.2K_1 = 0$$

and the imaginary roots are

$$s = \pm j\sqrt{14.2K_1} = \pm j\sqrt{520} = \pm j\,22.8$$

9. Additional points on the root locus are found by locating points that satisfy the angle condition

$$\angle s + \angle s+25 + \angle s+50-j10 + \angle s+50+j10 = (1+2m)180^\circ$$

The root locus is shown in Fig.10

10. The radial line for $\zeta = 0.5$ is drawn on the graph of Fig. 10 at the angle

$$\eta = \cos^{-1} 0.5 = 60^\circ$$

The dominant roots obtained from the graph are

$$s_{1,2} = -6.6 \pm j11.4$$

11. The gain is obtained from the expression

$$K = 65{,}000K_1 = |s| \cdot |s+25| \cdot |s+50-j10| \cdot |s+50+j10|$$

Inserting the value $s_1 = -6.6 + j11.4$ into this equation yields

$$K = 65{,}000K_1 = 598{,}800$$
$$K_1 = 9.25$$

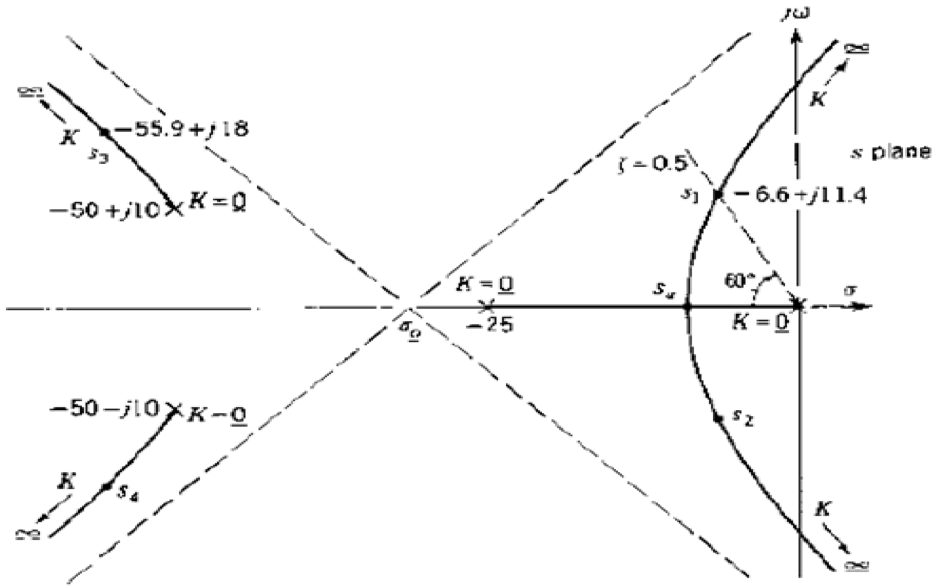


Fig 10. Root locus for $G(s)H(s) = \dfrac{65{,}000K_1}{s(s+25)(s^2+100s+2600)}$.

12. The other roots are evaluated to satisfy the magnitude condition $K = 598{,}800$. The remaining roots of the characteristic equation are

$$s_{3,4} = -55.9 \pm j18.0$$

The real part of the additional roots can also be determined by using the rule from Eq. (7.68):

$$0 - 25 + (-50 + j10) + (-50 - j10)$$
$$= (-6.6 + j11.4) + (-6.6 - j11.4) + (\sigma + j\omega_d) + (\sigma - j\omega_d)$$

This gives $\sigma = -55.9$

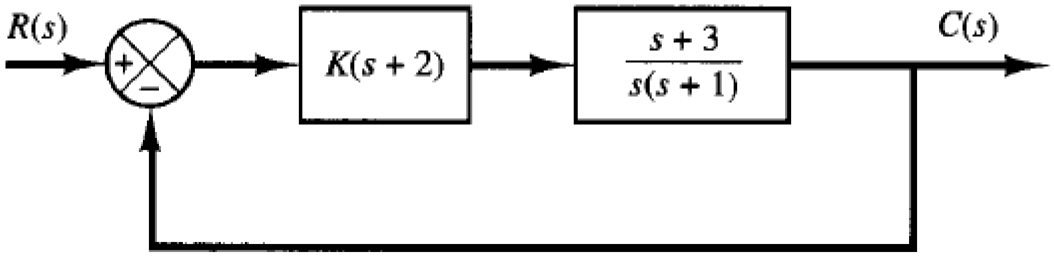
By using this value, the roots can be determined from the root locus as $-55.9 \pm j18.0$.

13. The control ratio, using values of the roots obtained in steps 10 and 12, is

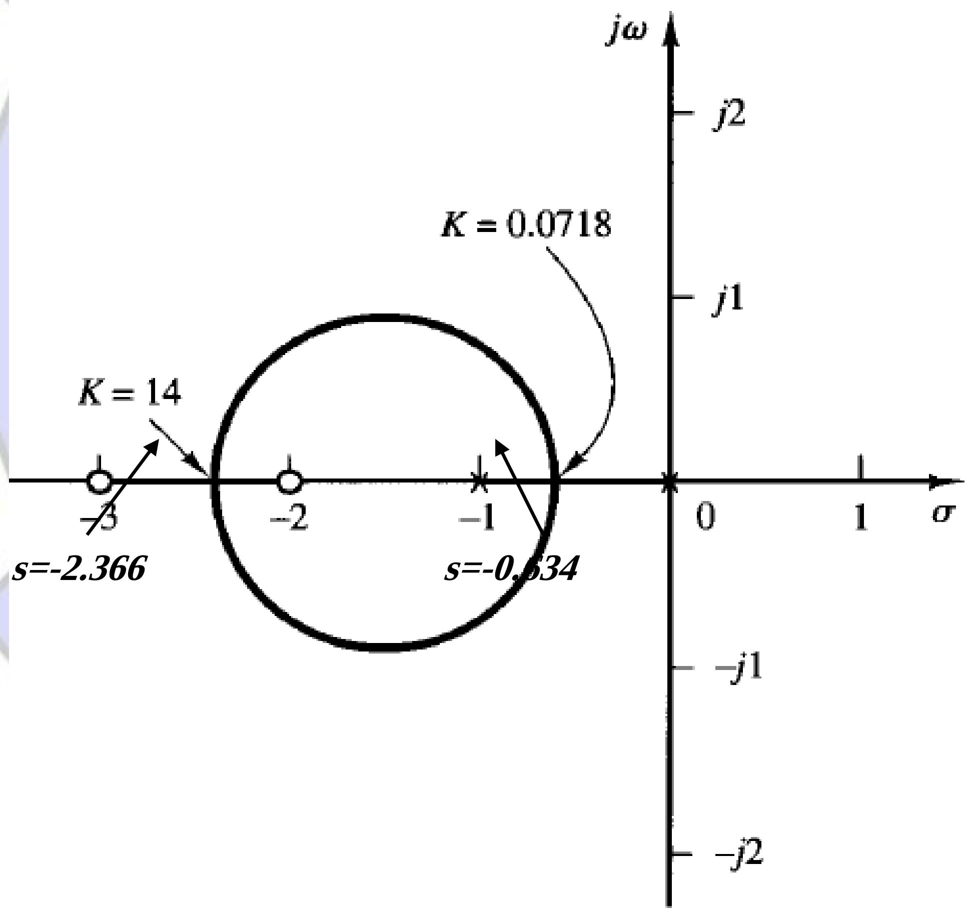
$$\frac{C(s)}{R(s)} = \frac{N_1 D_2}{\text{factors determined from root locus}}$$

$$= \frac{1}{(s+6.6+j11.4)(s+6.6-j11.4)} \times \frac{24{,}040(s+25)}{(s+55.9+j18)(s+55.9-j18)}$$
$$= \frac{24{,}040(s+25)}{(s^2+13.2s+173.5)(s^2+111.8s+3450)}$$

Ex.(2). Plot Root Loci for system shown below:



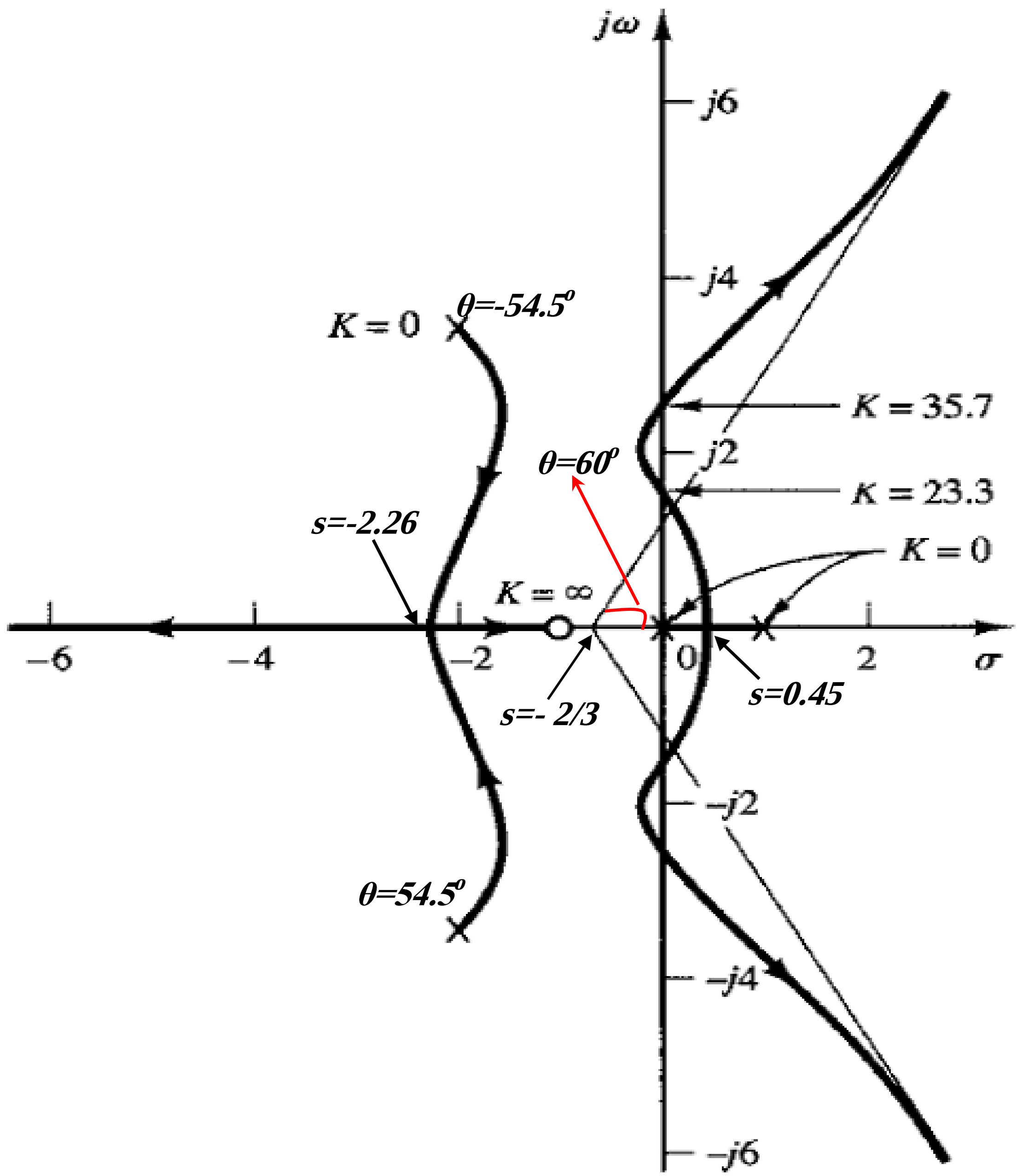
Solution:



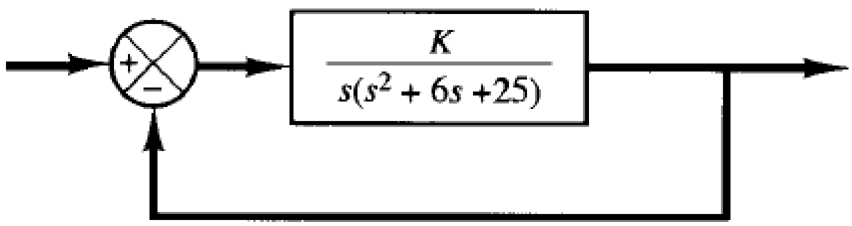
Ex.(3). Plot the root loci for the system has the following T.F:

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

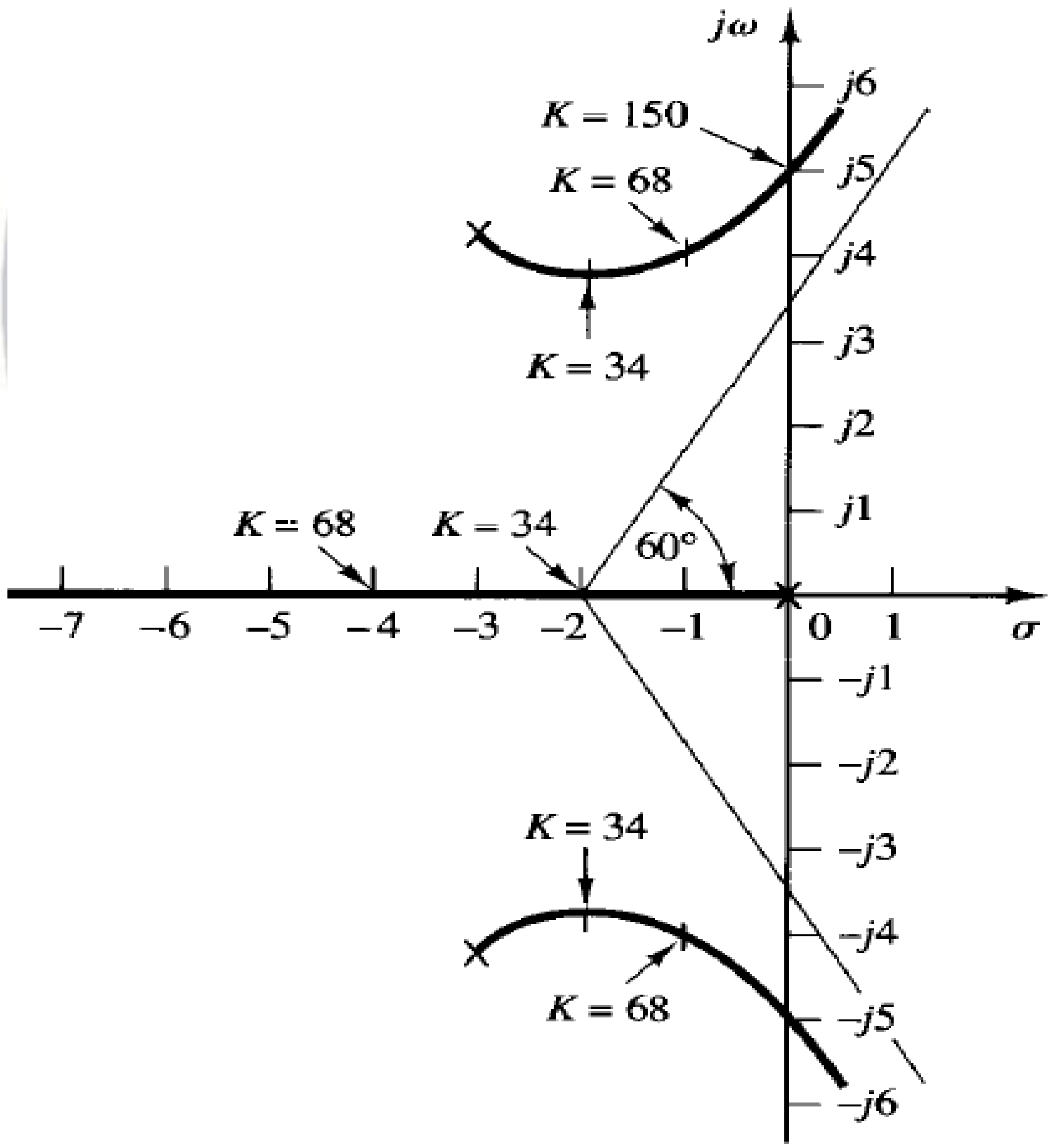
Solution:



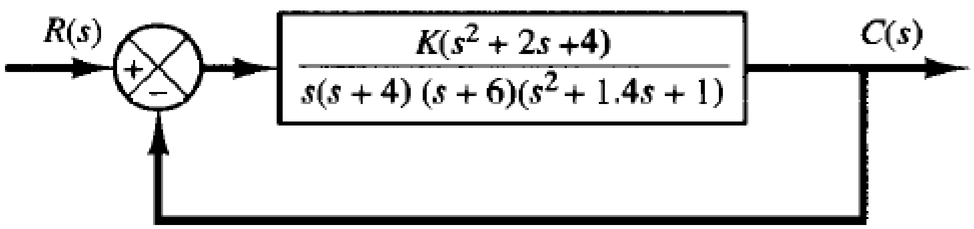
Ex.(4). Plot Root Loci for system shown below:



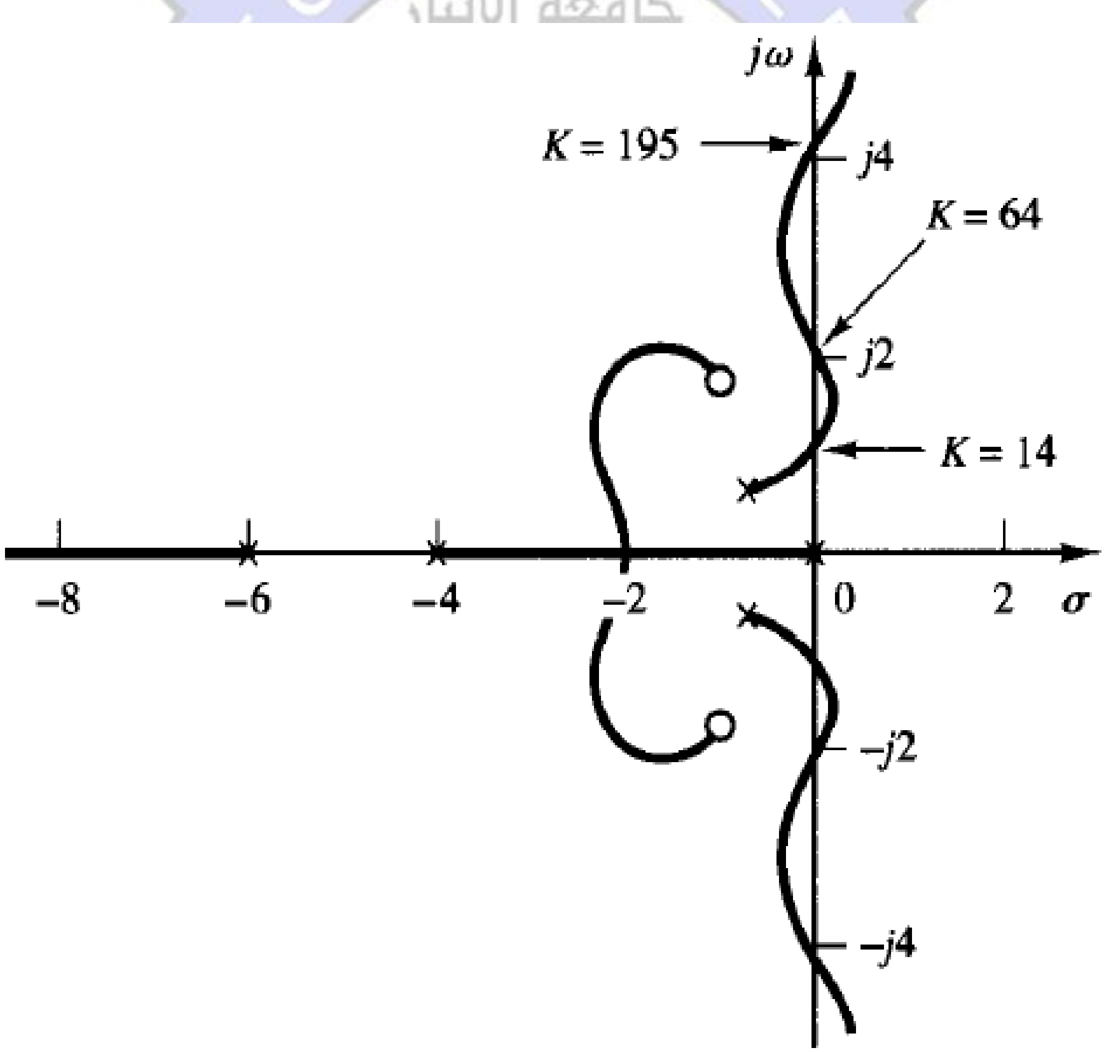
Solution:



Ex.(5). Plot Root Loci for system shown below:



Solution:



Ex.(6). Plot the root loci for the system has the following T.F:

$$G(s)\,H(s) = \frac{K(s^2 - 1)}{(s^2 + 1)(s^2 + 4)}$$

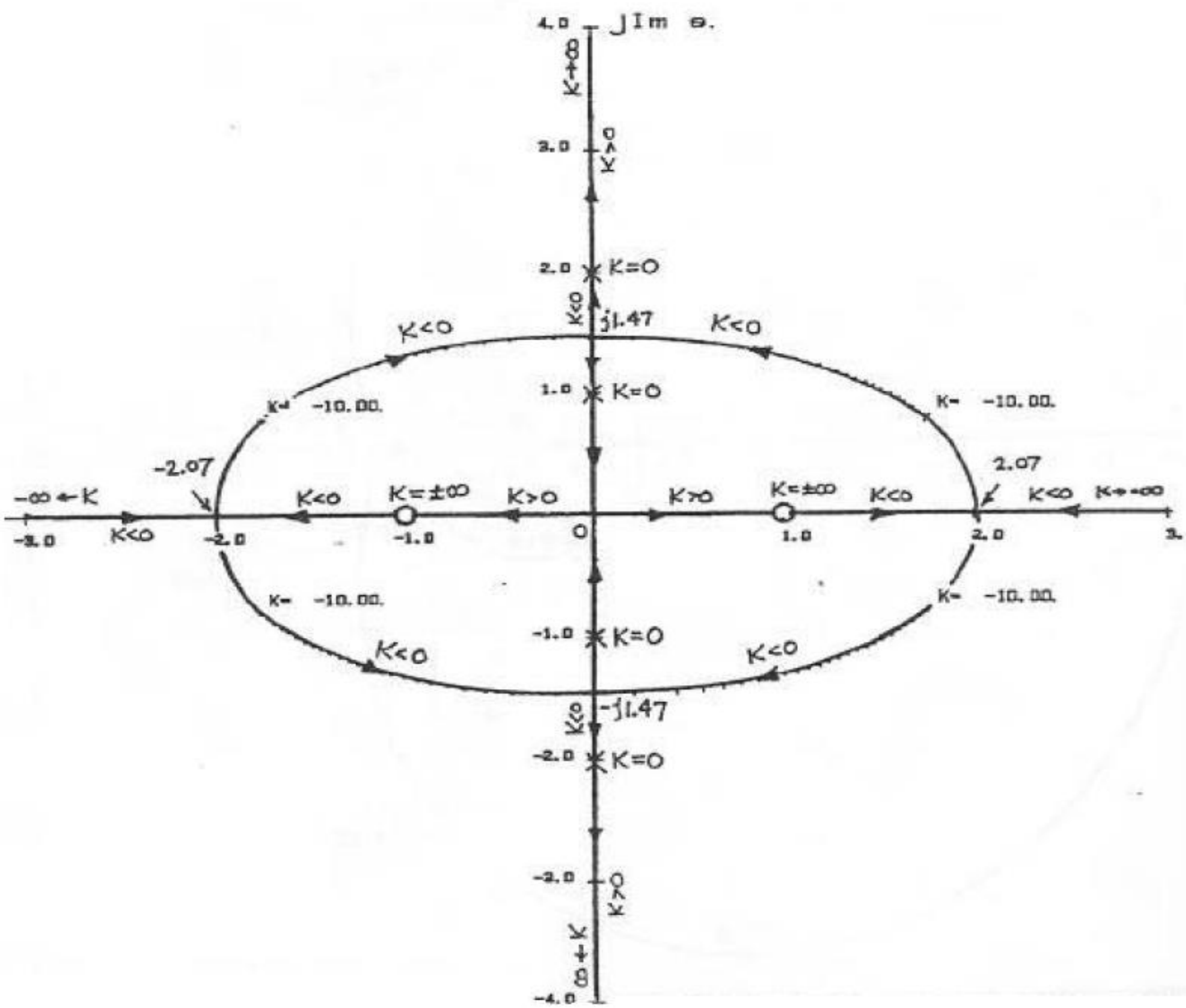
Solution:

Asymptotes: $K > 0$: 90^o, 270^o $K < 0$: 0^o, 180^o

Intersect of Asymptotes: $\sigma_1 = \dfrac{-1+1}{4-2} = 0$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: -2.07, 2.07, $-j1.47$, $j1.47$



Ex.(7). Plot the root loci for the system has the following T.F:

Solution:

$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Asymptotes: $K > 0$: 180^o $K < 0$: 0^o

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: $-1.21,\quad -2.4,\quad -9.07,\quad 0.683,\quad 0,\quad 0$