COURSE TOPICS:

- *Introduction of Types of Structures and Loads*
- *Concepts of Determinacy and Stability of Structures*
- *Analysis of Statically Determinate Shear and Moment Diagrams for a Frame*
- *Analysis of Statically Determinate Trusses*
- *Classification of Coplanar Trusses*
- *Influence Lines for Statically Determinate Beams and Trusses*
- *Maximum Influence at a Point due to a Series of Concentrated Loads*
- *Absolute Maximum Shear and Moment*
- *Deflections: The Double Integration Method*
- *Deflections : Conjugate-Beam Method*
- *Deflections Using Energy Methods*

REFERENCES:

✓ *Kenneth M. Leet, Chia-Ming Uang, Anne M. Gilbert.,* ***(2010), "Fundamentals of Structural Analysis"***, 4th edition, McGraw-Hill Education.

✓ *Hibbeler R. C.,* ***(2012), "Structural Analysis",*** 8th edition, Pearson Prentice Hall.

✓ *Igor A. Karnovsky , Olga Lebed*, ***(2006), "Advanced Methods of Structural Analysis"***, 5th Edition, Elsevier Ltd.

✓ *Alan Williams,* ***(2009), "Structural Analysis: In Theory and Practice",*** International Codes Council.

✓ *William M.C.McKenzi.,* ***(2006), "Examples in Structural Analysis",*** Taylor & Francis, New York, USA.

✓ *Mau S. T.,* ***(2003), "Fundamentals of Structural Analysis",*** The Library of Congress, United States.

✓ *Morgson T. H. G.,* ***(2005), "Structural and Stress Analysis",*** 2nd edition, Butterworth-Heinemann.

1 INTRODUCTION OF TYPES OF STRUCTURES AND LOADS

1.1 Introduction:

A *structure* refers to a system of connected parts used to support a load. Important examples related to civil engineering include buildings, bridges, and towers; and in other branches of engineering, ship and aircraft frames, tanks, pressure vessels, mechanical systems, and electrical supporting structures are important.

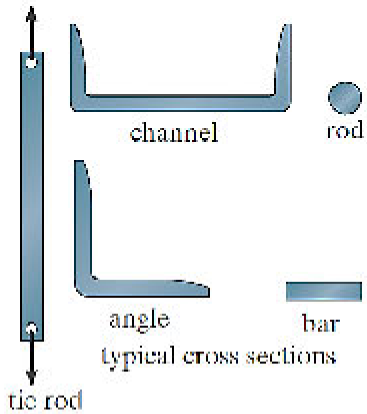
When designing a structure to serve a specified function for public use, the engineer must account for its safety, esthetics, and serviceability, while taking into consideration economic and environmental constraints.

1.2 Classification of Structures

It is important for a structural engineer to recognize the various types of elements composing a structure and to be able to classify structures as to their form and function.

1.2.1 Structural Elements.

Some of the more common elements from which structures are composed are as follows.

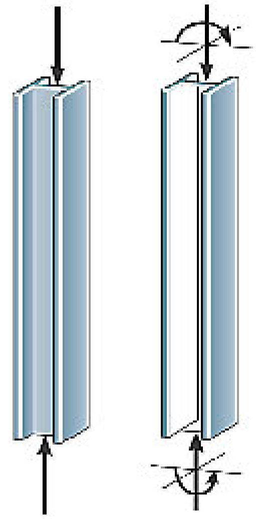


Tie Rods.

- ✓Structural members subjected to a *tensile force* are often referred to as *tie rods* or *bracing struts*.
- ✓Due to the nature of this load, these members are rather slender, and are often chosen from rods, bars, angles, or channels.

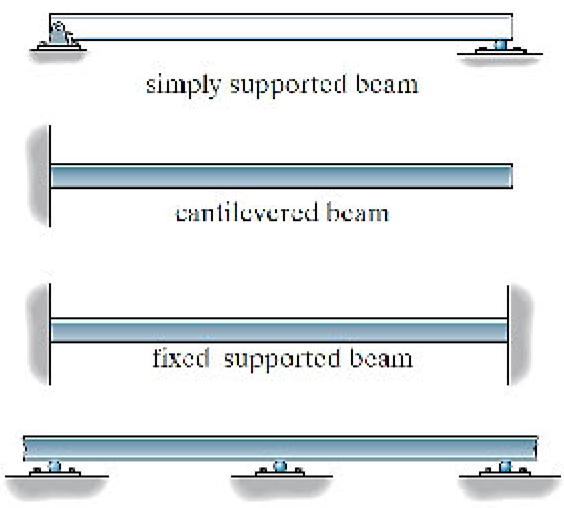
Columns.

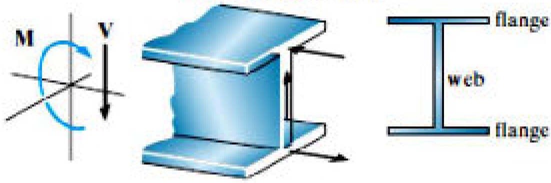
- ✓Members that are generally vertical and resist axial compressive loads are referred to as *columns*.
- ✓Tubes and wide-flange cross sections are often used for metal columns
- ✓Circular and square cross sections with reinforcing rods are used for those made of concrete.
- ✓Occasionally, columns are subjected to both an axial load and a bending moment as shown in the figure. These members are referred to as *beam columns*.



Beams.

- ✓ Beams are usually straight horizontal members used primarily to carry vertical loads.
- ✓ Often they are classified according to the way they are supported
- ✓ When the cross section varies the beam is referred to as tapered or haunched. Beam cross sections may also be "built up" by adding plates to their top and bottom.
- ✓ Beams are primarily designed to resist *bending moment*, however, if they are short and carry large loads, the internal shear force may become quite large and this force may govern their design.
- ✓ When the material used for a beam is a metal such as steel or aluminum, the cross section is most efficient.
- ✓ The forces developed in the top and bottom *flanges* of the beam form the necessarycouple used to resist the applied moment ***M***, whereas the *web* is effective in resisting the applied shear *V*.
- ✓ The cross section is commonly referred to as a "*wide flange*", and it is normally formed as a single unit in a rolling mill in lengths up to **75 ft** (**23 m**).
- ✓ When the beam is required to have a very large span and the loads applied are rather large, the cross section may take the form of a *plate girder*. This member is fabricated by using a large plate for the web and welding or bolting plates to its ends for flanges.
- ✓ The girder is often transported to the field in segments, and the segments are designed to be





The prestressed concrete girders are simply supported

spliced or joined together at points where the girder carries a small internal moment.

- ✓ Concrete beams generally have rectangular cross sections, since it is easy to construct this form directly in the field.
- ✓ Because concrete is rather weak in resisting tension, steel "*reinforcing rods*" are cast into the beam within regions of the cross section subjected to tension.
- ✓ Precast concrete beams or girders are fabricated at a shop or yard in the same manner and then transported to the job site.
- ✓ Beams made from timber may be sawn from a solid piece of wood or laminated. *Laminated* beams are constructed from solid sections of wood, which are fastened together using high-strength glues.

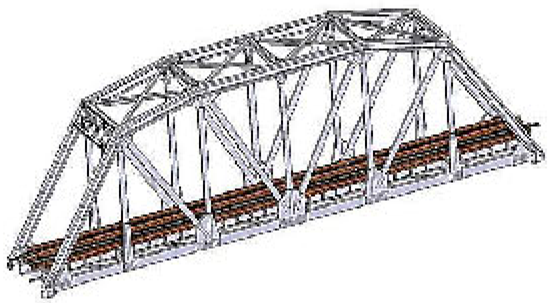
Beams made from timber

1.2.2 Types of Structures.

The combination of structural elements and the materials from which they are composed is referred to as a *structural system*.

Trusses.

- ✓ When the span of a structure is required to be large and its depth is not an important criterion for design, a truss may be selected. *Trusses* consist of slender elements, usually arranged in triangular fashion.
- ✓ *Planar trusses* are composed of members that lie in the same plane and are frequently used for bridge and roof support.
- ✓ Whereas *space trusses* have members extending in three dimensions and are suitable for derricks and towers.
- ✓ Due to the geometric arrangement of its members, loads that cause the entire truss to bend are converted into tensile or compressive forces in the members.
- ✓ One of the primary advantages of a truss, compared to a beam, is that it uses less material to support a given load.



- ✓ Also, a truss is constructed from *long and slender elements,* which can be arranged in various ways to support a load.
- ✓ Most often it is economically feasible to use a truss to cover sp*+ans ranging from **30 ft (9 m)** to **400 ft (122 m)**, although trusses have been used on occasion for spans of greater lengths.

Cables and Arches.

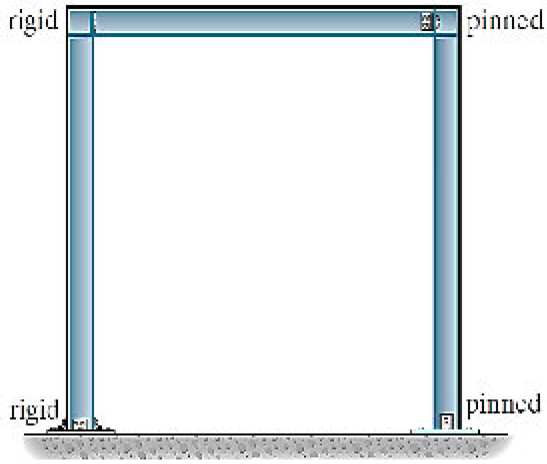
- ✓ Two other forms of structures used to span long distances are the cable and the arch.
- ✓ *Cables* are usually flexible and carry their loads in tension.
- ✓ They are commonly used to support bridges, and building roofs.
- ✓ The cable has an advantage over the beam and the truss, especially for spans that are greater than **150 ft (46 m)**. Because they are always in tension, cables will not become unstable and suddenly collapse, as may happen with beams or trusses.
- ✓ Furthermore, the truss will require added costs for construction and increased depth as the span increases.
- ✓ Use of cables, on the other hand, is limited only by their sag, weight, and methods of anchorage.
- ✓ The *arch* achieves its strength in compression, since it has a reverse curvature to that of the cable.
- ✓ The arch must be rigid, however, in order to maintain its shape, and this results in secondary loadings involving shear and moment, which must be considered in its design.
- ✓ Arches are frequently used in bridge structures, dome roofs, and for openings in masonry walls.

Cables support their loads in tension.

Arches support their loads in compression

Frames.

- ✓ Frames are often used in buildings and are composed of beams and columns that are either pin or fixed connected. Like trusses, frames extend in two or three dimensions.
- ✓ The loading on a frame causes bending of its members, and if it has rigid joint connections, this structure is generally "indeterminate" from a standpoint of analysis.
- ✓ The strength of such a frame is derived from the moment interactions between the beams and the columns at the rigid joints.



Frame members are subjected to internal axial, shear, and moment loadings.

Surface Structures.

- ✓ A *surface structure* is made from a material having a very small thickness compared to its other dimensions.
- ✓ Sometimes this material is very flexible and can take the form of a tent or air-inflated structure.
- ✓ In both cases the material acts as a membrane that is subjected to pure tension.
- ✓ Surface structures may also be made of rigid material such as reinforced concrete. As such they may be shaped as folded plates, cylinders, or hyperbolic paraboloids, and are referred to as *thin plates* or *shells*.
- ✓ These structures act like cables or arches since they support loads primarily in tension or compression, with very little bending.

The roof of the "Georgia Dome" in Atlanta, Georgia can be considered as a thin membrane.

1.3 Loads

In order to design a structure, it is therefore necessary to first specify the loads that act on it. The design loading for a structure is often specified in codes.

In general, the structural engineer works with two types of codes: ***general building codes*** and ***design codes***.

General building codes specify the requirements of governmental bodies for minimum design loads on structures and minimum standards for construction.

Design codes provide detailed technical standards and are used to establish the requirements for the actual structural design.

Table 1–1 lists some of the important codes used in practice. It should be realized, however, that codes provide only a general guide for design.
The ultimate responsibility for the design lies with the structural engineer.

TABLE 1–1 Codes
General Building Codes
Minimum Design Loads for Buildings and Other Structures, ASCE/SEI 7-10, American Society of Civil Engineers
International Building Code
Design Codes
Building Code Requirements for Reinforced Concrete, Am. Conc. Inst. (ACI)
Manual of Steel Construction, American Institute of Steel Construction (AISC)
Standard Specifications for Highway Bridges, American Association of State Highway and Transportation Officials (AASHTO)
National Design Specification for Wood Construction, American Forest and Paper Association (AFPA)
Manual for Railway Engineering, American Railway Engineering Association (AREA)

Dead Loads.

- ✓ *Dead loads* consist of the weights of the various structural members and the weights of any objects that are permanently attached to the structure.
- ✓ In some cases, a structural dead load can be estimated satisfactorily from simple formulas based on the weights and sizes of similar structures.
- ✓ Through experience one can also derive a "feeling" for the magnitude of these loadings. For example, the average weight for timber buildings is **40 – 50 lb/ft^2 (1.9 – 2.4 Kn/m^2)**, for steel framed buildings it is **60 – 75 lb/ft^2 (2.9 – 3.6 kN/m^2)**, and for reinforced concrete buildings it is **110 – 130 lb/ft^2 (5.3 – 6.2 kN/m^2)**.

✓ The densities of typical materials used in construction are listed in Table 1–2, and a portion of a table listing the weights of typical building components is given in Table 1–3.

TABLE 1–2 Minimum Densities for Design Loads from Materials*

	lb/ft^3	kN/m^3
Aluminum	170	26.7
Concrete, plain cinder	108	17.0
Concrete, plain stone	144	22.6
Concrete, reinforced cinder	111	17.4
Concrete, reinforced stone	150	23.6
Clay, dry	63	9.9
Clay, damp	110	17.3
Sand and gravel, dry, loose	100	15.7
Sand and gravel, wet	120	18.9
Masonry, lightweight solid concrete	105	16.5
Masonry, normal weight	135	21.2
Plywood	36	5.7
Steel, cold-drawn	492	77.3
Wood, Douglas Fir	34	5.3
Wood, Southern Pine	37	5.8
Wood, spruce	29	4.5

*Reproduced with permission from American Society of Civil Engineers *Minimum Design Loads for Buildings and Other Structures*, ASCE/SEI 7-10. Copies of this standard may be purchased from ASCE at www.pubs.asce.org.

TABLE 1–3 Minimum Design Dead Loads*

Walls	psf	kN/m^2
4-in. (102 mm) clay brick	39	1.87
8-in. (203 mm) clay brick	79	3.78
12-in. (305 mm) clay brick	115	5.51
Frame Partitions and Walls		
Exterior stud walls with brick veneer	48	2.30
Windows, glass, frame and sash	8	0.38
Wood studs 2 × 4 in., (51 × 102 mm) unplastered	4	0.19
Wood studs 2 × 4 in., (51 × 102 mm) plastered one side	12	0.57
Wood studs 2 × 4 in., (51 × 102 mm) plastered two sides	20	0.96
Floor Fill		
Cinder concrete, per inch (mm)	9	0.017
Lightweight concrete, plain, per inch (mm)	8	0.015
Stone concrete, per inch (mm)	12	0.023
Ceilings		
Acoustical fiberboard	1	0.05
Plaster on tile or concrete	5	0.24
Suspended metal lath and gypsum plaster	10	0.48
Asphalt shingles	2	0.10
Fiberboard, $\frac{1}{2}$-in. (13 mm)	0.75	0.04

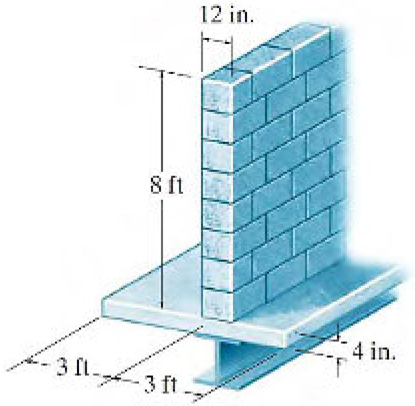
*Reproduced with permission from American Society of Civil Engineers *Minimum Design Loads for Buildings and Other Structures*, ASCE/SEI 7-10.

Even if the material is specified, the unit weights of elements reported in codes may vary from those given by manufacturers, and later use of the building may include some changes in dead loading. As a result, estimates of dead loadings can be in error by **15%** to **20%** or more.

Normally, the dead load is not large compared to the design load for simple structures such as a beam or a single-story frame; however, for multistory buildings it is important to have an accurate accounting of all the dead loads in order to properly design the columns, especially for the lower floors.

Example 1.3.1

The floor beam in the figure is used to support the **6-ft** width of a lightweight plain concrete slab having a thickness of **4 in**. The slab serves as a portion of the ceiling for the floor below, and therefore its bottom is coated with plaster. Furthermore, an **8-ft**-high, **12-in**.-thick lightweight solid concrete block wall is directly over the top flange of the beam. Determine the loading on the beam measured per foot of length of the beam.



Solution

Using the data in Tables 1–2 and 1–3, we have

Concrete slab: $[8\ \text{lb}/(\text{ft}^2.\text{in})](4\ \text{in})(6\ \text{ft}) = 192\ \text{lb/ft}$

Plaster ceiling: $(5\ \text{lb/ft}^2)(6\ \text{ft}) = 30\ \text{lb/ft}$

Block wall: $(105\ \text{lb/ft}^3)(8\ \text{ft})(1\ \text{ft}) = \underline{840\ \text{lb/ft}}$

Total load $1062\ \text{lb/ft} = 1.06\ \text{k/ft}$ *Ans.*

Live Loads.

- ✓ *Live Loads* can vary both in their magnitude and location. They may be caused by the weights of objects temporarily placed on a structure, moving vehicles, or natural forces. The minimum live loads specified in codes are determined from studying the history of their effects on existing structures.
- ✓ Usually, these loads include additional protection against excessive deflection or sudden overload.

TABLE 1–4 Minimum Live Loads*

Occupancy or Use	Live Load psf	Live Load kN/m^2
Assembly areas and theaters		
Fixed seats	60	2.87
Movable seats	100	4.79
Garages (passenger cars only)	50	2.40
Office buildings		
Lobbies	100	4.79
Offices	50	2.40
Storage warehouse		
Light	125	6.00
Heavy	250	11.97
Residential		
Dwellings (one- and two-family)	40	1.92
Hotels and multifamily houses		
Private rooms and corridors	40	1.92
Public rooms and corridors	100	4.79
Schools		
Classrooms	40	1.92
Corridors above first floor	80	3.83

*Reproduced with permission from *Minimum Design Loads for Buildings and Other Structures*, ASCE/SEI 7-10.

Building Loads.

- ✓ The floors of buildings are assumed to be subjected to *uniform live loads,* which depend on the purpose for which the building is designed.
- ✓ These loadings are generally tabulated in local, state, or national codes.
- ✓ A representative sample of such *minimum live loadings,* taken from the ASCE 7-10 Standard, is shown in Table 1–4.
- ✓ The values are determined from a history of loading various buildings.
- ✓ They include some protection against the possibility of overload due to emergency situations, construction loads, and serviceability requirements due to vibration.
- ✓ In addition to uniform loads, some codes specify *minimum concentrated live loads,* caused by hand carts, automobiles, etc., which must also be applied anywhere to the floor system.
- ✓ For example, both uniform and concentrated live loads must be considered in the design of an automobile parking deck.
- ✓ For some types of buildings having very large floor areas, many codes will allow a *reduction* in the uniform live load for a *floor,* since it is unlikely that the prescribed live load will occur simultaneously throughout the entire structure at any one time. For example, ASCE 7-10 allows a reduction of live load on a member having an *influence area* (**$K_{LL} A_T$**) of **400 ft^2 (37.2 m^2)** or more. This reduced live load is calculated using the following equation:

$$L = L_o\left(0.25 + \frac{15}{\sqrt{K_{LL}A_T}}\right) \quad \text{(FPS units)}$$

... (1- 1)

$$L = L_o\left(0.25 + \frac{4.57}{\sqrt{K_{LL}A_T}}\right) \quad \text{(SI units)}$$

where

L = reduced design live load per square foot or square meter of area supported by the member.

L_o = unreduced design live load per square foot or square meter of area supported by the member (see Table 1–4).

K_{LL} = live load element factor. For interior columns $K_{LL} = 4$

A_T = tributary area in square feet or square meters.

The reduced live load defined by these equation is limited to not less than **50%** of L_o for members supporting one floor, or not less than **40%** of L_o for members supporting more than one floor. No reduction is allowed for loads exceeding **100 lb/ft^2 (4.79 kN/m^2)**, or for structures used for public assembly, garages, or roofs.

Highway Bridge Loads.

- ✓ The primary live loads on bridge spans are those due to traffic, and the heaviest vehicle loading encountered is that caused by a series of trucks.
- ✓ Specifications for truck loadings on highway bridges are reported in the *LRFD* Bridge Design Specifications* of the American Association of State and Highway Transportation Officials (AASHTO).

* load-and-resistance factor design

Impact Loads.

- ✓ Moving vehicles may bounce or sidesway as they move over a bridge, and therefore they impart an impact to the deck.The percentage increase of the live loads due to impact is called the *impact factor*, ***I***. This factor is generally obtained from formulas developed from experimental evidence. For example, for highway bridges the AASHTO specifications require that

$$I = \frac{50}{L + 125} \quad \textbf{but not larger than 0.3}$$

where L is the length of the span in feet that is subjected to the live load.

- ✓ In some cases provisions for impact loading on the structure of a building must also be taken into account. For example, the ASCE 7-10 Standard requires the weight of elevator machinery to be increased by 100%, and the loads on any hangers used to support floors and balconies to be increased by 33%.

Wind Loads.

- ✓ When structures block the flow of wind, the wind's kinetic energy is converted into potential energy of pressure, which causes a wind loading.

- ✓ The effect of wind on a structure depends upon the density and velocity of the air, the angle of incidence of the wind, the shape and stiffness of the structure, and the roughness of its surface.
- ✓ For design purposes, wind loadings can be treated using either a static or a dynamic approach.
- ✓ For the static approach, the fluctuating pressure caused by a constantly blowing wind is approximated by a mean velocity pressure that acts on the structure. This pressure q is defined by its kinetic energy $q = 0.5\,\rho V^2$, where ρ is the density of the air and V is its velocity.
- ✓ According to the ASCE* 7-10 Standard, this equation is modified to account for the importance of the structure, its height, and the terrain in which it is located. It is represented as

$$q_z = 0.00256 K_z\ K_{zt}\ K_d V^2\ \text{(lb/ft}^2\text{)}$$
$$q_z = 0.613 K_z\ K_{zt}\ K_d V^2\ \text{(N/m}^2\text{)} \qquad \ldots(1\text{-}2)$$

V = the velocity in mi/h (m/s) of a 3-second gust of wind measured 33 ft (10 m) above the ground. Specific values depend upon the "category" of the structure obtained from a wind map.

K_z = the velocity pressure exposure coefficient, which is a function of height and depends upon the ground terrain.

K_{zt} = a factor that accounts for wind speed increases due to hills and escarpments. For flat ground K_{zt} = 1.0.

K_d = a factor that accounts for the direction of the wind. It is used only when the structure is subjected to combinations of loads. For wind acting alone, K_d = 1.0.

*American Society of Civil Engineers

Snow Loads.

- ✓ Design loadings typically depend on the building's general shape and roof geometry, wind exposure, location, its importance, and whether or not it is heated.
- ✓ Like wind, snow loads in the ASCE 7-10 Standard are generally determined from a zone map reporting 50-year recurrence intervals of an extreme snow depth. For example, on the relatively flat elevation throughout the mid-section of Illinois and Indiana, the ground snow loading is 20 lb/ft^2 (0.96 kN/m^2).
- ✓ Specifications for snow loads are covered in the ASCE 7-10 Standard, although no single code can cover all the implications of this type of loading.

Earthquake Loads.

- ✓ Earthquakes produce loadings on a structure through its interaction with the ground and its response characteristics.
- ✓ These loadings result from the structure's distortion caused by the ground's motion and the lateral resistance of the structure.
- ✓ Their magnitude depends on the amount and type of ground accelerations and the

Hydrostatic and Soil Pressure.

- ✓ When structures are used to retain water, soil, or granular materials, the pressure developed by these loadings becomes an important criterion for their design. Examples of such types of structures include tanks, dams, ships, bulkheads, and retaining walls.
- ✓ The laws of hydrostatics and soil mechanics are applied to define the intensity of the loadings on the structure.mass and stiffness of the structure.

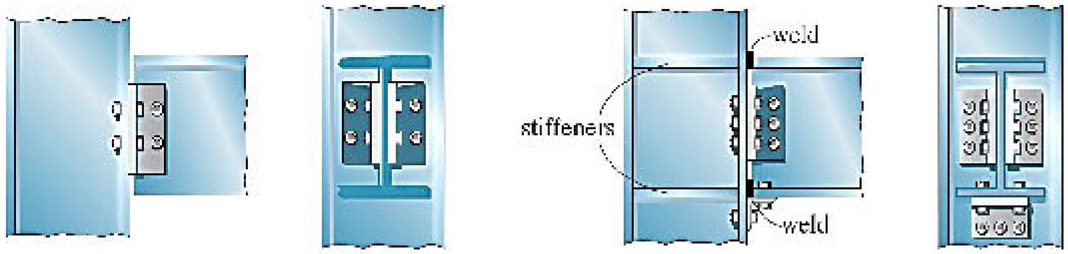
Other Natural Loads.

- ✓ Several other types of live loads may also have to be considered in the design of a structure, depending on its location or use.
- ✓ These include the effect of blast, temperature changes, and differential settlement of the foundation.

2 DETERMINACY AND STABILITY

2.1 Support Connections:

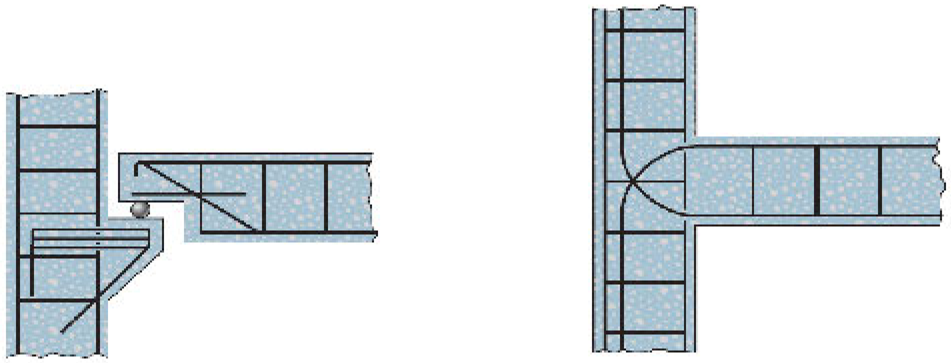
- ✓ Structural members are joined together in various ways depending on the intent of the designer.
- ✓ The three types of joints most often specified are the *pin connection, the roller support, and the fixed joint.*
- ✓ A ***pin-connected*** joint and a ***roller support*** allow some freedom for **slight rotation**.
- ✓ ***Fixed joint*** allows no relative rotation between the connected members and is consequently more expensive to fabricate. Examples of these joints, fashioned in metal and concrete, are shown in Figs. 2–1 and 2–2, respectively.
- ✓ For most timber structures, the members are assumed to be pin connected, since bolting or nailing them will not sufficiently restrain them from rotating with respect to each other.
- ✓ In reality, however, all connections exhibit some stiffness toward joint rotations, owing to friction and material behavior. In this case a more appropriate model for a support or joint might be that shown in Fig. 2–3c. If the torsional spring constant $k = 0$, the joint is a pin, and if $k = \infty$, the joint is fixed.



typical "pin-supported" connection (metal)
(a)

typical "fixed-supported" connection (metal)
(b)

Fig. 2–1



typical "roller-supported" connection (concrete)
(a)

typical "fixed-supported" connection (concrete)
(b)

Fig. 2–2

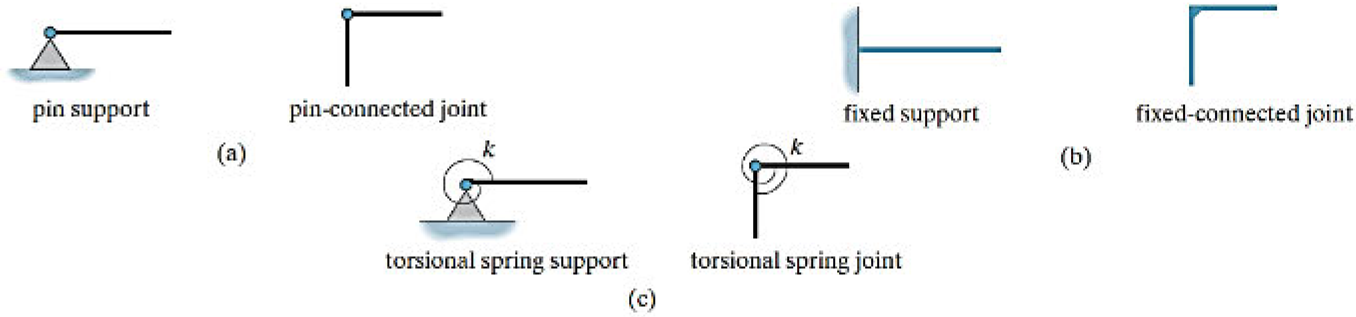
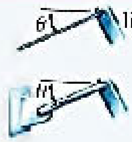
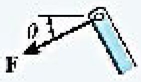
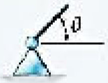
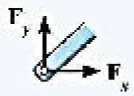
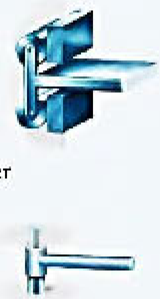
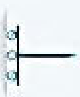
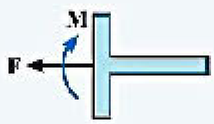
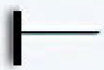
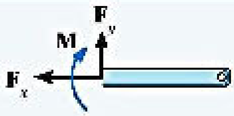


Fig. 2–3

- ✔ In reality, all supports actually exert *distributed surface loads* on their contacting members. The concentrated forces and moments shown in Table 2–1 represent the *resultants* of these load distributions. This representation is, of course, an idealization; however, it is used here since the surface area over which the distributed load acts is considerably *smaller* than the *total* surface area of the connecting members.

TABLE 2–1 Supports for Coplanar Structures

Type of Connection	Idealized Symbol	Reaction	Number of Unknowns
(1)  light cable, weightless link			One unknown. The reaction is a force that acts in the direction of the cable or link.
(2)  rollers, rocker			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.
(3)  smooth contacting surface			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.
(4)  smooth pin-connected collar			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact.
(5)  smooth pin or hinge			Two unknowns. The reactions are two force components.
(6)  slider, fixed-connected collar			Two unknowns. The reactions are a force and a moment.
(7)  fixed support			Three unknowns. The reactions are the moment and the two force components.

2.2 Determinacy and Stability

Determinate Structure

The structure is said to be determinate if,

Number of Unknowns = Total Number of Equilibrium Equations.

Indeterminate Structure

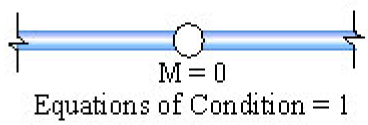
The structure is said to be indeterminate if,

Number of Unknowns > Total Number of Equilibrium Equations.

Equations of Condition (C)

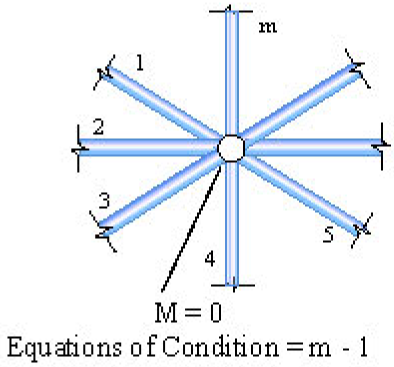
1. Interior hinge connecting two members

C = 1



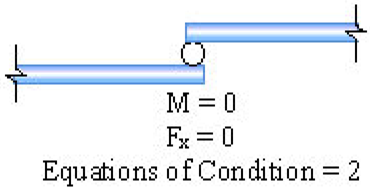
2. Interior hinge connecting (m) members

C = m – 1



3. Interior roller

C = 2



Basic Equations of Equilibrium

In plan structures, the basic equations of equilibrium are three, which are:-

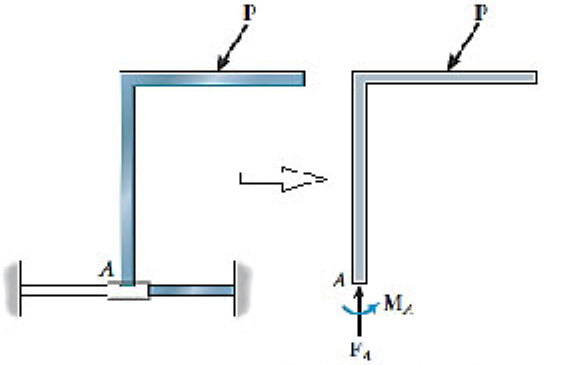
$$\sum F_x = 0 \;;\; \sum F_y = 0 \;;\; \sum M = 0$$

✔ It must be emphasized her that for one structural element, only three unknowns can be evaluated by using these equations. (no matter how several times the equation $\sum M = 0$)

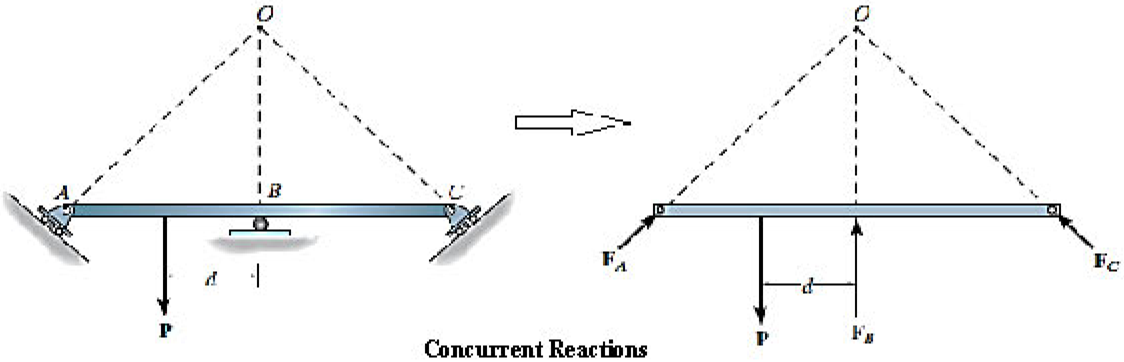
Unstable Structure

The structure is said to be unstable if any one of the following four conditions available

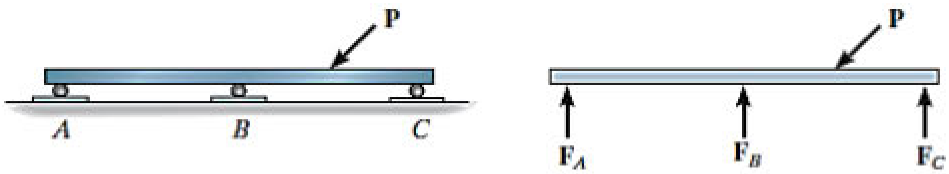
1. When number of unknowns < Total Number of equilibrium equations.
2. When all reaction are concurrent (meeting at one point).
3. When all reaction are parallel.
4. When the structure is geometrically unstable.
   (ex. Three hinges in one span makes a mechanism unstable)



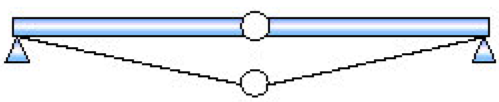
Partial Constraints
Number of Unknowns < Total Number of Equilibrium Equations



Concurrent Reactions



parallel reactions



Structure is Geometrically Unstable

2.2.1 Stability and Determinacy of Beams

Let r = Number of Reactions (Unknowns).

Total Number of Equilibriums Equations = 3 + C

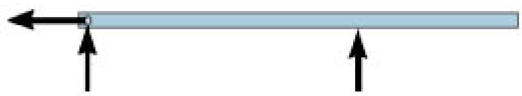
Therefore,

If	$r < 3 + C$	The beam is unstable
If	$r = 3 + C$	The beam is determinate <u>if</u> stable
If	$r > 3 + C$	The beam is indeterminate <u>if</u> stable

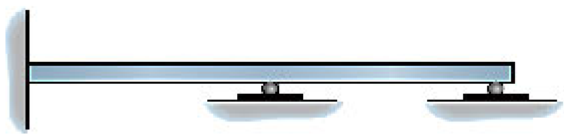
EXAMPLE 2.2.1.1

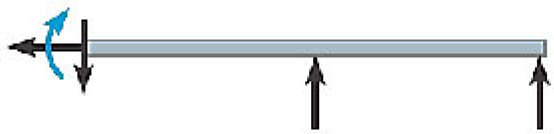
Classify each of the beams as statically determinate, statically indeterminate, or unstable. If indeterminate, specify the degree of indeterminacy. The supports or connections are to be assumed as stated.

Solution

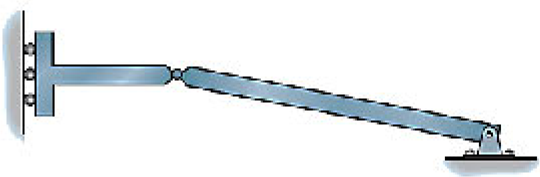


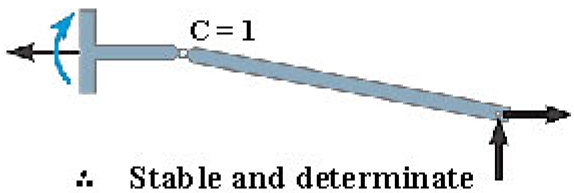
$r = 3$, $3 + C = 3 + 0 = 3$ ⇨ $r = 3 + C$ $(3 = 3)$ ∴ Stable and determinate



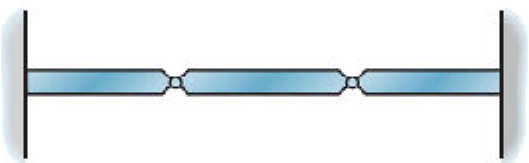


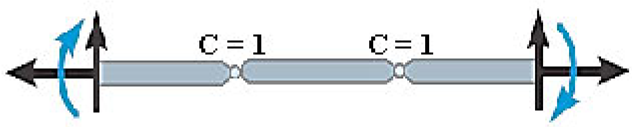
$r = 5$, $3 + C = 5$ ⇨ $r > 3 + C$ $(5 > 3)$ ∴ Stable and indeterminate to 2nd degree





$r = 4$, $3 + C = 3 + 1 = 4$ ⇨ $r = 3 + C$ $(4 = 4)$ ∴ Stable and determinate

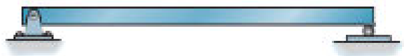
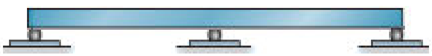
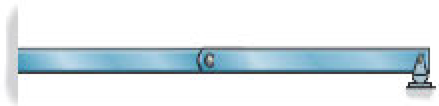
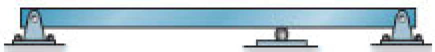
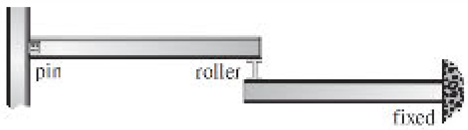
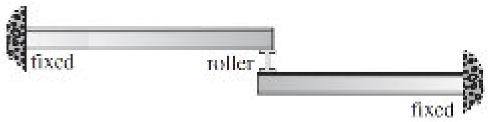
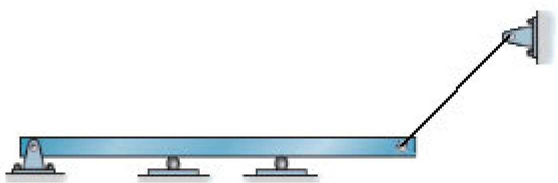




$r = 6$, $3 + C = 3 + 2 = 5$ ⇨ $r > 3 + C$ $(6 = 5)$ ∴ Stable and indeterminate to 1st degree

EXAMPLE 2.2.1.2

Classify each of the beams as statically determinate, statically indeterminate, or unstable. If indeterminate, specify the degree of indeterminacy. The supports or connections are to be assumed as stated.

Beam	r	C	3 + C	< r = 3 + C >	Stability and Determinacy
	3	0	3	3 = 3	Stable and determinate
	3	0	3	3 = 3	Unstable (parallel reactions)
	4	1	4	4 = 4	Stable and determinate
	5	0	3	5 > 3	Stable and indeterminate 2nd degree
	5	1	4	5 > 4	Stable and indeterminate 1st degree
	5	2	5	5 =5	Unstable (geometrically unstable)
	6	2	5	6 > 5	Stable and indeterminate 1st degree
	5	2	5	5 = 5	Stable and determinate
	6	2	5	6 > 5	Stable and indeterminate 1st degree
	5	0	3	5 > 3	Stable and indeterminate 2nd degree

2.2.2 Stability and Determinacy of Trusses

Let r = Number of Reactions.
b = Number of Bars.
J = Number of Joints.

Number of Unknowns = b + r

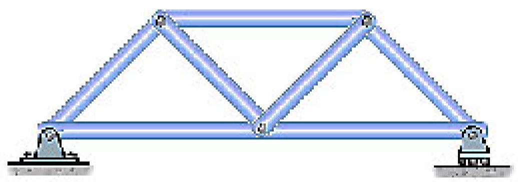
Total Number of Equilibriums Equations = 2J

Therefore,

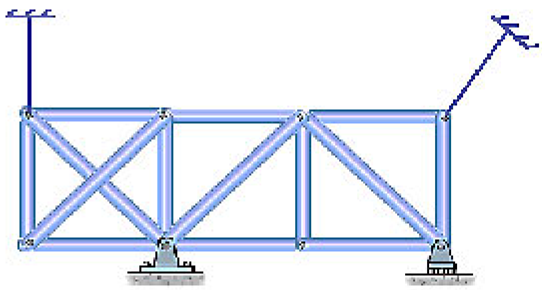
If	**$b + r < 2J$**	**The truss is unstable**
If	**$b + r = 2J$**	**The truss is determinate <u>if</u> stable**
If	**$b + r > 2J$**	**The truss is indeterminate <u>if</u> stable**

EXAMPLE 2.2.2.1
Classify each of the trusses as statically determinate, statically indeterminate, or unstable. If indeterminate, specify the degree of indeterminacy. The supports or connections are to be assumed as stated.

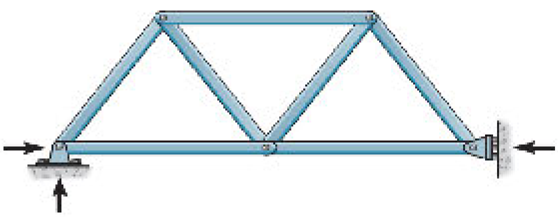
Solution



$r = 3$, $b = 7$, $2J = 2(5)$ ⇨ **$r + b = 2J$ (10 = 10)**
∴ **Stable and determinate**



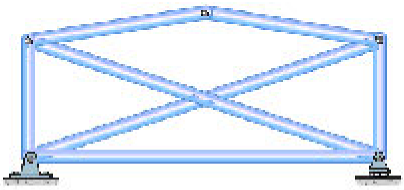
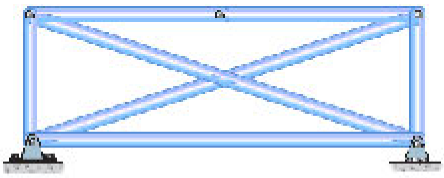
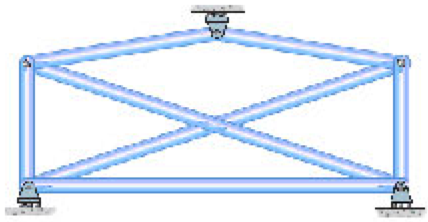
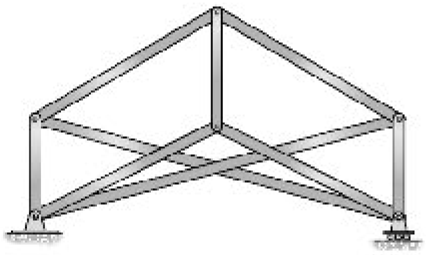
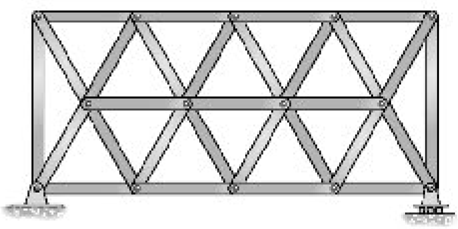
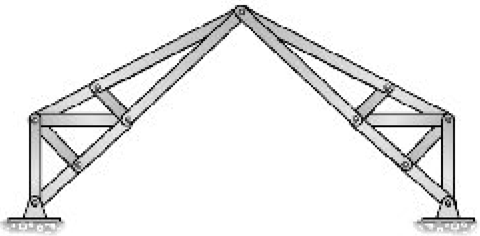
$r = 5$, $b = 14$, $2J = 2(8)$ ⇨ **$r + b > 2J$ (19 > 16)**
∴ **Stable and indeterminate to 3^{rd} degree**



$r = 3$, $b = 7$, $2J = 2(5)$ ⇨ **$r + b = 2J$ (10 = 10)**
∴ **Unstable (concurrent reactions)**

EXAMPLE 2.2.2.2

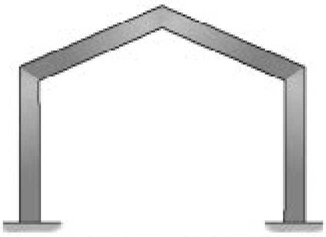
Classify each of the trusses as statically determinate, statically indeterminate, or unstable. If indeterminate, specify the degree of indeterminacy. The supports or connections are to be assumed as stated.

Truss	r	b	J	$r + b \overset{<}{\underset{>}{=}} 2J$	Stability and Determinacy
	3	7	5	10 = 10	Stable and determinate
	3	7	5	10 = 10	Unstable (geometrically unstable)
	3	7	5	10 = 10	Unstable (parallel reactions)
	3	9	6	12 = 12	Stable and determinate
	3	29	14	$32 > 28$	Stable and indeterminate 4^{th} degree
	4	18	11	22 = 22	Stable and determinate

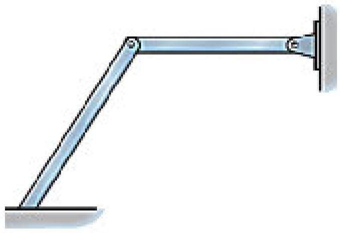
2.2.3 Stability and Determinacy of Frames and Arches

1. Open Frames and Arches

Can be treated similar to beams by apply $r \lesseqgtr 3 + C$



r = 6 , 3 + C = 3 + 0 = 3
r > 3 + C (6 > 3)
∴ Stable and indeterminate 3rd degree



r = 5 , 3 + C = 3 + 1= 4
r > 3 + C (5 > 4)
∴ Stable and indeterminate 1st degree

2. Closed Frames and Arches

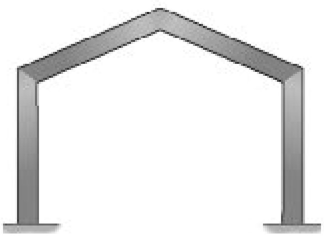
Let r = Number of Reactions.
b = Number of Members.
J = Number of Joints.

Number of Unknowns = 3b + r

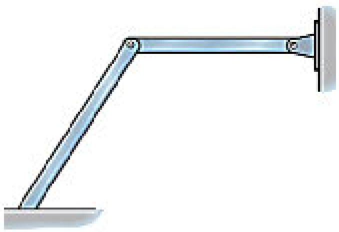
Total Number of Equilibriums Equations = 3J + C

Therefore,

If	$3b + r < 3J + C$	The frame is unstable
If	$3b + r = 3J + C$	The frame is determinate <u>if</u> stable
If	$3b + r > 3J + C$	The frame is indeterminate <u>if</u> stable



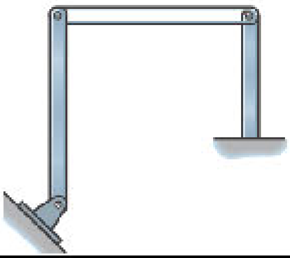
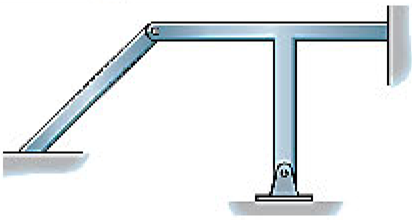
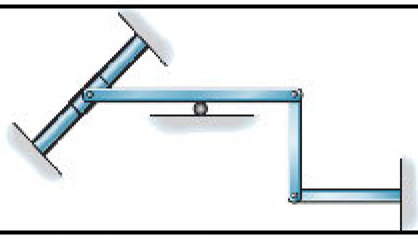
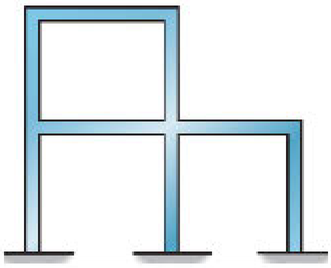
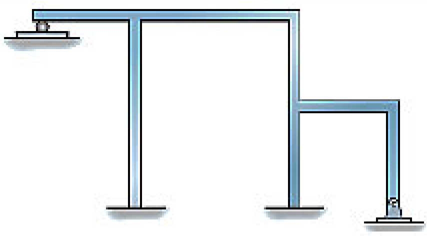
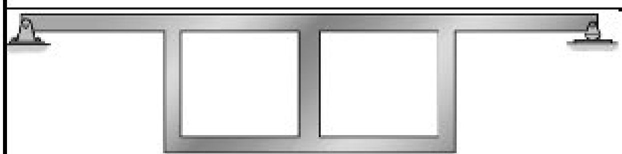
b = 4 , r = 6 , J = 5 , C = 0
3b + r = 3(4) + 6 = 18 , 3J = 15
3b + r > 3J + C (18 > 15)
∴ Stable and indeterminate 3rd degree



b = 2 , r = 5 , J = 3 , C = 1
3b + r = 3(2) + 5 = 11 , 3J = 9
3b + r > 3J + C (11 > 10)
∴ Stable and indeterminate 1st degree

EXAMPLE 2.2.3.1

Classify each of the Frames as statically determinate, statically indeterminate, or unstable. If indeterminate, specify the degree of indeterminacy. The supports or connections are to be assumed as stated.

Frame	b	r	C	J	< 3b+r =3J+C >	Stability and Determinacy
	3	5	2	4	14 = 14	Stable and determinate
	4	8	1	5	20 = 16	Stable and indeterminate 4^{th} degree
	4	10	4	6	22 = 22	Stable and determinate
	8	9	0	8	33 > 24	Stable and indeterminate 9^{th} degree
	7	9	0	8	30 > 24	Stable and indeterminate 6^{th} degree
	9	3	0	8	30 > 24	Stable and indeterminate 6^{th} degree

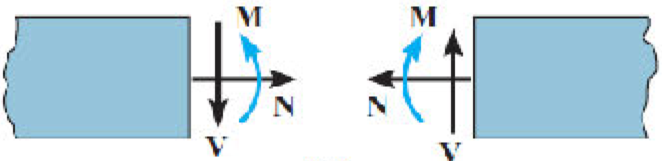
3 INTERNAL LOADINGS DEVELOPED IN STRUCTURAL MEMBERS

3.1 Internal Loadings at a Specified Point:

The internal load at a specified point in a member can be determined by using the *method of sections*. In general, this loading for a coplanar structure will consist of a normal force **N**, shear force **V**, and bending moment **M**, (Three-dimensional frameworks can also be subjected to a torsional moment, which tends to twist the member about its axis). It should be realized, however, that these loadings actually represent the *resultants* of the *stress distribution* acting over the member's cross-sectional area at the cut section. Once the resultant internal loadings are known, the magnitude of the stress can be determined provided an assumed distribution of stress over the cross-sectional area is specified.

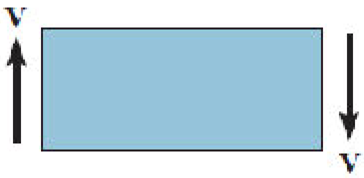
3.2 Sign Convention:

Before presenting a method for finding the internal normal force, shear force, and bending moment, we will need to establish a sign convention to define their "positive" and "negative" values. Although the choice is arbitrary, the sign convention to be adopted here has been widely accepted in structural engineering practice, and is illustrated in **Fig.*a***. On the ***left-hand face*** of the cut member the normal force **N** acts to the **right**, the internal shear force **V** acts **downward**, and the moment **M** acts **counterclockwise**. In accordance with Newton's third law, an equal but opposite normal force, shear force, and bending moment must act on the ***right-hand face*** of the member at the section. Perhaps an easy way to remember this sign convention is to isolate a small segment of the member and note that ***positive normal force tends to elongate the segment***, **Fig.*b***; ***positive shear tends to rotate the segment clockwise***, **Fig. *c*** ; **and *positive bending moment tends to bend the segment concave upward*** **Fig. *d***.

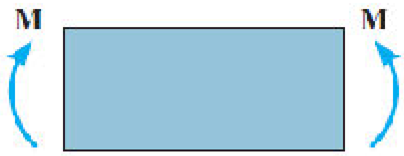


(a)

(b)



(c)



(d)

3.3 Procedure for Analysis:

Support Reactions

✓ Before the member is "cut" or sectioned, it may be necessary to determine the member's support reactions so that the equilibrium equations are used only to solve for the internal loadings when the member is sectioned.

Free-Body Diagram

✓ Keep all distributed loadings, couple moments, and forces acting on the member in their *exact location,* then pass an imaginary section through the member, perpendicular to its axis at the point where the internal loading is to be determined.

✓ After the section is made, draw a free-body diagram of the segment that has the least number of loads on it. At the section indicate the unknown resultants **N**, **V**, and **M** acting in their *positive* directions.

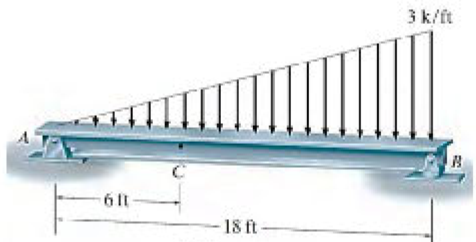
Equations of Equilibrium

✓ Moments should be summed at the section about axes that pass through the *centroid* of the member's cross-sectional area, in order to eliminate the unknowns **N** and **V** and thereby obtain a direct solution for **M**.

✓ If the solution of the equilibrium equations yields a quantity having a negative magnitude, the assumed directional sense of the quantity is opposite to that shown on the free-body diagram.

EXAMPLE 3.3.1

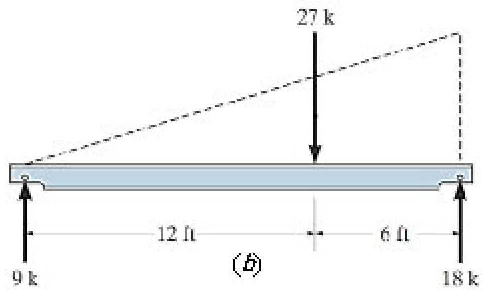
Determine the internal shear and moment acting at a section passing through point *C* in the beam shown in fig. *a*



(*a*)

Solution

Support Reactions. Replacing the distributed load by its resultant force and computing the reactions yields the results shown in fig *b*.



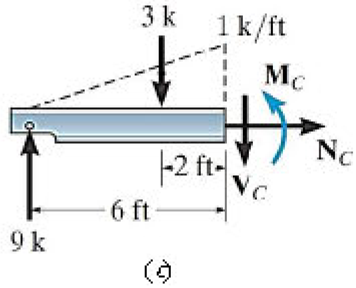
Free-Body Diagram. Segment *AC* will be considered since it yields the simplest solution, the figure. The distributed load intensity at *C* is computed by proportion, that is,

$$\frac{w_c}{6\text{ ft}} = \frac{3\text{ k/ft}}{18\text{ ft}} \Rightarrow \; w_c = 6\text{ ft} \times \frac{3\text{ k/ft}}{18\text{ ft}} = 1\text{ k/ft}$$

Equations of Equilibrium.

$+\uparrow \sum F_y = 0; \quad 9 - 3 - V_C = 0 \qquad V_C = 6\text{ k}$

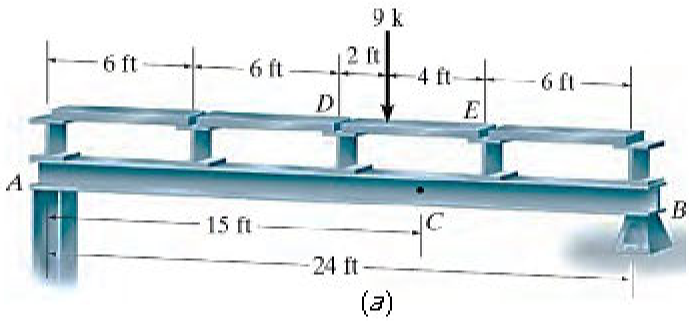
$+\circlearrowleft \sum M_C = 0; \quad -9(6) - 3(2) + M_C = 0 \qquad M_C = 48\text{ k.ft}$



Note: This problem illustrates the importance of *keeping* the distributed loading on the beam until *after* the beam is sectioned. If the beam in **Fig. *b*** were sectioned at *C*, the effect of the distributed load on segment *AC* would not be recognized, and the result $V_C = \mathbf{9\text{ k}}$ and $M_C = \mathbf{54\text{ k . ft}}$ **<u>would be wrong.</u>**

EXAMPLE 3.3.2

The **9-k** force in Fig. *a* is supported by the floor panel ***DE***, which in turn is simply supported at its ends by floor beams. These beams transmit their loads to the simply supported girder ***AB***. Determine the internal shear and moment acting at point ***C*** in the girder.



Solution

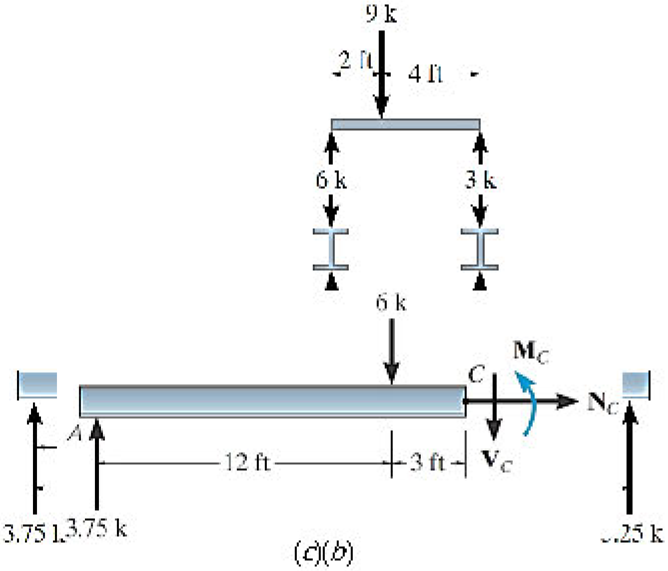
Support Reactions.

Equilibrium of the floor panel, floor beams, and girder is shown in **Fig.*b***. It is advisable to check these results.

Free-Body Diagram.

The free-body diagram of segment *AC* of the girder will be used since it leads to the simplest solution, **Fig. *c***.

Note that there are *no loads* on the floor beams supported by *AC*.



Equations of Equilibrium.

$+\uparrow \sum F_y = 0; \quad 3.75 - 6 - V_C = 0 \quad V_C = -2.25 \text{ k}$

$+\circlearrowleft \sum M_C = 0; \quad -3.75(15) - 6(3) + M_C = 0 \quad M_C = 38.25 \text{ k.ft}$

3.4 Shear and Moment Functions

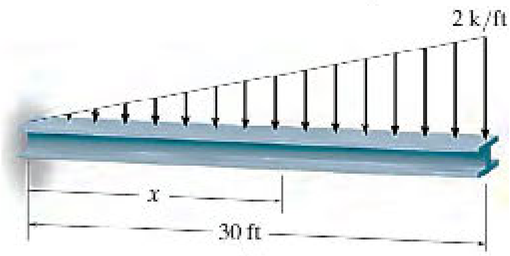
The design of a beam requires a detailed knowledge of the variations of the internal shear force ***V*** and moment ***M*** acting at each point along the axis of the beam.
The internal normal force is generally not considered for two reasons:

1. In most cases the loads applied to a beam act perpendicular to the beam's axis and hence produce only an internal shear force and bending moment.
2. For design purposes the beam's resistance to shear, and particularly to bending, is more important than its ability to resist normal force.

The variations of ***V*** and ***M*** as a function of the position ***x*** of an arbitrary point along the beam's axis can be obtained by using the method of sections. It is necessary to locate the imaginary section or cut at an arbitrary distance ***x*** from one end of the beam rather than at a specific point. In general, the internal shear and moment functions will be discontinuous, or their slope will be discontinuous, at points where the type or magnitude of the distributed load changes or where concentrated forces or couple moments are applied. Because of this, shear and moment functions must be determined for each region of the beam located *between* any two discontinuities of loading.

EXAMPLE 3.4.1

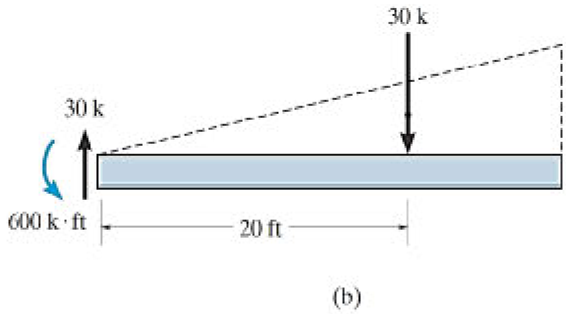
Determine the shear and moment in the beam shown in **Fig. *a*** as a function of ***x***.



(a)

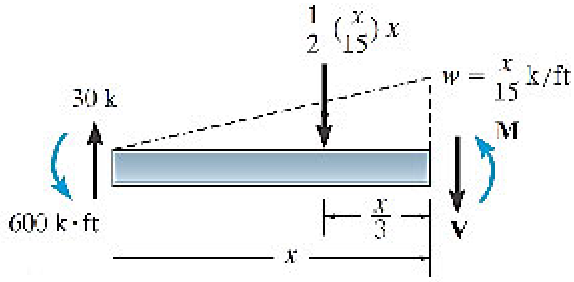
Solution

Support Reactions. For the purpose of computing the support reactions, the distributed load is replaced by its resultant force of **30 k, Fig. *b***. It is important to remember, however, that this resultant is not the actual load on the beam.



(b)

Shear and Moment Functions. A free-body diagram of the beam segment of length x is shown in **Fig. c**. Note that the intensity of the triangular load at the section is found by proportion; that is, $w/x = 2/30$, $w = x/15$.
With the load intensity known, the resultant of the distributed loading is found in the usual manner as shown in the figure. Thus,



$$+\uparrow \sum F_y = 0; \quad 30 - \frac{1}{2}\left(\frac{x}{15}\right)x - V = 0$$

$$V = 30 - 0.0333x^2$$

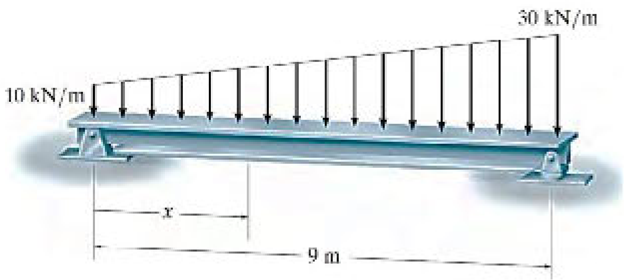
$$+\circlearrowleft \sum M_{Section} = 0; \quad 600 - 30x + \left[\frac{1}{2}\left(\frac{x}{15}\right)x\right]\frac{x}{3} + M = 0$$

$$M = -600 + 30x - 0.0111x^3$$

Note that $dM/dx = V$ and $dV/dx = -x/15 = w$ which serves as a check of the results.

EXAMPLE 3.4.2

Determine the shear and moment in the beam shown in **Fig.*a*** as a function of ***x***.

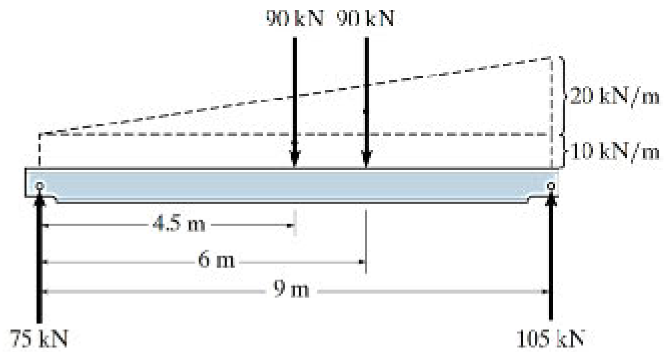


(a)

Solution

Support Reactions.

To determine the support reactions, the distributed load is divided into a triangular and rectangular loading, and these loadings are then replaced by their resultant forces. These reactions have been computed and are shown on the beam's free body diagram, **Fig.*b***.



(b)

Shear and Moment Functions.

A free-body diagram of the cut section is shown in **Fig.*c***. As above, the trapezoidal loading is replaced by rectangular and triangular distributions. Note that the intensity of the triangular load at the cut is found by proportion. The
resultant force of each distributed loading and its location are indicated. Applying the equilibrium equations, we have

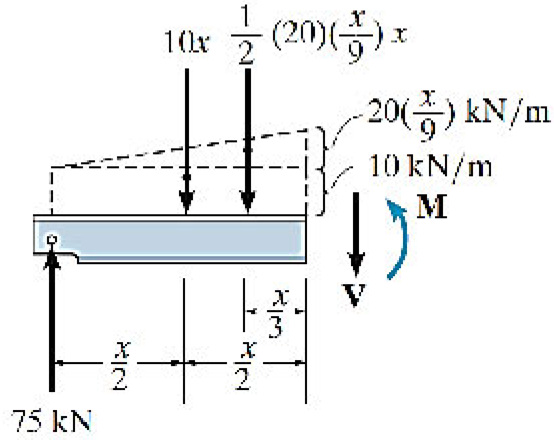
$$+\uparrow \sum F_y = 0; \quad 75 - 10x - \left[\frac{1}{2}(20)\left(\frac{x}{9}\right)x\right] - V = 0$$

$$V = 75 - 10x - 1.11x^2$$

$$+\circlearrowleft \sum M_{Section} = 0;$$

$$-75x + (10x)\left(\frac{x}{2}\right) + \left[\frac{1}{2}(20)\left(\frac{x}{9}\right)x\right]\frac{x}{3} + M = 0$$

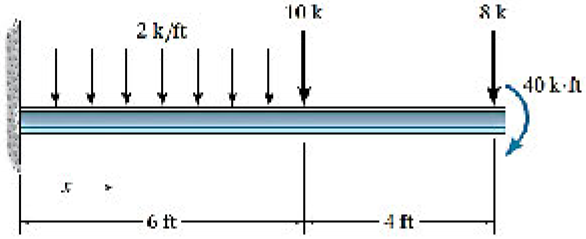
$$M = 75x - 5x^2 - 0.370x^3$$



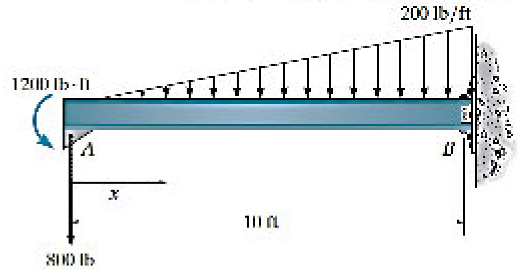
(c)

Hw.1

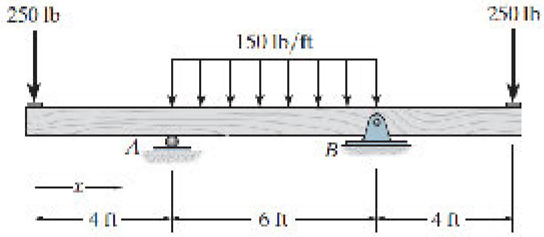
Determine the shear and moment throughout the beams as functions of x. And draw the shear and moment diagrams



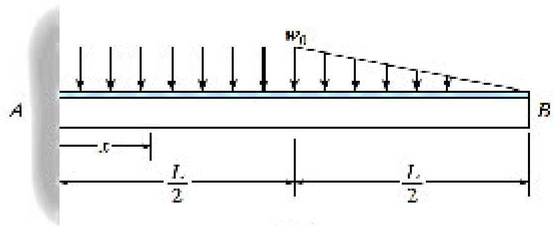
(1)



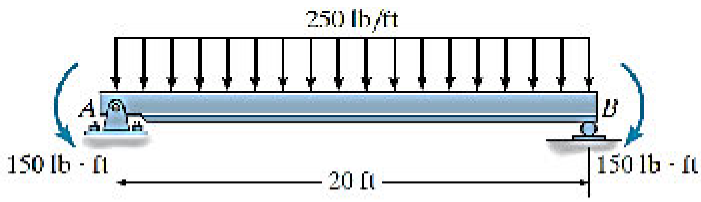
(2)



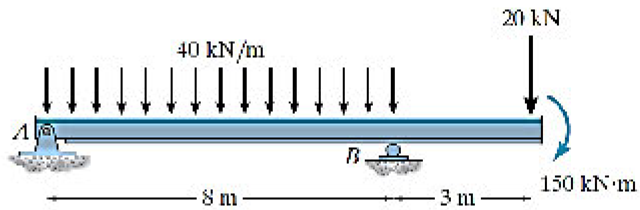
(3)



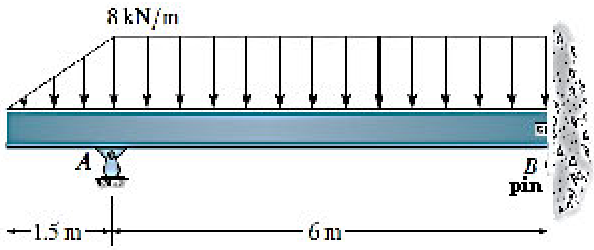
(4)



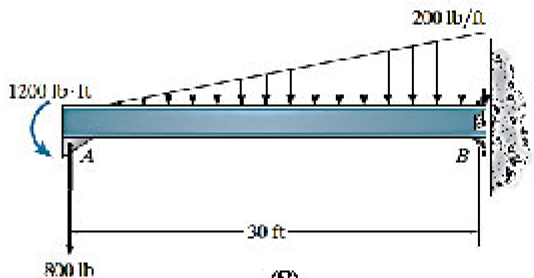
(5)



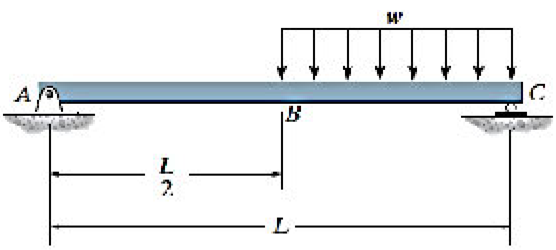
(6)



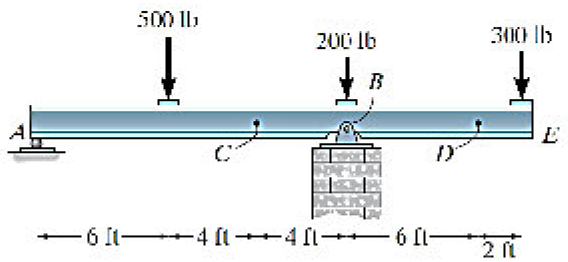
(7)



(8)



(9)



(10)

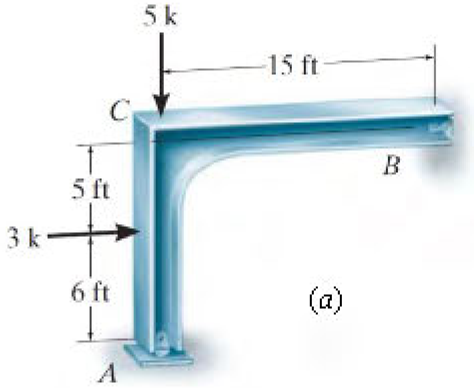
3.5 Shear and Moment Diagrams for a Frame

Frame is composed of several connected members that are either fixed or pin connected at their ends. The design of these structures often requires drawing the shear and moment diagrams for each of the members.

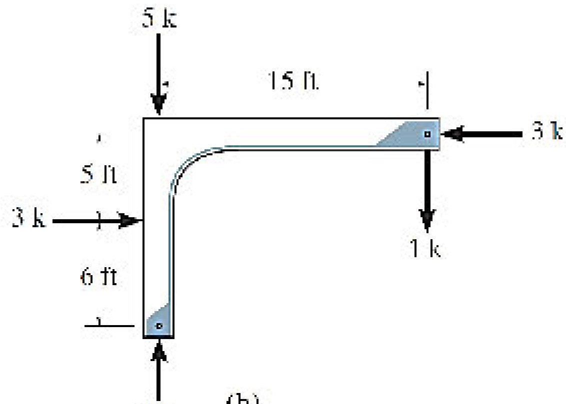
To analyze any problem first determining the reactions at the frame supports. Then, using the method of sections, we find the axial force, shear force, and moment acting at the ends of each member.

EXAMPLE 3.5.1

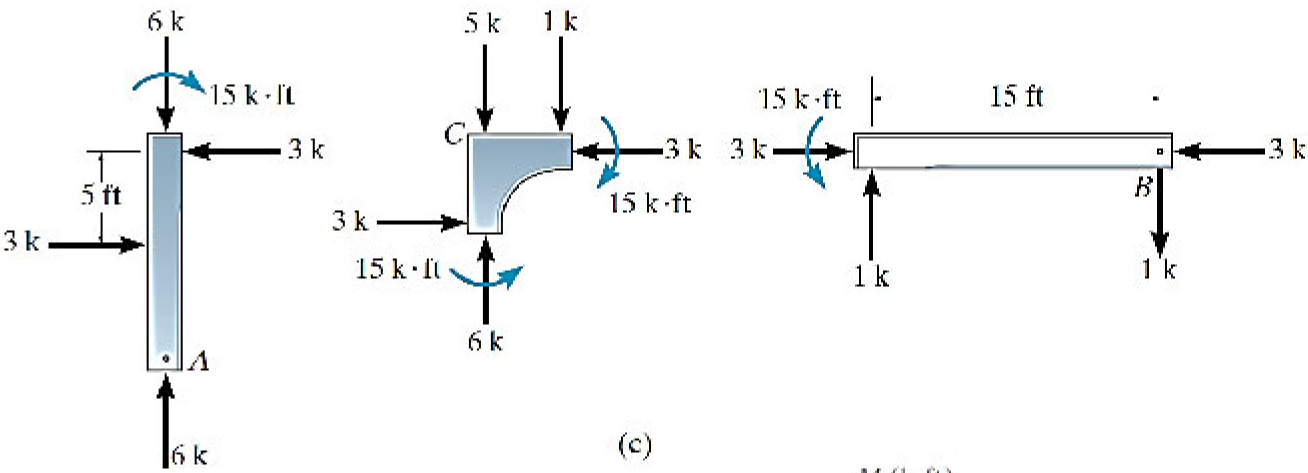
Draw the moment diagram for the tapered frame shown in **Fig. *a***. Assume the support at ***A*** is a roller and ***B*** is a pin.



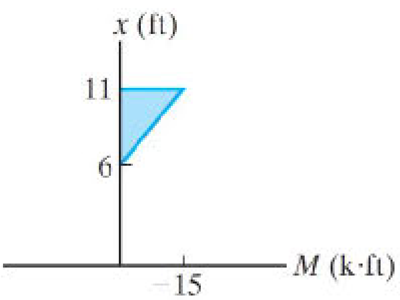
(*a*)



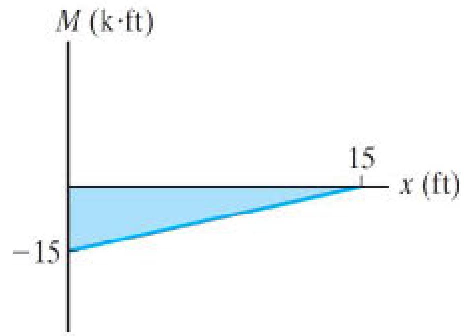
(b)



(c)



member *AC*



member *CB*

EXAMPLE 3.5.2

Draw the shear and moment diagrams for the frame shown in **Fig. *a***. Assume A is a pin, C is a roller, and B is a fixed joint. Neglect the thickness of the members.

Solution

Notice that the distributed load acts over a length of

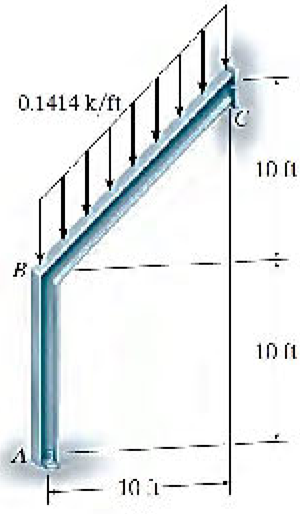
$$10\text{ ft}\sqrt{2} = 14.14\text{ ft}$$

The reactions on the entire frame are calculated and shown on its free-body diagram, **Fig. *b***. From this diagram the free-body diagrams of each member are drawn, **Fig. *c***.

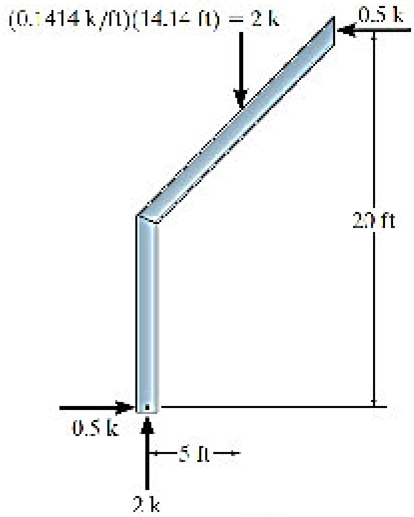
The distributed loading on BC has components along BC and perpendicular to its axis of

$$(0.1414\text{ k/ft})\cos 45° = (0.1414\text{ k/ft})\sin 45° = 0.1\text{ k/ft}$$

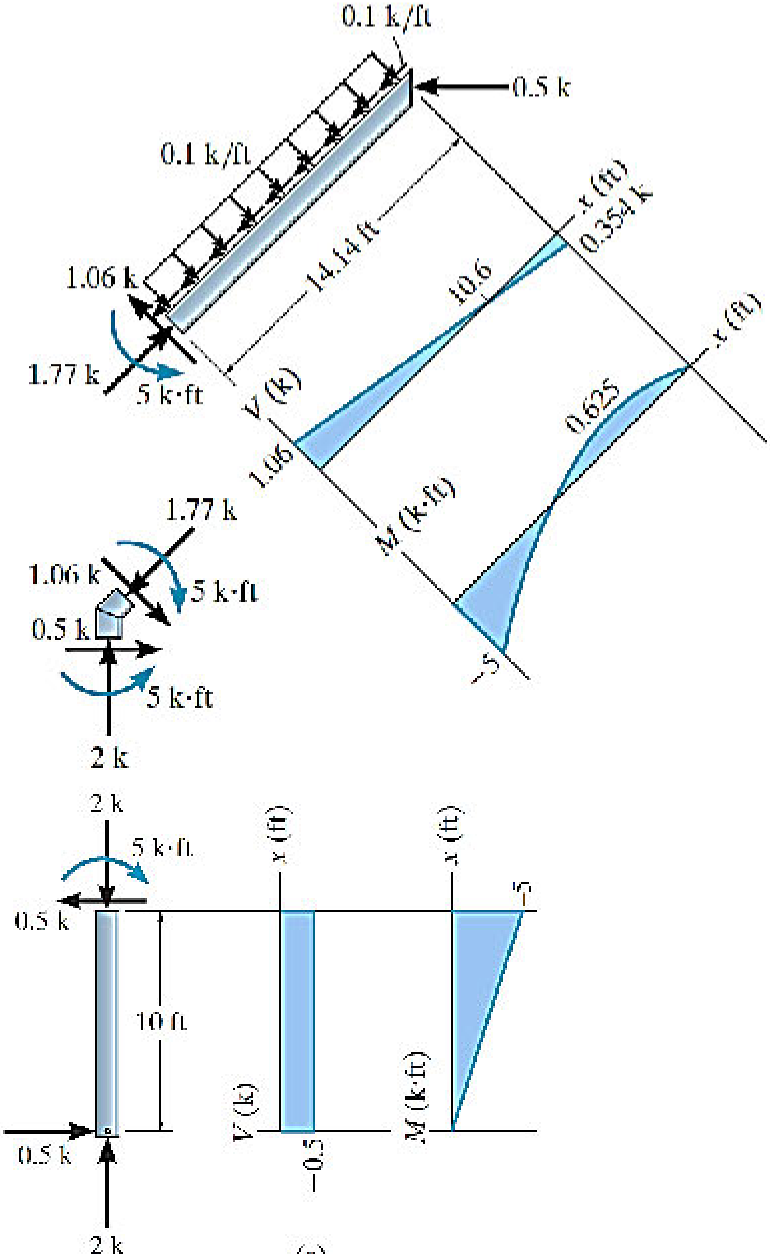
the shear and moment diagrams are also shown in **Fig. *c***.



(a)



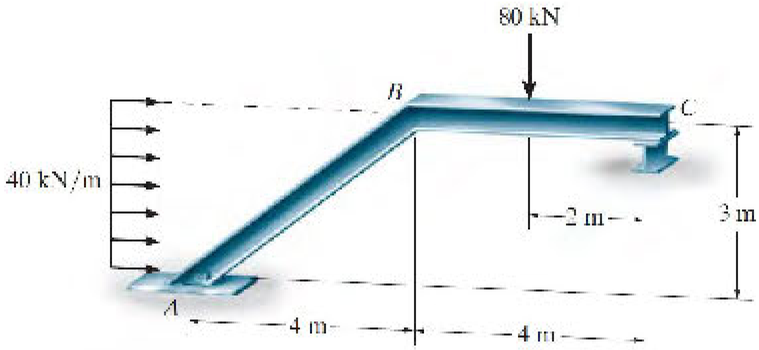
(b)



(c)

EXAMPLE 3.5.3

Draw the shear and moment diagrams for the frame shown in ***Fig. a***. Assume ***A*** is a pin, ***C*** is a roller, and ***B*** is a fixed joint.

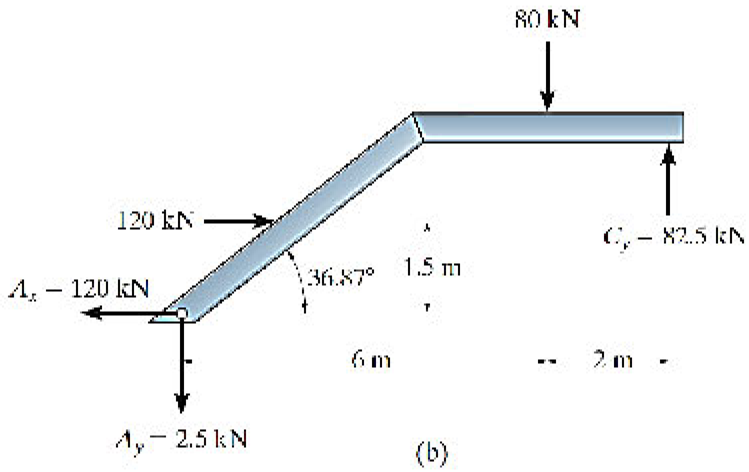


(a)

Solution

Support Reactions.

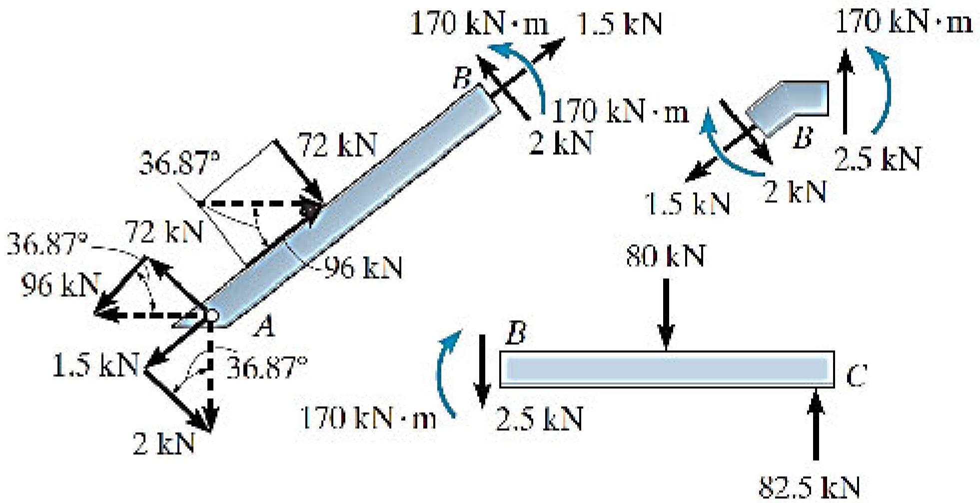
The free-body diagram of the entire frame is shown in **Fig.*b***. Here the distributed load, which represents wind loading, has been replaced by its resultant, and the reactions have been computed. The frame is then sectioned at joint ***B*** and the internal loadings at ***B*** are determined, **Fig.*c***. As a check, equilibrium is satisfied at joint ***B***, which is also shown in the figure.



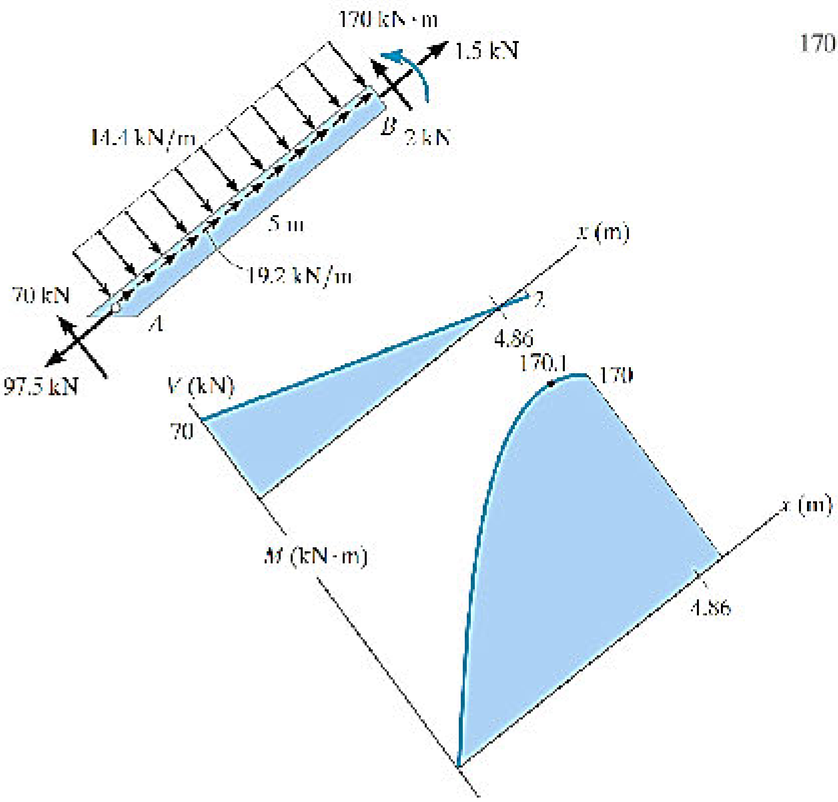
(b)

Shear and Moment Diagrams.

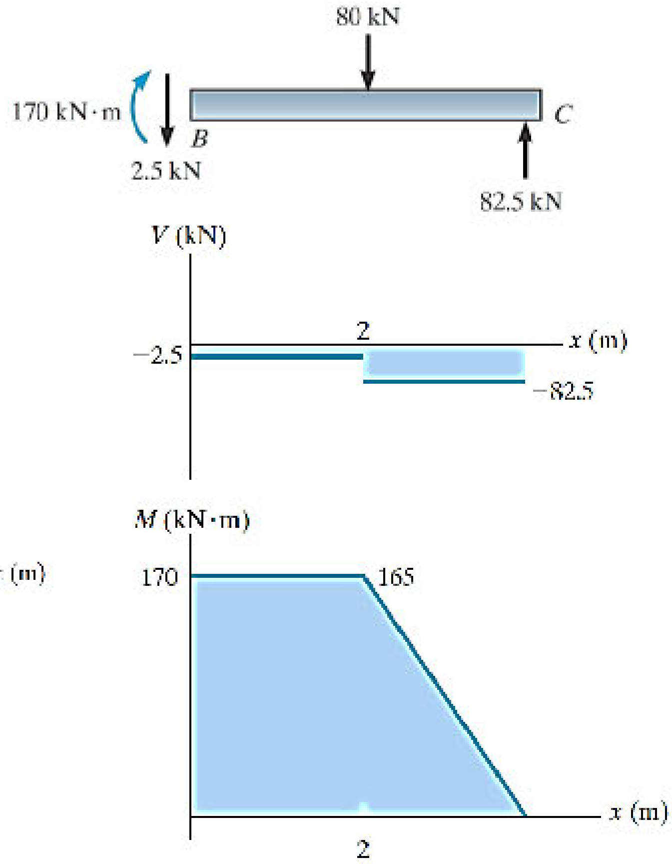
The components of the distributed load, **(72 kN)/(5 m) = 14.4 kN/m** and **(96 kN)/(5 m) = 19.2 kN/m**, are shown on member ***AB*****, Fig. *d***. The associated shear and moment diagrams are drawn for each member as shown in **Figs. *d*** and ***e***.



(c)

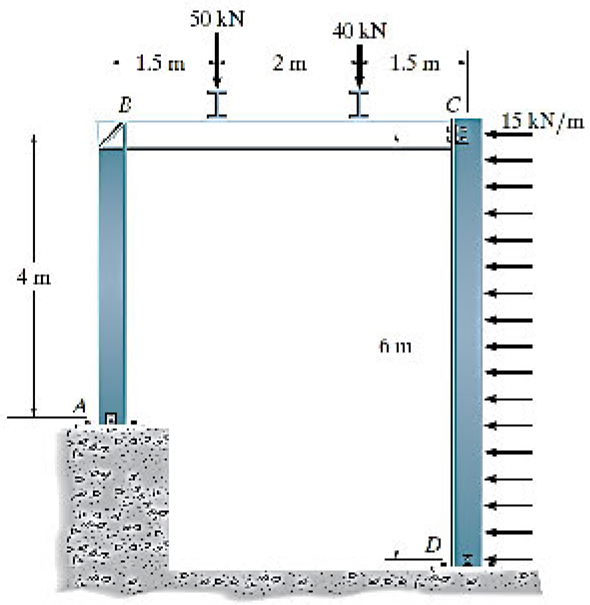


(d)

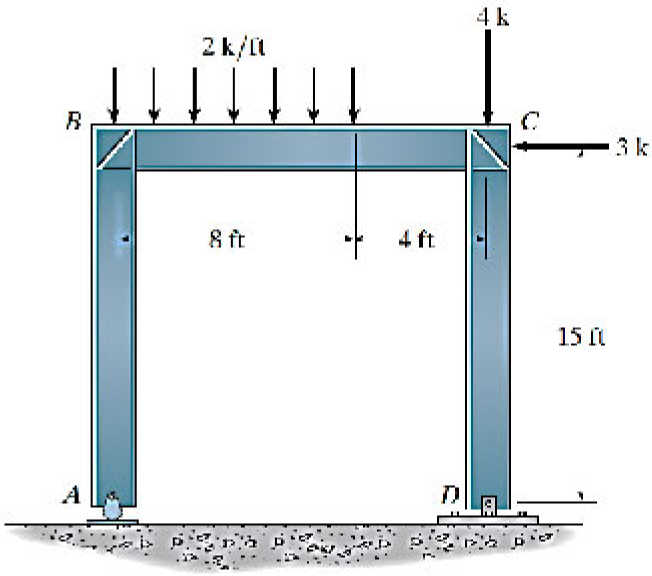


(e)

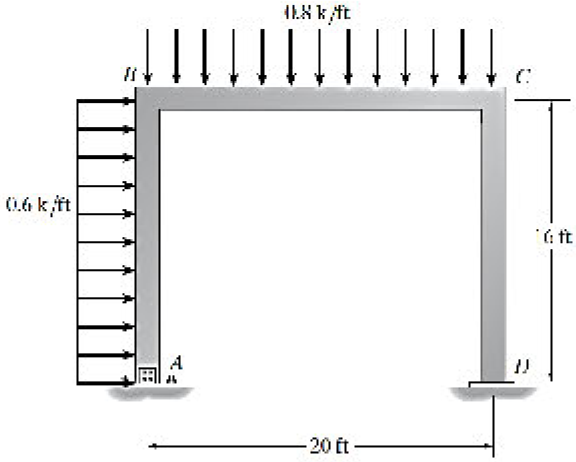
Hw.2 Draw the shear and moment diagrams for each of the three members of the frame. Assume the frame is pin connected at *A*, *C*, and *D* and there is a fixed joint at *B*.



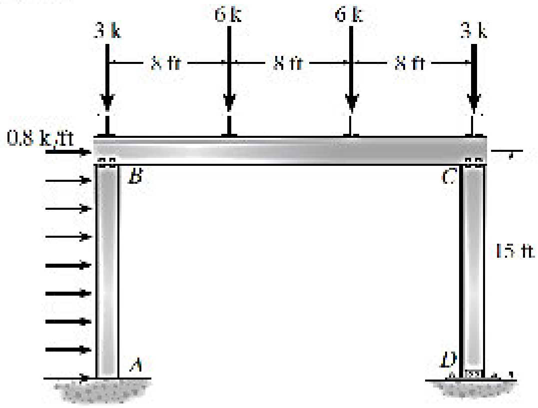
Hw.3 Draw the shear and moment diagrams for each member of the frame. Assume *A* is a rocker, and *D* is pinned.



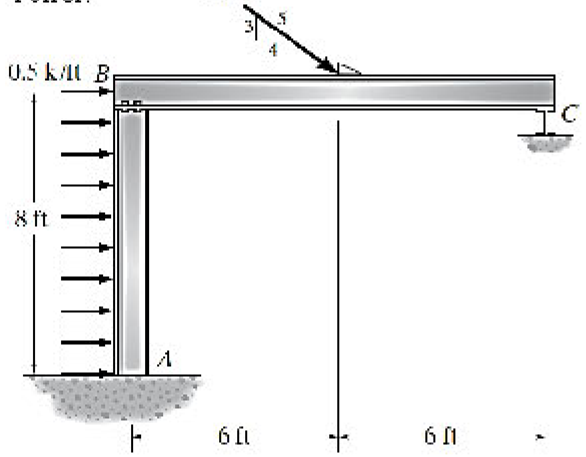
Hw.4 Draw the shear and moment diagrams for each member of the frame. Assume the support at *A* is a pin and *D* is a roller.



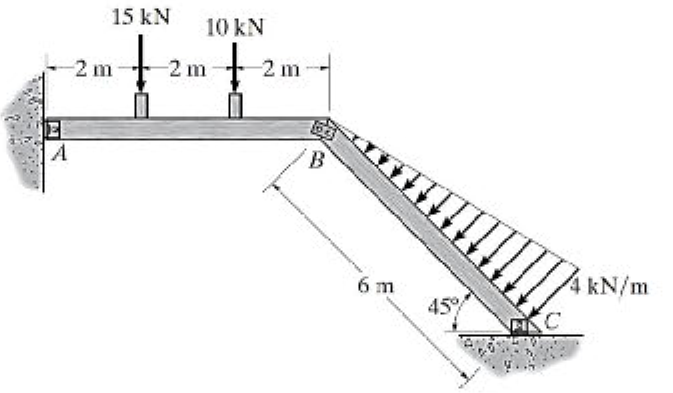
Hw.5 Draw the shear and moment diagrams for each member of the frame. Assume the frame is pin connected at *B*, *C*, and *D* and *A* is fixed.



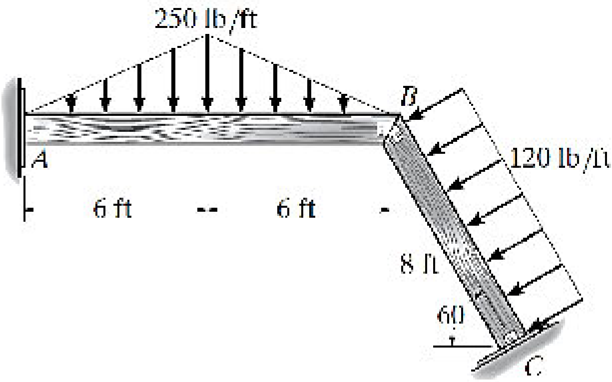
Hw.6 Draw the shear and moment diagrams for each member of the frame. Assume *A* is fixed, the joint at *B* is a pin, and support *C* is a roller.



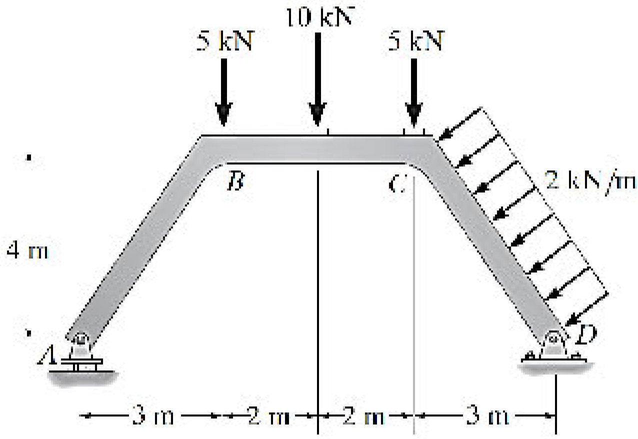
Hw.7 Draw the shear and moment diagrams for each member of the frame. The members are pin connected at *A*, *B*, and *C*.



Hw.8 Draw the shear and moment diagrams for each member of the frame. The joints at *A*, *B*, and *C* are pin connected.



Hw.9 Draw the shear and moment diagrams for each member of the frame.



4 ANALYSIS OF STATICALLY DETERMINATE TRUSSES

4.1 Common Types of Trusses

A ***truss*** is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts, metal bars, angles, or channels. The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called a *gusset plate,* as shown in Fig. 4–1, or by simply passing a large bolt or pin through each of the members. Planar trusses lie in a single plane and are often used to support roofs and bridges.

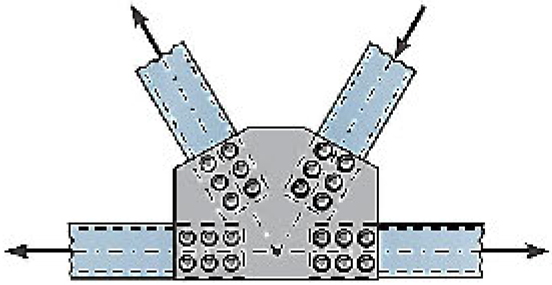


Fig. 4-1

Roof Trusses. Roof trusses are often used as part of an industrial building frame, such as the one shown in Fig. 4–2. Here, the roof load is transmitted to the truss at the joints by means of a series of *purlins*. The roof truss along with its supporting columns is termed a *bent*. Ordinarily, roof trusses are supported either by columns of wood, steel, or reinforced concrete, or by masonry walls. To keep the bent rigid, and thereby capable of resisting horizontal wind forces, knee braces are sometimes used at the supporting columns.

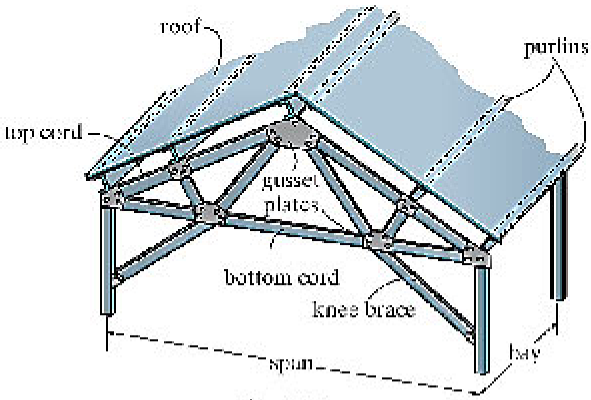


Fig. 4-2

Trusses used to support roofs are selected on the basis of the span, the slope, and the roof material. Some of the more common types of trusses used are shown in Fig. 4–3. In particular, the scissors truss, Fig. 4–3a, can be used for short spans that require overhead clearance. The Howe and Pratt trusses, Figs. 4–3b and 4–3c, are used for roofs of moderate span, about 60 ft (18 m) to 100 ft (30 m). If larger spans are required to support the roof, the fan truss or Fink truss may be used, Figs. 4–3d and 4–3e. These trusses may be built with a cambered bottom cord such as that shown in Fig. 4–3f. If a flat roof or nearly flat roof is to be selected, the Warren truss, Fig. 4–3g, is often used. Also, the Howe and Pratt trusses may be modified for flat roofs. Sawtooth trusses, Fig. 4–3h, are often used where column spacing is not objectionable and uniform lighting is important. A textile mill would be an example. The bowstring truss, Fig. 4–3i, is sometimes selected for garages and small airplane hangars; and the arched truss, Fig. 4–3j, although relatively expensive, can be used for high rises and long spans such as field houses, gymnasiums, and so on.

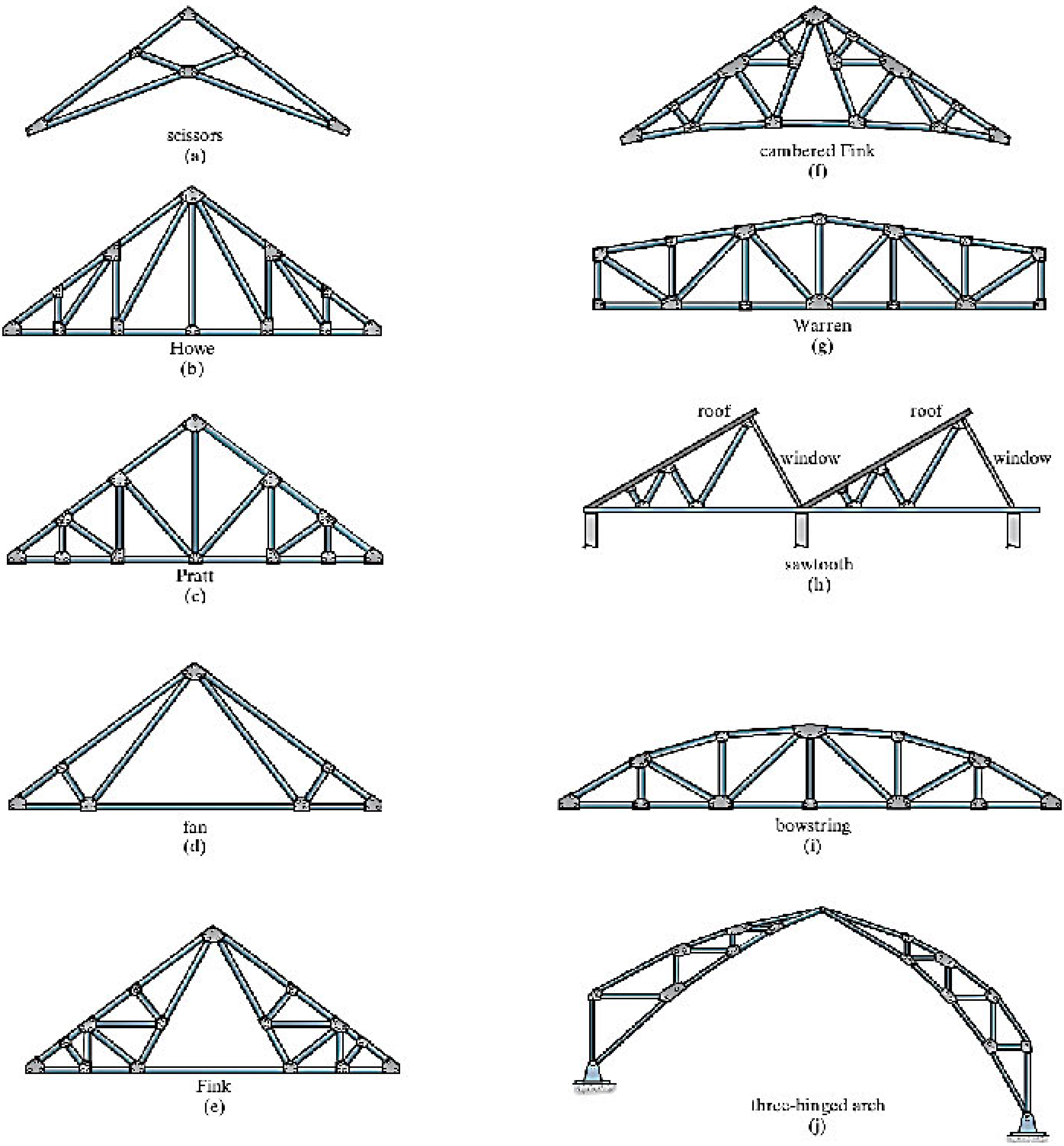

scissors
(a)
cambered Fink
(f)
Howe
(b)
Warren
(g)
roof
window
roof
window
Pratt
(c)
sawtooth
(h)
fan
(d)
bowstring
(i)
Fink
(e)
three-hinged arch
(j)

Fig. 4-3

Bridge Trusses. The main structural elements of a typical bridge truss are shown in Fig. 4–4. Here it is seen that a load on the *deck* is first transmitted to *stringers,* then to *floor beams,* and finally to the joints of the two supporting side trusses.The top and bottom cords of these side trusses are connected by top and bottom *lateral bracing,* which serves to resist the lateral forces caused by wind and the sidesway caused by moving vehicles on the bridge. Additional stability is provided by the *portal* and *sway bracing*. As in the case of many long-span trusses, a roller is provided at one end of a bridge truss to allow for thermal expansion.

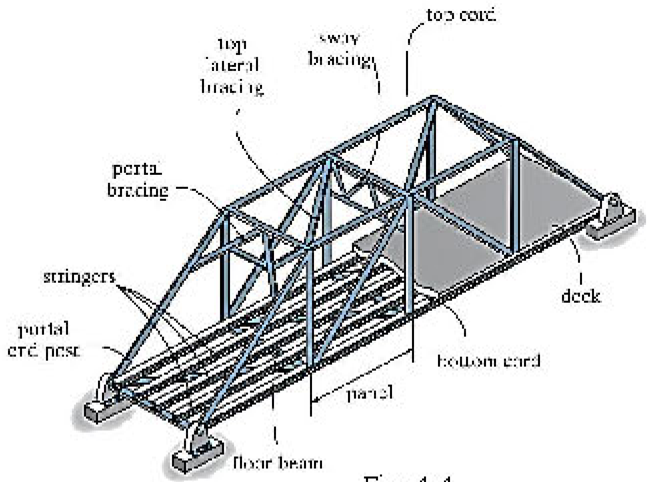


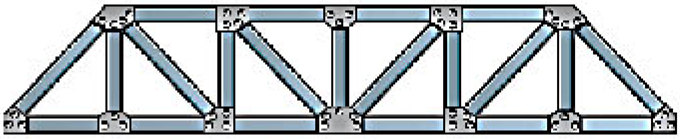
Fig. 4-4

A few of the typical forms of bridge trusses currently used for single spans are shown in Fig. 4–5. In particular, the Pratt, Howe, and Warren trusses are normally used for spans up to 200 ft (61 m) in length. The most common form is the Warren truss with verticals, Fig. 4–5*c*. For larger spans, a truss with a polygonal upper cord, such as the Parker truss, Fig. 4–5*d*, is used for some savings in material. The Warren truss with verticals can also be fabricated in this manner for spans up to 300 ft (91 m). The greatest economy of material is obtained if the diagonals have a slope between 45° and 60° with the horizontal. If this rule is maintained, then for spans greater than 300 ft (91 m), the depth of the truss must increase and consequently the panels will get longer. This results in a heavy deck system and, to keep the weight of the deck within tolerable limits, *subdivided* trusses have been developed.Typical examples include the Baltimore and subdivided Warren trusses, Figs. 4–5*e* and 4–5*f*. Finally, the K-truss shown in Fig. 4–5g can also be used in place of a subdivided truss, since it accomplishes the same purpose.

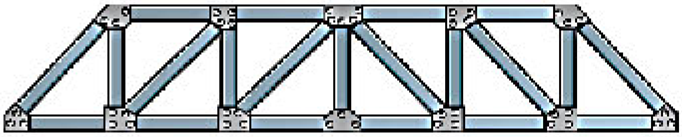
Assumptions for Design. To design both the members and the connections of a truss, it is first necessary to determine the *force* developed in each member when the truss is subjected to a given loading. In this regard, two important assumptions will be made in order to idealize the truss.

1 *The members are joined together by smooth pins*. In cases where bolted or welded joint connections are used, this assumption is generally satisfactory provided the center lines of the joining members are concurrent at a point, as in Fig. 4–1.

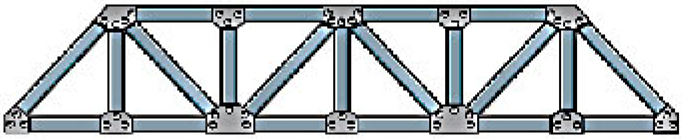
2 *All loadings are applied at the joints*. In most situations, such as for bridge and roof trusses, this assumption is true. Frequently in the force analysis, the weight of the members is neglected, since the force supported by the members is large in comparison with their weight.



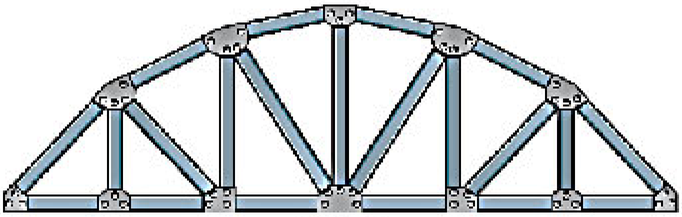
Pratt
(a)



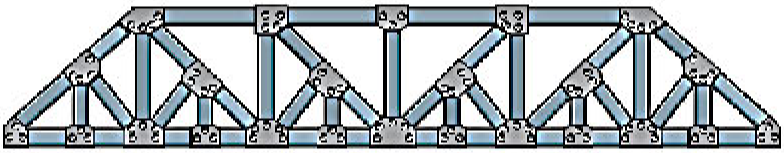
Howe
(b)



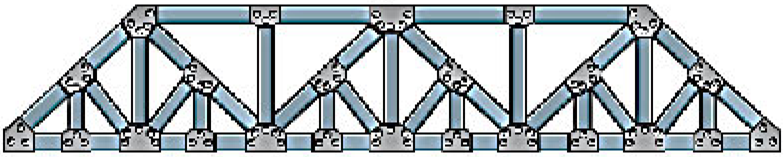
Warren (with verticals)
(c)



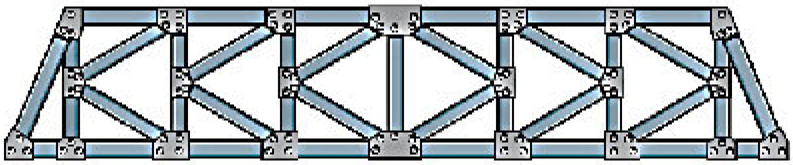
Parker
(d)



Baltimore
(e)



subdivided Warren
(f)



K-truss
(g)

Fig. 4-5

4.2 Classification of Coplanar Trusses

Before beginning the force analysis of a truss, it is important to classify the truss as simple, compound, or complex, and then to be able to specify its determinacy and stability.

Simple Truss.
To prevent collapse, the framework of a truss must be rigid. Obviously, the four-bar frame ***ABCD*** in Fig. 4–6 will collapse unless a diagonal, such as ***AC***, is added for support. The simplest framework that is rigid or stable is a triangle. Consequently, a simple truss is constructed by starting with a basic triangular element, such as ***ABC*** in Fig. 4–7, and connecting two members (***AD*** and ***BD***) to form an additional element. Thus it is seen that as each additional element of two members is placed on the truss, the number of joints is increased by one.

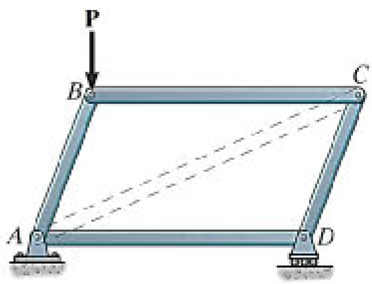


Fig. 4–6

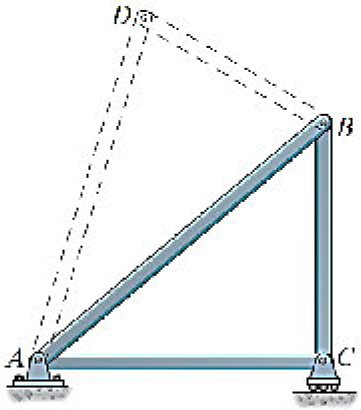
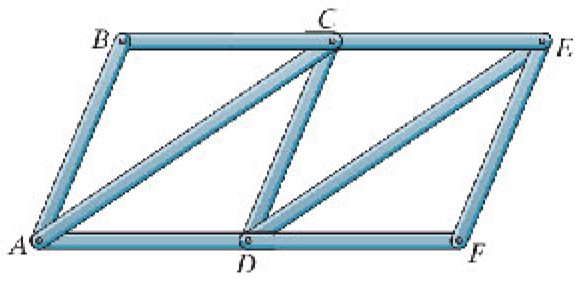


Fig. 4–7



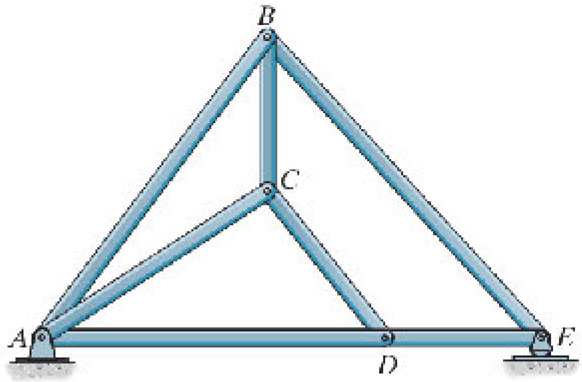
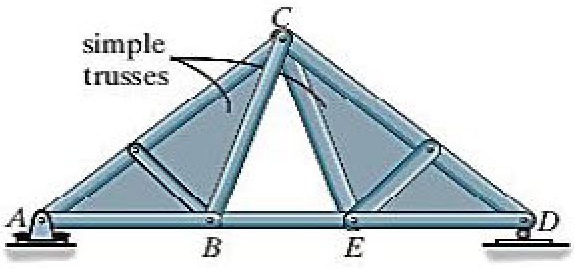


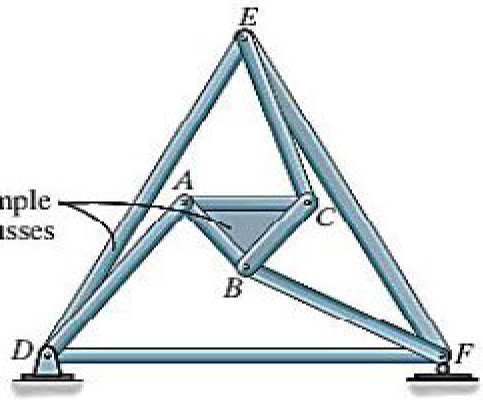
Fig. 4–8

Compound Truss.
A *compound truss* is formed by connecting two or more simple trusses together. Quite often this type of truss is used to support loads acting over a *large span,* since it is cheaper to construct a somewhat lighter compound truss than to use a heavier single simple truss.
The trusses may be connected by a common **joint** and **bar**. An example is given in Fig. 4–9*a*, where the shaded truss ***ABC*** is connected to the shaded truss ***CDE*** in this manner. The trusses

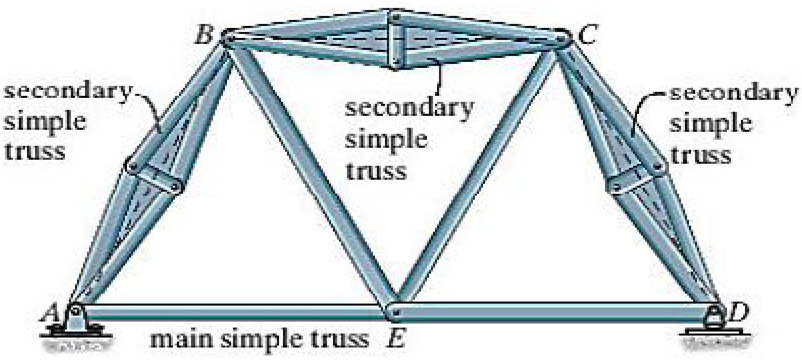
may be joined by three bars, as in the case of the shaded truss *ABC* connected to the larger truss ***DEF***, Fig. 4–9*b*. And finally, the trusses may be joined where bars of a large simple truss, called the *main truss,* have been *substituted* by simple trusses, called *secondary trusses*. An example is shown in Fig. 4–9*c*, where dashed members of the main truss ***ABCDE*** have been *replaced* by the secondary shaded trusses.



(a)



(b)



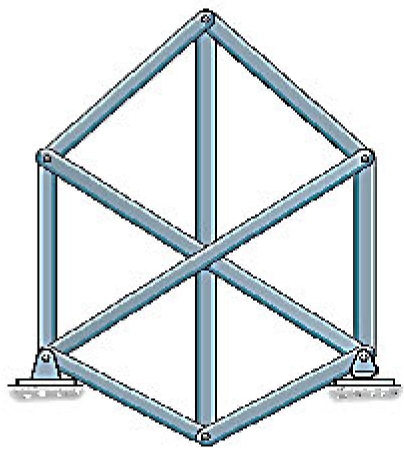
(c)

Various types of compound trusses

Fig. 4–9

Complex Truss.

A *complex truss* is one that cannot be classified as being either simple or compound. The truss in Fig. 4–10 is an example.



Complex truss

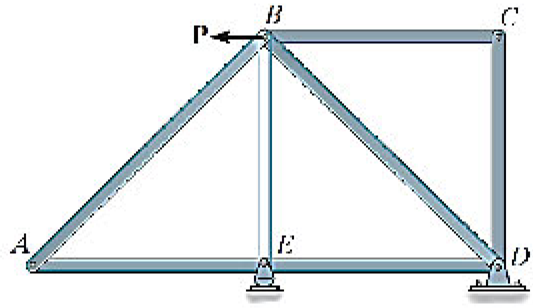
Fig. 4–10

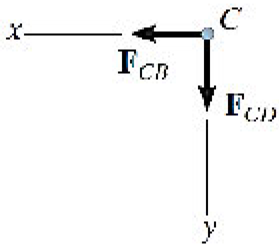
4.3 Analysis of Trusses

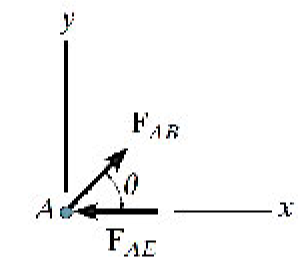
4.3.1 Zero-Force Members

Truss analysis using the method of joints is greatly simplified if one is able to first determine those members that support ***no loading***. These ***zero-force members*** may be necessary for the stability of the truss during construction and to provide support if the applied loading is changed. The zero-force members of a truss can generally be determined by inspection of the joints, and they occur in two cases.

- ✓ If **only two non-collinear** members form a truss joint and no external load or support reaction is applied to the joint, the members must be zero-force members.







$\overset{+}{\leftarrow} \Sigma F_x = 0;\ F_{CB} = 0$

$+\downarrow \Sigma F_y = 0;\ F_{CD} = 0$

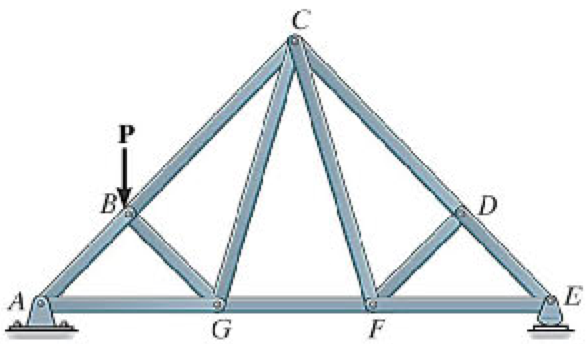
$+\uparrow \Sigma F_y = 0;\ F_{AB} \sin\theta = 0$

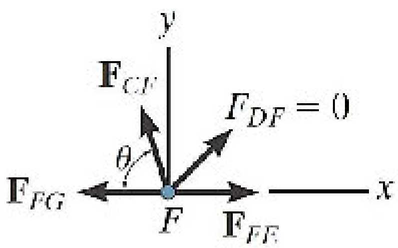
$F_{AB} = 0 \ \ (\text{since } \sin\theta \neq 0)$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\ -F_{AE} + 0 = 0$

$F_{AE} = 0$

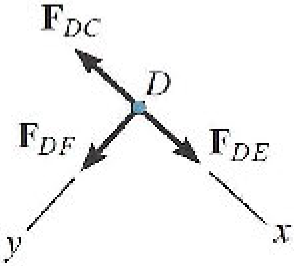
- ✓ If **three members** form a truss joint for which **two of the members are collinear**, the **third member is a zero-force member**, provided no external force or support reaction is applied to the joint.





$+\uparrow \Sigma F_y = 0;\ F_{CF} \sin\theta + 0 = 0$

$F_{CF} = 0 \ (\text{since } \sin\theta \neq 0)$

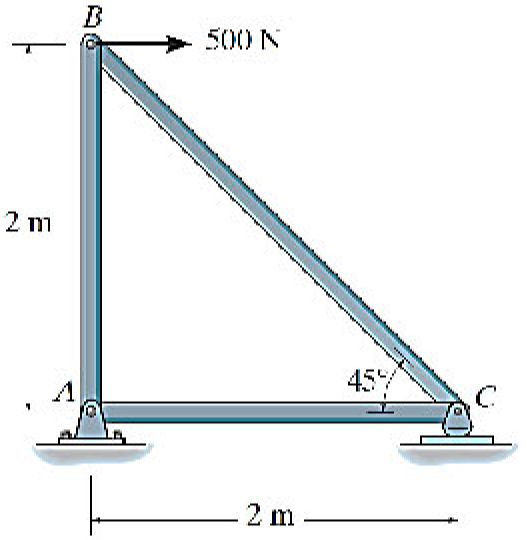


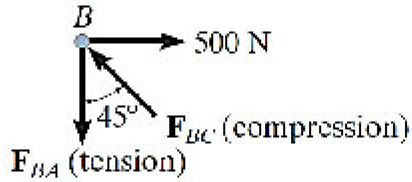
$+\swarrow \Sigma F_y = 0;\ F_{DF} = 0$

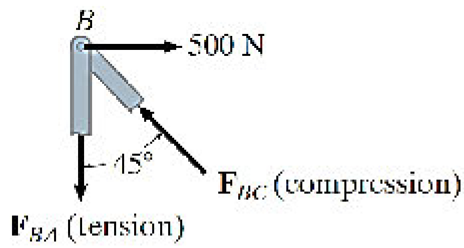
4.3.2 The Method of Joints

If a truss is in equilibrium, then each of its joints must also be in equilibrium. Hence, the method of joints consists of satisfying the equilibrium conditions $\sum F_x = 0$ and $\sum F_y = 0$ for the forces exerted *on the pin* at each joint of the truss.

- ✓ The joint analysis should start at a joint having at **least *one known*** force and at **most two unknown** forces, as in Figure.







- ✓ Draw the free-body diagram of a joint having at least one known force and at most two unknown forces. ***(If this joint is at one of the supports, it may be necessary to calculate the external reactions at the supports by drawing a free-body diagram of the entire truss.)***

- ✓ ***Always assume the unknown member forces acting on the joint's free-body diagram to be in tension, i.e., "pulling" on the pin.*** **this is done**, then numerical solution of the equilibrium equations will yield ***positive scalars for members in tension and negative scalars for members in compression.***

- ✓ Once an unknown member force is found, use its *correct* magnitude and sense (**T** or **C**) on subsequent joint free-body diagrams.

- ✓ Continue to analyze each of the other joints, where again it is necessary to choose a joint having at most two unknowns and at least one known force.

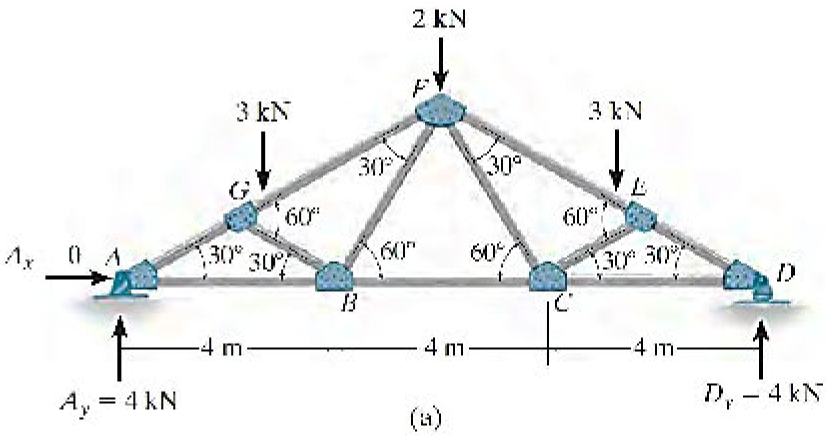
4.3.3 The Method of Sections

If the forces in only a few members of a truss are to be found, the method of sections generally provides the most direct means of obtaining these forces.

- ✓ It may first be necessary to determine the truss's ***external*** **reactions**,
- ✓ Try to select a section that, in general, passes through *not more than* ***three*** *members in which the forces are unknown*.
- ✓ Draw the free-body diagram of that part of the sectioned truss which has the least number of forces on it.
- ✓ Moments ($\sum M = 0$) should be summed about a point that lies ***at the intersection of the lines*** of action of ***two unknown*** forces; in this way, ***the third unknown force is determined*** directly from the equation.
- ✓ If two of the unknown forces are ***parallel***, forces may be summed ***perpendicular*** to the direction of these unknowns to determine ***directly*** **the third unknown** force.

EXAMPLE 4.3.1

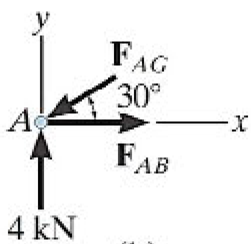
Determine the force in each member of the roof truss shown in the photo. The dimensions and loadings are shown in **Fig. *a***. State whether the members are in tension or compression.



(a)

Solution

Only the forces in half the members have to be determined, since the truss is symmetric with respect to *both* loading and geometry.



(b)

Joint A, Fig. b.

The free-body diagram is shown in **Fig. *b***.

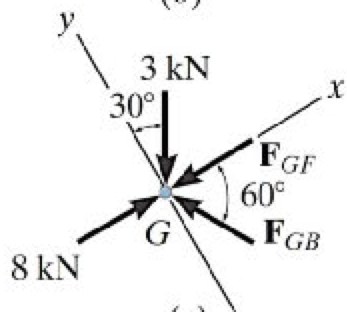
$$+\uparrow \sum F_y = 0;\ \ 4 - F_{AG}\sin 30^\circ = 0 \quad \Rightarrow \quad F_{AG} = 8\text{ kN (C)}$$

$$+\rightarrow \sum F_x = 0;\ \ F_{AB} - 8\cos 30^\circ = 0 \quad \Rightarrow \quad F_{AB} = 6.928\text{ kN (T)}$$

Joint G, Fig. c.

$$+\nwarrow \sum F_y = 0;\ \ F_{GB}\sin 60^\circ - 3\cos 30^\circ = 0 \quad \Rightarrow \quad F_{GB} = 3\text{ kN (C)}$$

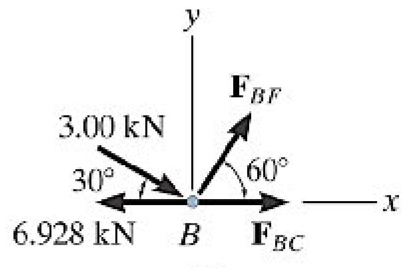
$$+\nearrow \sum F_x = 0;\ \ 8 - 3\sin 30^\circ - 3\cos 60^\circ - F_{GF} = 0 \quad \Rightarrow \quad F_{GF} = 5\text{ kN (C)}$$



(c)

Joint B, Fig. d.

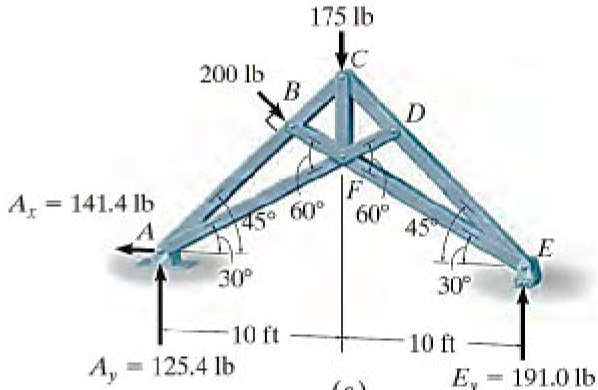
$$+\uparrow\sum F_y = 0;\ \ F_{BF}\sin 60^\circ - 3\sin 30^\circ = 0$$
$$\Rightarrow\ \ F_{BF} = 1.73\ \text{kN (T)}$$
$$+\rightarrow\sum F_x = 0;\ \ F_{BC} + 1.73\cos 60^\circ + 3\cos 30^\circ - 6.928 = 0$$
$$\Rightarrow\ \ F_{BC} = 3.46\ \text{kN (T)}$$



(d)

EXAMPLE 4.3.2

Determine the force in each member of the scissors truss shown in **Fig. *a***. State whether the members are in tension or compression. The reactions at the supports are given.

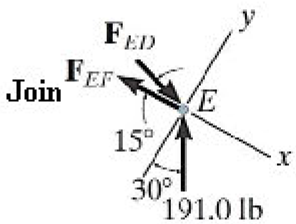


(a)

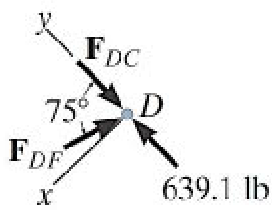
Solution

Joint E, Fig. b.

$$+\nearrow\sum F_y = 0;\ \ 191.0\cos 30^\circ - F_{ED}\sin 15^\circ = 0$$
$$\Rightarrow\ \ F_{ED} = 639.1\ \text{lb (C)}$$
$$+\searrow\sum F_x = 0;\ \ 639.1\cos 15^\circ - F_{EF} - 191.0\sin 30^\circ = 0$$
$$\Rightarrow\ \ F_{EF} = 521.8\ \text{lb (T)}$$
$$+\swarrow\sum F_x = 0;\ \ -F_{DF}\sin 75 = 0\ \ \Rightarrow\ \ F_{DF} = 0$$
$$+\nwarrow\sum F_y = 0;\ \ -F_{DC} + 639.1 = 0\ \ \Rightarrow\ \ F_{DC} = 639.1\ \text{lb (C)}$$



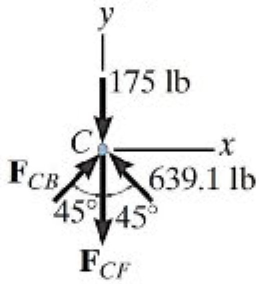
(b)



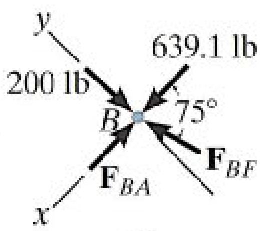
(c)

Joint C, Fig. d.

$$+\rightarrow\sum F_x = 0;\ \ F_{CB}\sin 45^\circ - 639.1\sin 45^\circ = 0$$
$$\Rightarrow\ \ F_{CB} = 639.1\ \text{lb (C)}$$
$$+\uparrow\sum F_y = 0;\ \ -F_{CF} - 175 + 2(639.1)\cos 45^\circ = 0$$
$$\Rightarrow\ \ F_{DC} = 728.8\ \text{lb (T)}$$



(d)



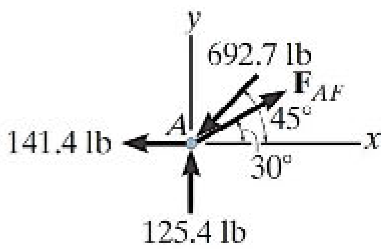
(e)

Joint B, Fig. e.

$$+\nwarrow\sum F_y = 0;\ \ F_{BF}\sin 75^\circ - 200 = 0\ \ \Rightarrow\ \ F_{BF} = 207.1\ \text{lb (C)}$$
$$+\swarrow\sum F_x = 0;\ \ 639.1 + 207.1\cos 75^\circ - F_{BA} = 0\ \Rightarrow\ F_{BA} = 692.7\ \text{lb (C)}$$

Joint A, Fig. f.

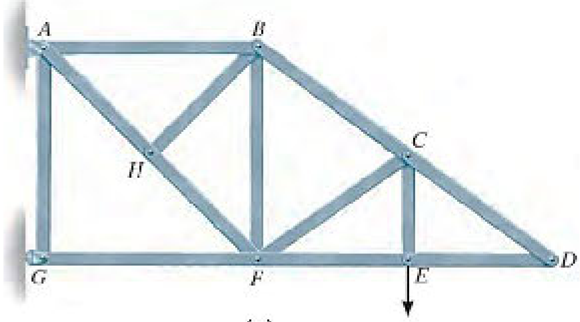
$$+\rightarrow\sum F_x = 0;\ \ F_{AF}\cos 30^\circ - 692.7\cos 45^\circ - 141.4 = 0$$
$$\Rightarrow F_{AF} = 728.9\ \text{lb (T)}$$
$$+\uparrow\sum F_y = 0;\ \ 125.4 - 692.7\sin 45^\circ + 728.9\sin 30^\circ = 0\ \ \text{Check}$$



(f)

EXAMPLE 4.3.3

Using the method of joints, indicate all the members of the truss shown in **Fig. *a*** that have zero force.



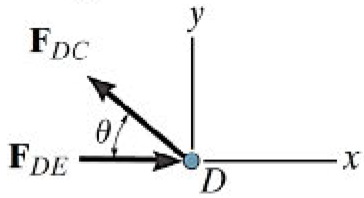
(a)

Solution

Joint D, Fig. b.

$+\uparrow \sum F_y = 0; \quad F_{DC} \sin\theta = 0 \quad F_{DC} = 0$

$+\rightarrow \sum F_x = 0; \quad F_{DE} + 0 = 0 \quad F_{DE} = 0$



(b)

Joint E, Fig. .

$+\leftarrow \sum F_x = 0; \quad F_{EF} = 0$

Note that $F_{EC} = P$ and an analysis of joint *C* would yield a force in member *CF*.

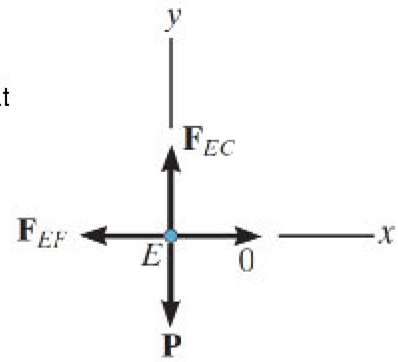
Joint H, Fig. d.

$+\nearrow \sum F_y = 0; \quad F_{HB} = 0$

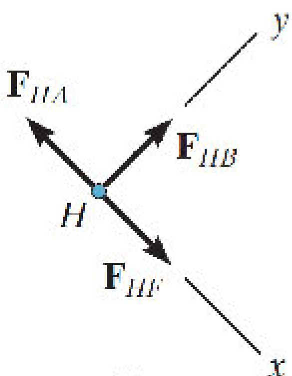
Joint G, Fig. e.

The rocker support at *G* can only exert an *x* component of force on the joint

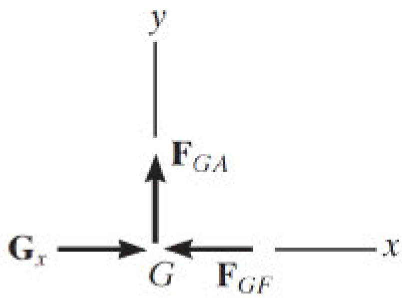
$+\uparrow \sum F_y = 0; \quad F_{GA} = 0$



(c)



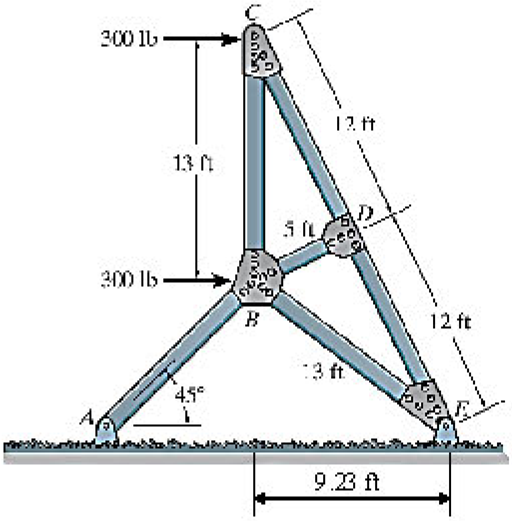
(d)



(e)

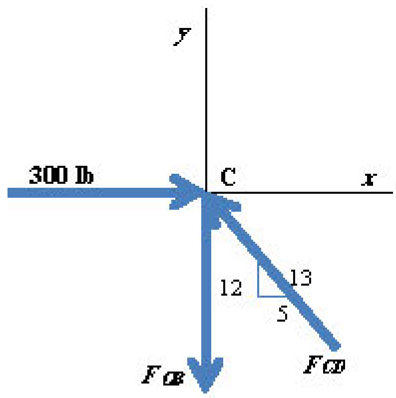
EXAMPLE 4.3.4

A sign is subjected to a wind loading that exerts horizontal forces of **300 lb** on joints *B* and *C* of one of the side supporting trusses. Determine the force in each member of the truss and state if the members are in tension or compression.



Solution

Joint ***C*****: Fig** ***a*****.**



$$+\rightarrow\sum F_x = 0;\quad 300 - F_{CD}\left(\frac{5}{13}\right) = 0 \quad \Rightarrow F_{CD} = 780 \text{ lb (C)}$$

$$+\uparrow\sum F_y = 0;\quad 780\left(\frac{12}{13}\right) - F_{CB} = 0 \quad \Rightarrow F_{CB} = 720 \text{ lb (T)}$$

Joint ***D*****: Fig** ***b*****.**

$$+\nearrow\sum F_x = 0;\quad F_{DB} = 0$$

$$+\nwarrow\sum F_y = 0;\quad F_{DE} - 780 = 0 \quad \Rightarrow F_{DE} = 780 \text{ lb (C)}$$

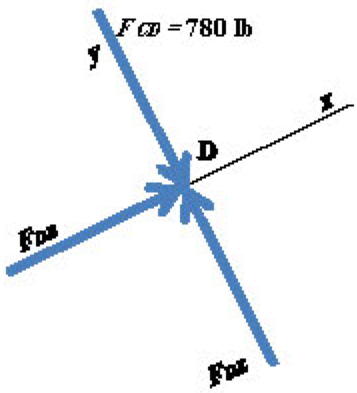
Joint ***B*****: Fig** ***b*****.**

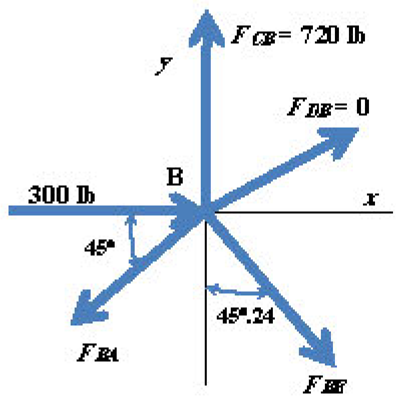
$$+\rightarrow\sum F_x = 0;\quad 300 + F_{BE}\sin 45.24^\circ - F_{BA}\cos 45^\circ = 0$$

$$+\uparrow\sum F_y = 0;\quad 720 - F_{BE}\cos 45.24^\circ - F_{BA}\sin 45^\circ = 0$$

Solving

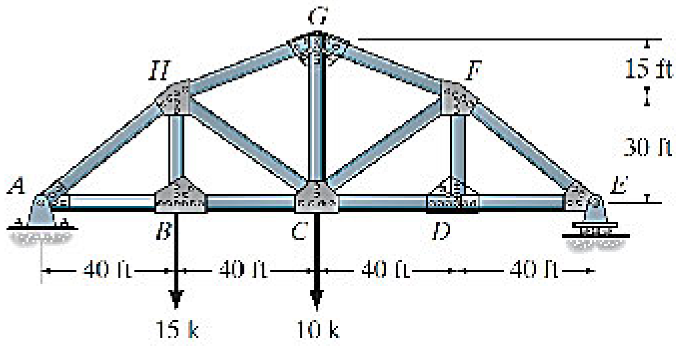
$$F_{BE} = 296.99 \text{ lb} = 297 \text{ lb (T)} \qquad F_{BA} = 722.49 \text{ lb (T)}$$





EXAMPLE 4.3.5

Determine the force in members ***GF***, ***FC***, and ***CD*** of the bridge truss. State if the members are in tension of compression. Assume all members are pin connected.



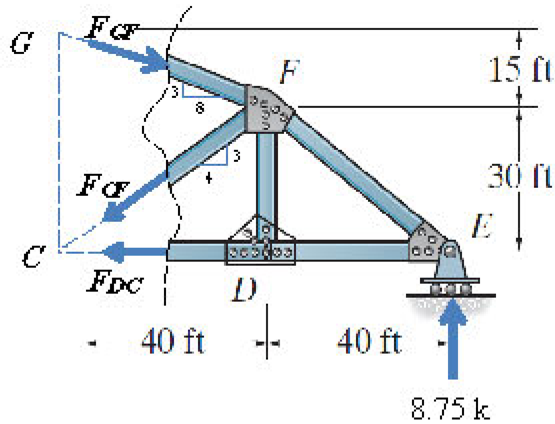
Solution

$$+\circlearrowleft \sum M_A = 0; \quad -15(40) - 10(80) + R_E(160) = 0$$

$$\Rightarrow R_E = 8.75\ k$$

$$+\circlearrowleft \sum M_F = 0; \quad -F_{DC}(30) + 8.75(40) = 0$$

$$\Rightarrow F_{DC} = 11.7\ \mathbf{k}\ \ (\mathrm{T})$$



$$+\circlearrowleft \sum M_C = 0; \quad -F_{FG}\left(\frac{8}{\sqrt{73}}\right)(45) + 8.75(80) = 0$$

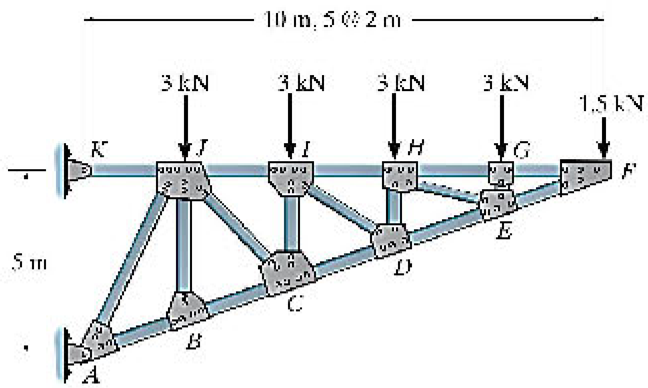
$$\Rightarrow F_{FG} = \mathbf{16.6\ k}\ \ \mathbf{(C)}$$

$$+\uparrow \sum F_y = 0; \quad 8.75 - 16.6\left(\frac{3}{\sqrt{73}}\right) - F_{FC}\left(\frac{3}{5}\right) = 0$$

$$\Rightarrow F_{FC} = 4.86\ \mathbf{k}\ \ (\mathrm{T})$$

EXAMPLE 4.3.6

Determine the force in members ***IH***, ***ID***, and ***CD*** of the truss. State if the members are in tension or compression. Assume all members are pin connected.



Solution

Referring to the FBD of the right segment of the truss sectioned through, **Fig.**,

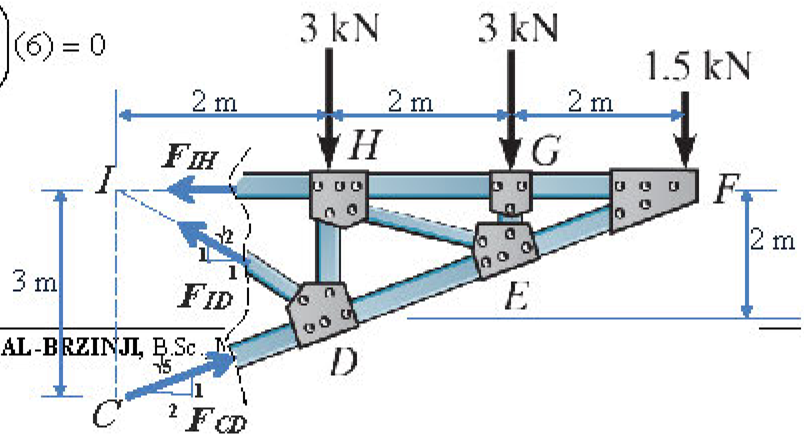
$$+\circlearrowleft \sum M_D = 0; \quad F_{IH}(2) - 3(2) - 1.5(4) = 0$$

$$\Rightarrow F_{IH} = \mathbf{6\ kN}\ \ \mathbf{(T)}$$

$$+\circlearrowleft \sum M_F = 0; \quad 3(2) + 3(4) - F_{ID}\left(\frac{1}{\sqrt{2}}\right)(6) = 0$$

$$\Rightarrow F_{ID} = 4.24\ \mathbf{kN}\ \ (\mathrm{T})$$

$$+\circlearrowleft \sum M_I = 0;$$

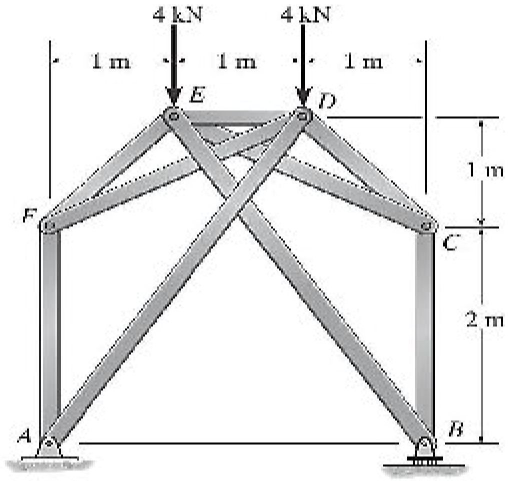


$$F_{CD}\left(\frac{2}{\sqrt{5}}\right)(3)-3(2)-3(4)-1.5(6)=0$$

$$\Rightarrow F_{CD}=10.06 \text{ kN (C)}$$

EXAMPLE 4.3.7

Determine the force in each member and state if the members are in tension or compression.



Solution

Reactions

$$\sum M_B=0, \quad \Rightarrow A_y = 4.00 \text{ kN}$$

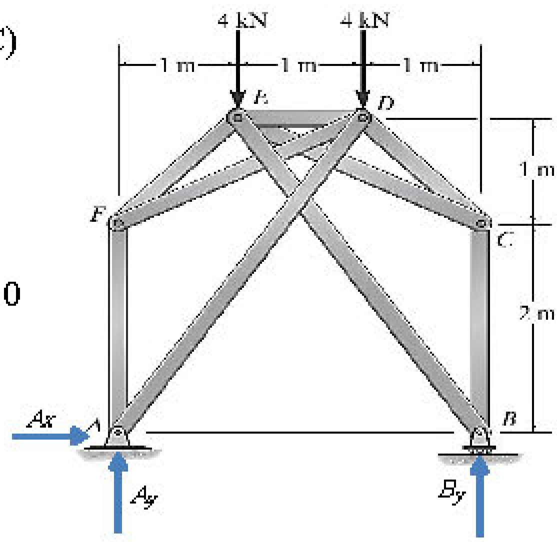
$$\sum F_y=0, \quad \Rightarrow B_y = 4.00 \text{ kN}$$

$$\sum F_x=0, \quad \Rightarrow A_x = 0$$

Joint *A*:

$$+\rightarrow\sum F_x = 0; \quad F_{AD} = 0$$

$$+\uparrow \sum F_y = 0; \quad 4.00 - F_{AF} = 0; \quad F_{AF} = 4.00 \text{ kN (C)}$$



Joint *F*:

$$+\nwarrow\sum F_y = 0; \quad 4.00 \sin 45° - F_{FD} \sin 18.43° = 0$$

$$F_{FD} = 8.944 \text{ kN} = 8.94 \text{ kN (T)}$$

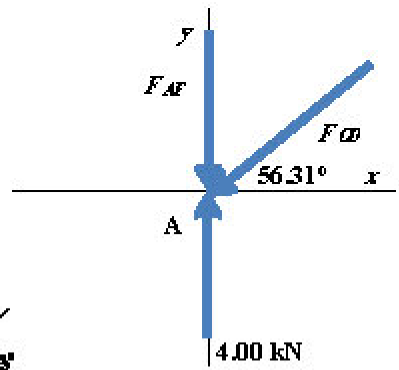
$$+\nearrow\sum F_x = 0; \quad 4.00 \cos 45° - 8.94 \cos 18.43° - F_{FE} = 0$$

$$F_{FE} = 11.313 \text{ kN} = 8.94 \text{ kN (C)}$$

Due to symmetrical loading and geometry

$$F_{BC} = 4.00 \text{ kN (C)}, \quad F_{CE} = 8.94 \text{ kN (T)}$$

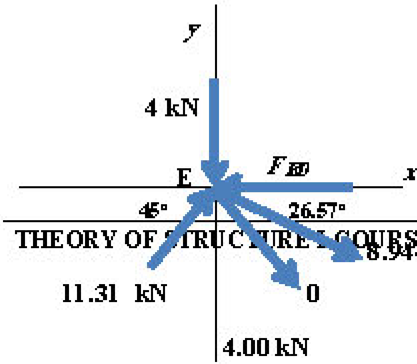
$$F_{BE} = 0, \quad F_{CD} = 11.3 \text{ kN (C)}$$

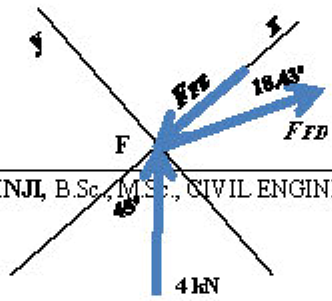


Joint *E*:

$$+\rightarrow\sum F_x = 0; \quad -F_{ED} + 8.944 \cos 26.56° + 11.31 \cos 45° = 0$$

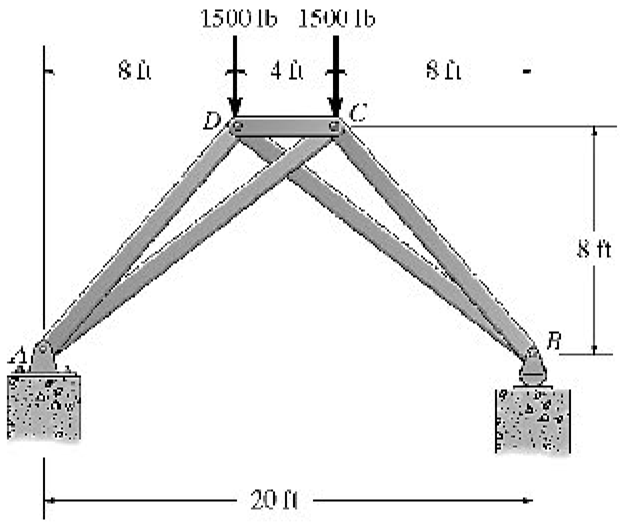
$$F_{ED} = 16.0 \text{ kN (C)}$$





EXAMPLE 4.3.8

Determine the force in each member and state if the members are in tension or compression.



Solution

Reactions

$\sum M_B = 0, \quad \Rightarrow A_y = 1500 \text{ Ib}$

$\sum F_y = 0, \quad \Rightarrow B_y = 1500 \text{ Ib}$

$\sum F_x = 0, \quad \Rightarrow A_x = 0$

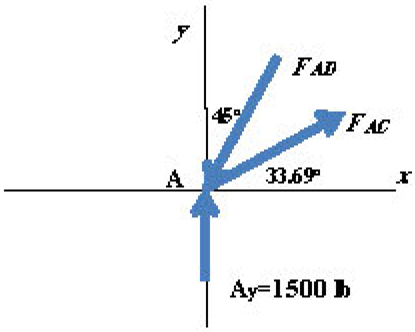
Joint *A*:

$+\rightarrow \sum F_x = 0; \quad F_{AC}\cos 33.69^\circ - F_{AD}\cos 45^\circ = 0$

$+\uparrow \sum F_y = 0; \quad 1500 - F_{AD}\sin 45^\circ + F_{AC}\sin 33.69^\circ = 0;$

$F_{AC} = 5408.3 \text{ lb} = 5.41 \text{ k (T)}$

$F_{AD} = 6363.9 \text{ lb} = 6.36 \text{ k (C)}$



Joint *D*:

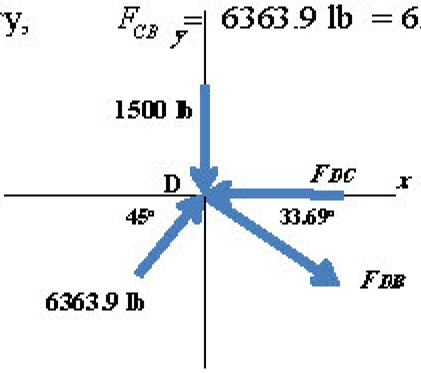
$+\uparrow \sum F_y = 0; \quad 6363.9\sin 45^\circ - 1500 - F_{DB}\sin 33.69^\circ = 0$

$F_{DB} = 5408.3 \text{ lb} = 5.41 \text{ k (T)}$

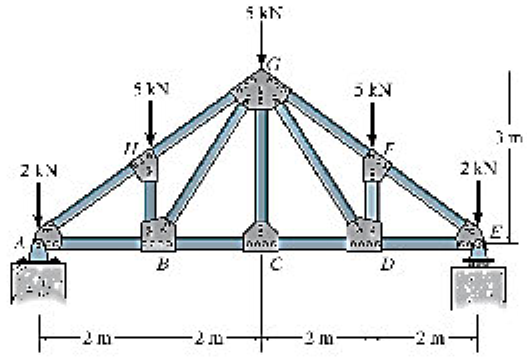
$+\rightarrow \sum F_x = 0; \quad 6363.9\cos 45^\circ - F_{DC} - F_{DB}\sin 33.69^\circ = 0;$

$F_{DC} = 9000 \text{ lb} = 9.00 \text{ k (C)}$

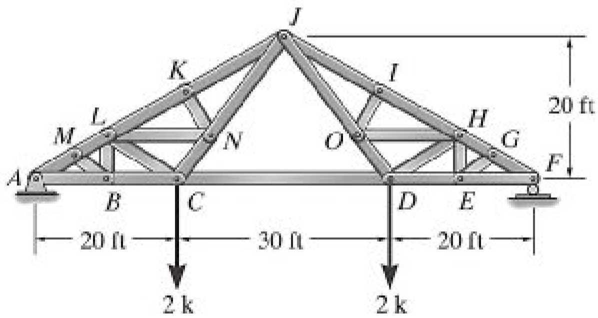
by symmetry, $F_{CB} = 6363.9 \text{ lb} = 6.36 \text{ k (C)}$



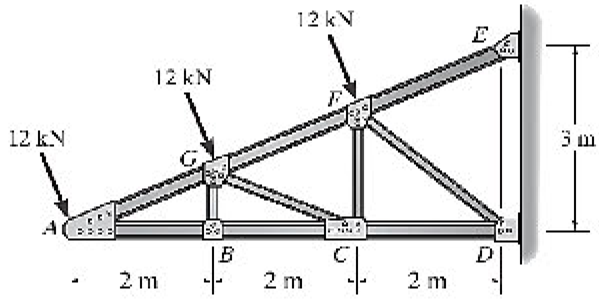
Hw.10 The Howe truss is subjected to the loading shown. Determine the forces in members **GF**, **CD**, and **GC**. State if the members are in tension or compression. Assume all members are pin connected.



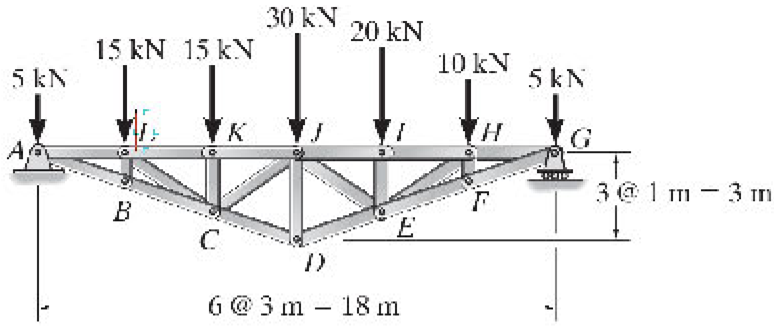
Hw.11
Determine the force in members *JK*, *JN*, and *CD*. State if the members are in tension or compression. Identify all the zero-force members.



Hw.12
Determine the force in members *GF*, *FC*, and *CD* of the cantilever truss. State if the members are in tension or compression. Assume all members are pin connected.

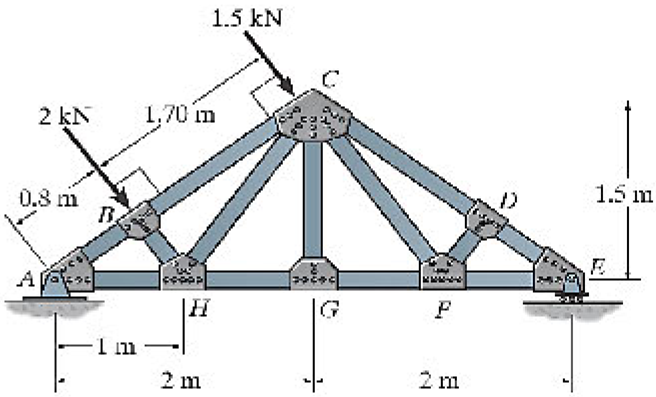


Hw.13
Determine the forces in members *KJ*, *CD*, and *CJ* of the truss. State if the members are in tension or compression.



Hw.14

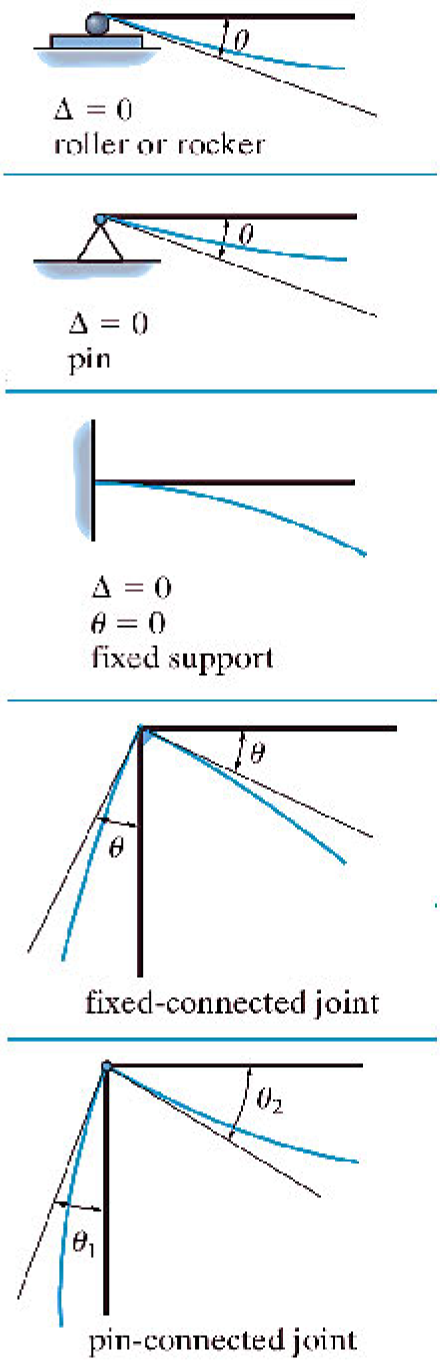
Determine the force in members ***GF***, ***CF***, and ***CD*** of the roof truss and indicate if the members are in tension or compression.



5 DEFLECTIONS

5.1 Deflection Diagrams and the Elastic Curve

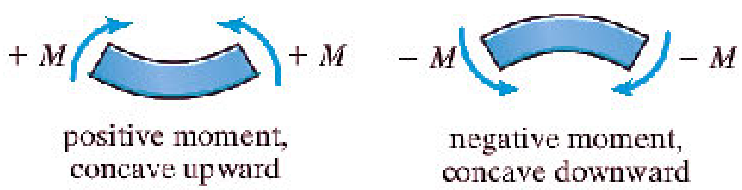
Deflections of structures can occur from various sources, such as loads, temperature, fabrication errors, or settlement. In design, deflections must be limited in order to provide integrity and stability of roofs, and prevent cracking of attached brittle materials such as concrete, plaster or glass.



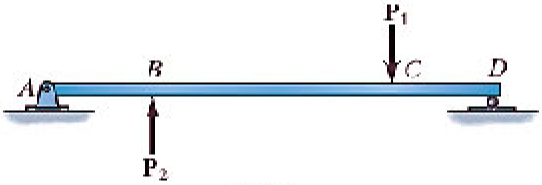
Furthermore, a structure must not vibrate or deflect severely in order to "appear" safe for its occupants. More important, though, deflections at specified points in a structure must be determined if one is to analyze statically indeterminate structures.

The deflections to be considered throughout this text apply only to structures having ***linear elastic material response***. Under this condition, a structure subjected to a load will return to its original undeformed position after the load is removed.

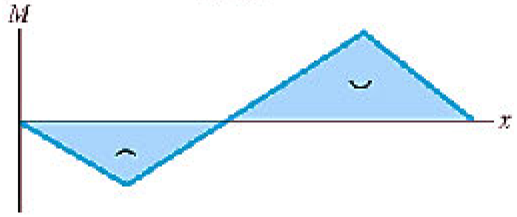
- ✓ The deflection of a structure is caused by its internal loadings such as normal force, shear force, or bending moment.
- ✓ For ***beams*** and ***frames***, however, the greatest deflections are most often caused by ***internal bending***, whereas ***internal axial forces*** cause the deflections of a ***truss***.
- ✓ Before the slope or displacement of a point on a beam or frame is determined, it is often helpful to sketch the deflected shape of the structure when it is loaded in order to partially check the results.
- ✓ This ***deflection diagram*** represents the ***elastic curve*** or locus of points which defines the displaced position of the centroid of the cross section along the members.
- ✓ If the elastic curve seems difficult to establish, it is suggested that the moment diagram for the beam or frame be drawn first.
- ✓ A ***positive*** moment tends to bend a beam or horizontal member ***concave upward***. Likewise, a ***negative moment*** tends to bend the beam or member ***concave downward***,



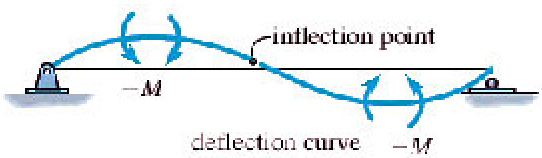
Deflection Diagrams and the Elastic Curve: The Double Integration Method



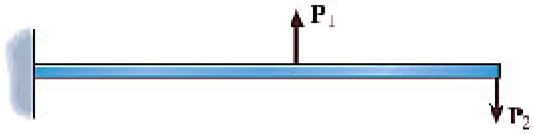
beam



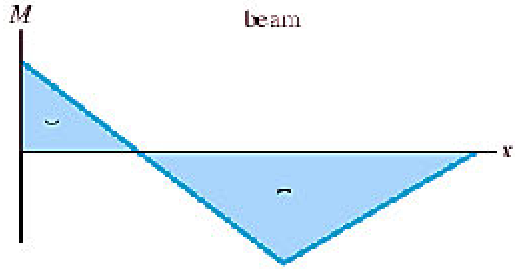
moment diagram



deflection curve



beam



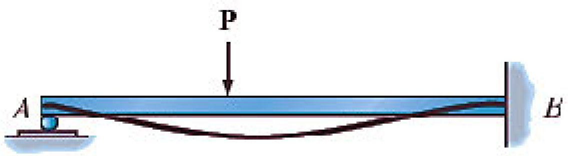
moment diagram

deflection curve

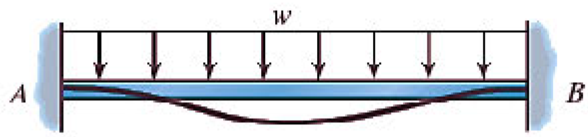
EXAMPLE 5.1.1

Draw the deflected shape of each of the beams shown in Figures.

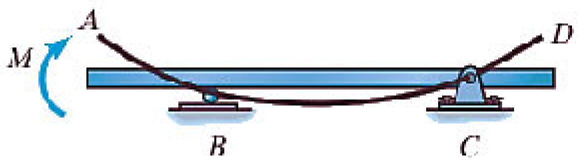
Solution



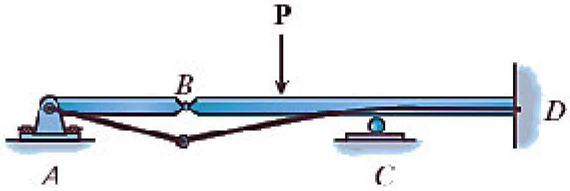
(a)



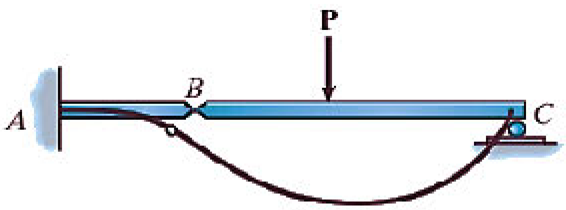
(b)



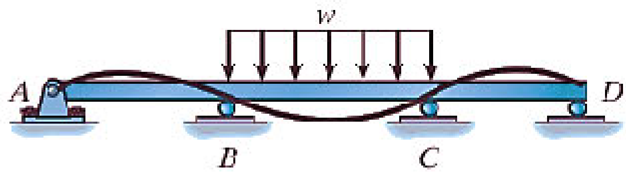
(c)



(d)



(e)

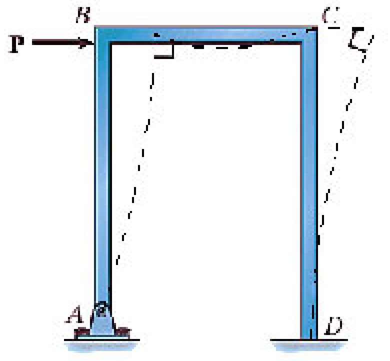


(f)

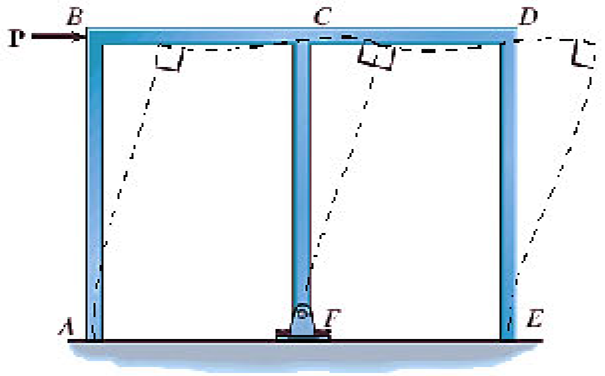
EXAMPLE 5.1.2

Draw the deflected shape of each of the frames shown in Figures.

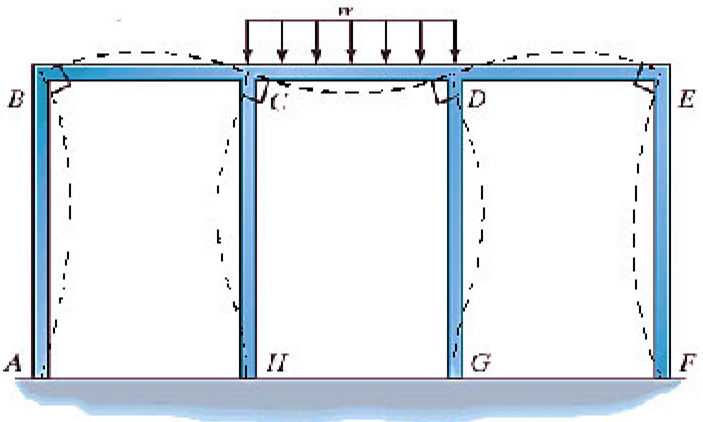
Solution



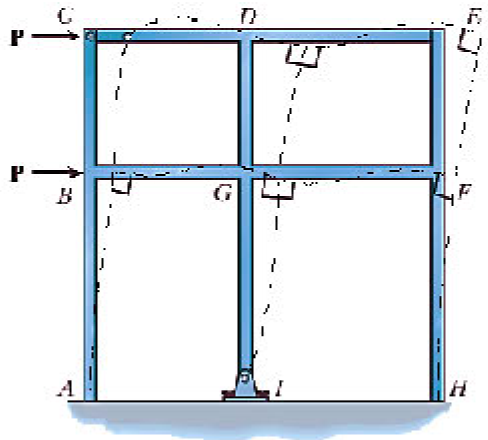
(a)



(b)



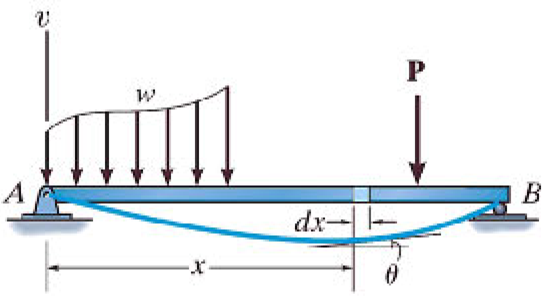
(c)



(d)

5.2 Elastic-Beam Theory

When the internal moment ***M*** deforms the element of the beam, each cross section remains plane and the angle between them becomes ***dθ***, **Fig. *b***. The arc ***dx*** that represents a portion of the elastic curve intersects the neutral axis for each cross section. The ***radius of curvature*** for this arc is defined as the distance , which is measured from ***the center of curvature O'*** to ***dx***. Any arc on the element other than ***dx*** is subjected to a normal strain.



(a)

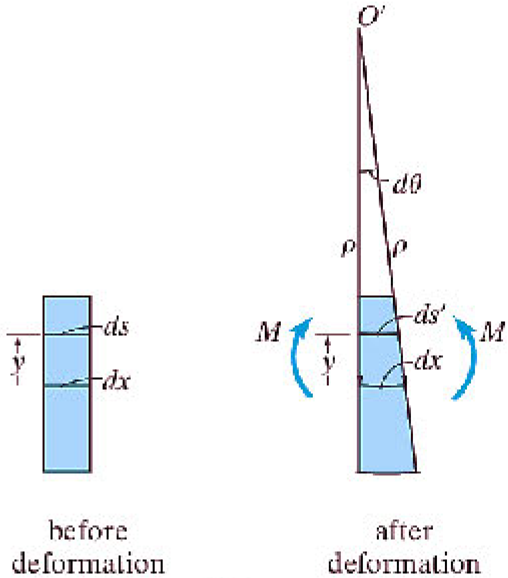
For example, the strain in arc ***ds***, located at a position ***y*** from the neutral axis, is

$$\epsilon = \left(ds' - ds\right)/ds$$

However,

$ds = dx = \rho d\theta$ and $ds' = (\rho - y)/d\theta$ and so

$$\epsilon = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} \quad \text{or} \quad \frac{1}{\rho} = \frac{\epsilon}{y}$$



(b)

If the material is homogeneous and behaves in a linear elastic manner, then Hooke's law applies, $\epsilon = \dfrac{\sigma}{E}$.

Also, since the flexure formula applies, $\sigma = -\dfrac{My}{I}$.

Combining these equations and substituting into the above equation, we have

$$\frac{1}{\rho} = \frac{M}{EI} \qquad ...(1)$$

Here

ρ = the radius of curvature at a specific point on the elastic curve ($1/\rho$ is referred to as the ***curvature***)
M = the internal moment in the beam at the point where ρ is to be determined
E = the material's modulus of elasticity
I = the beam's moment of inertia computed about the neutral axis

The product ***EI*** in this equation is referred to as the ***flexural rigidity,*** and it is always a positive quantity. Since $dx = \rho\, d\theta$, then from **Eq. 1**

$$d\theta = \frac{M}{EI}dx \qquad ...(2)$$

If we choose the ***v*** axis positive upward, **Fig. *a***, and if we can express the curvature **(1/ρ)** in terms of ***x*** and ***v***, we can then determine the elastic curve for the beam. The curvature relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

Therefore,

$$\frac{M}{EI} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \qquad ...(3)$$

This equation represents a nonlinear second-order differential equation. Its solution, ***v = f(x)***, gives the exact shape of the elastic curve-assuming, of course, that beam deflections occur only due to bending. In order to facilitate the solution of a greater number of problems, **Eq. 3** will be modified by making an important simplification. Since the slope of the elastic curve for most structures is very small, we will use <u>small deflection theory</u> and assume $dv/dx \approx 0$.

Consequently its square will be negligible compared to unity and therefore **Eq. 3** reduces to

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad ...(4)$$

It should also be pointed out that by assuming $dv/dx \approx 0$, the original length of the beam's axis ***x*** and the ***arc*** of its elastic curve will be approximately the same. In other words, ***ds*** in **Fig. *b*** is approximately equal to ***dx***, since

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (dv\ /\ dx)^2}\,dx \approx dx$$

This result implies that points on the elastic curve will only be displaced vertically and not horizontally.

5.3 The Double Integration Method

Once ***M*** is expressed as a function of position ***x***, then successive integrations of **Eq. 4** will yield the beam's slope, $\theta \approx \tan\theta = dv\ /\ dx = \int (M\ /\ EI)\, dx$ and the equation of the elastic curve, $v = f\,(x) = \iint (M\ /\ EI)\, dx$ respectively.

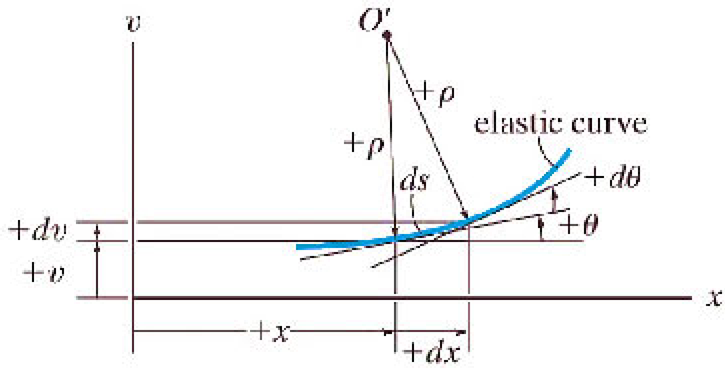
For each integration it is necessary to introduce a "constant of integration" and then solve for the constants to obtain a unique solution for a particular problem.

Sign Convention

When applying **Eq.4**, it is important to use the proper sign for ***M*** as established by the sign convention that was used in the derivation of this equation, **Fig.*a***.

(a)

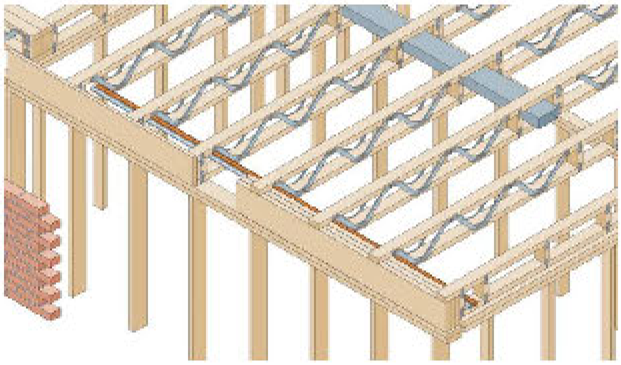
Furthermore, recall that positive deflection, ***v***, is upward, and as a result, the positive slope angle ***θ*** will be measured counterclockwise from the *x* axis. The reason for this is shown in **Fig.*b***. Here, positive increases ***dx*** and ***d*** in ***x*** and create an increase ***dθ*** that is counterclockwise. Also, since the slope angle will be very small, its value in radians can be determined directly from $\theta \approx \tan\theta = dv/dx$



(b)

EXAMPLE 5.3.1

Each simply supported floor joist shown in the photo is subjected to a uniform design loading of **4 kN/m**, **Fig.*a***. Determine the maximum deflection of the joist. ***EI*** is constant.

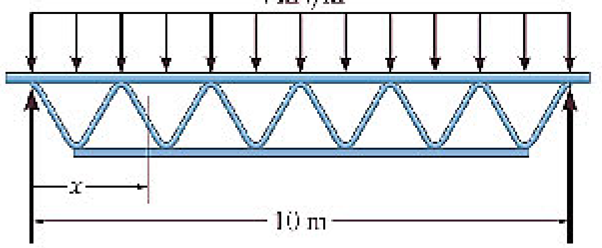


Solution

Elastic Curve.
Due to symmetry, the joist's maximum deflection will occur at its center.

Moment Function.
From the free-body diagram, **Fig.*b***, we have

$$M = 20x - 4x\left(\frac{x}{2}\right) = 20x - 2x^2$$



(a)

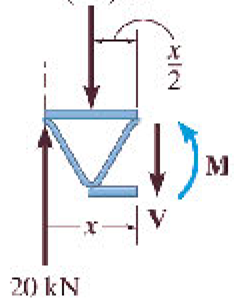
Slope and Elastic Curve.
Applying **Eq. 4** and integrating twice gives

$$EI\frac{d^2v}{dx^2} = 20x - 2x^2$$

$$EI\frac{dv}{dx} = 10x^2 - 0.6667x^3 + C_1$$

$$EIv = 3.333x^3 - 0.1667x^4 + C_1x + C_2$$

Here
$v = 0$ at $x = 0$ so that $C_2 = 0$,
and
$v = 0$ at $x = 10$, so that $C_1 = -166.7$.



(b)

The equation of the elastic curve is therefore

$$EIv = 3.333x^3 - 0.1667x^4 - 1.667x$$

At $x = 5$ **m** note that $dv/dx = 0$ The maximum deflection is therefore

$$v_{max} = -\frac{521}{EI}$$

EXAMPLE 5.3.2

The cantilevered beam shown in **Fig.*a*** is subjected to a couple moment M_0 at its end. Determine the equation of the elastic curve. ***EI*** is constant.

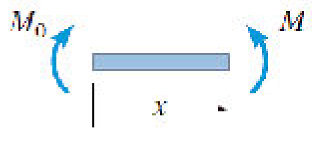
(a)

Solution

Elastic Curve.
The load tends to deflect the beam as shown in **Fig.*a***. By inspection, the internal moment can be represented throughout the beam.

Moment Function.

From the free-body diagram, with M acting in the ***positive direction***, **Fig.b** we have



(b)

$$M = M_0$$

Slope and Elastic Curve.

Applying **Eq.4** and integrating twice yields

$$EI\frac{d^2v}{dx^2} = M_0$$

$$EI\frac{dv}{dx} = M_0x + C_1$$

$$EIv = \frac{M_0x^2}{2} + C_1x + C_2$$

Using the boundary conditions a $dv/dx = 0$ at $x = 0$ and $v = 0$ at $x = 0$, then $C_1 = C_2 = 0$.

$$\theta = \frac{dv}{dx} = \frac{M_0x}{EI}$$

$$v = \frac{M_0x^2}{2EI}$$

Maximum slope and displacement occur at **A ($x = L$)**, for which

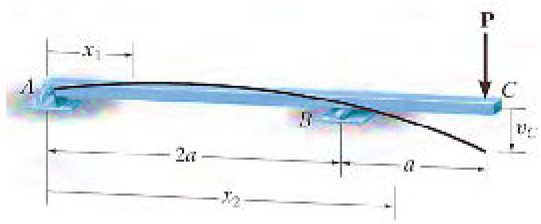
$$\theta_A = \frac{M_0L}{EI}$$

$$v_A = \frac{M_0L^2}{2EI}$$

The ***positive*** result for θ_A indicates ***counterclockwise*** rotation and the ***positive*** result for v_A indicates that v_A is ***upward***. This agrees with the results sketched in **Fig. *a***.

EXAMPLE 5.3.3

The beam in **Fig.*a*** is subjected to a load **P** at its end. Determine the displacement at C. EI is constant.



(a)

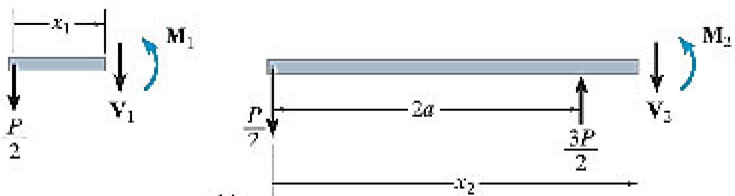
Solution 1

Elastic Curve.

The beam deflects into the shape shown in **Fig.*a***. Due to the loading, two x coordinates must be considered.

Moment Functions.

Using the free-body diagrams shown in **Fig.*b***, we have



(b)

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_1 \le 3a$$

Slope and Elastic Curve.

Applying **Eq.4**,

for x_1

$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

For x_2

$$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_1} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \qquad (3)$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \qquad (4)$$

The *four* constants of integration are determined using *three* boundary conditions, namely $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$ and $v_2 = 0$ at $x_2 = 2a$, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$. Applying these four conditions yields

$v_1 = 0$ at $x_1 = 0$; $\qquad 0 = 0 + 0 + C_2$

$v_1 = 0$ at $x_1 = 2a$; $\qquad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$

$v_2 = 0$ at $x_2 = 2a$; $\qquad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$

$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}$; $\qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$

Solving, we obtain

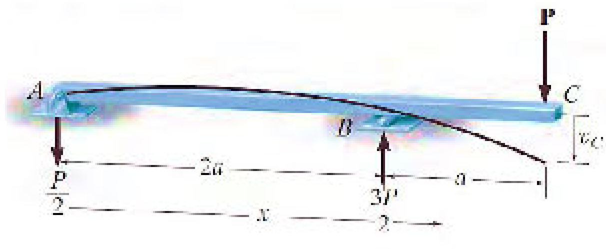
$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq.(4) gives

$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

The displacement at C is determined by setting $x_2 = 3a$ We get

$$v_C = -\frac{Pa^3}{EI}$$



(a)

Solution 2
Moment Functions.

$$M = -\frac{P}{2}x + \frac{3P}{2}\langle x - 2a \rangle$$

Slope and Elastic Curve.

$$EI\frac{d^2v}{dx^2} = -\frac{P}{2}x + \frac{3P}{2}\langle x - 2a \rangle$$

$$EI\frac{dv}{dx} = -\frac{P}{4}x^2 + \frac{3P}{4}\langle x - 2a \rangle^2 + C_1$$

$$EIv = -\frac{P}{12}x^3 + \frac{3P}{12}\langle x - 2a \rangle^3 + C_1x + C_2$$

$\boldsymbol{v = 0}$ at $\boldsymbol{x = 0}$,

$$0 = -\frac{P}{12}(0)^3 + 0 + C_1(0) + C_2 \quad \Rightarrow \quad C_2 = 0$$

$\boldsymbol{v = 0}$ at $\boldsymbol{x = 2a}$

$$0 = -\frac{P}{12}(2a)^3 + \frac{3P}{12}\langle 2a - 2a \rangle^3 + C_1(2a)$$

$$0 = -\frac{8Pa^3}{12} + C_1(2a) \quad \Rightarrow \quad C_1 = \frac{1}{3}Pa^2$$

$$EIv = -\frac{P}{12}x^3 + \frac{3P}{12}\langle x - 2a \rangle^3 + \frac{1}{3}Pa^2x$$

The displacement at $\boldsymbol{C}$ is determined by setting $\boldsymbol{x = 3a}$ We get

$$EIv_C = -\frac{P}{12}(3a)^3 + \frac{3P}{12}\langle 3a - 2a \rangle^3 + \frac{1}{3}Pa^2(3a)$$

$$EIv_C = -\frac{27Pa^3}{12} + \frac{3Pa^3}{12} + Pa^3$$

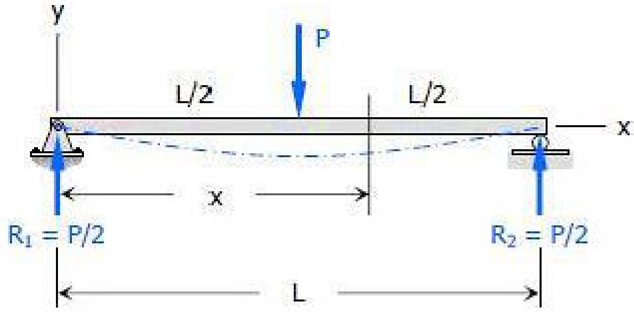
$$EIv_C = -\frac{Pa^3}{12} + \frac{3Pa^3}{12} + \frac{12}{12}Pa^3$$

$$EIv_C = -\frac{12Pa^3}{12}$$

$$\therefore \quad v_C = -\frac{Pa^3}{EI}$$

EXAMPLE 5.3.4

Determine the maximum deflection δ in a simply supported beam of length L carrying a concentrated load P at midspan.



Solution

$$EIv'' = \frac{1}{2}Px - P\left\langle x - \frac{1}{2}L\right\rangle$$

$$EIv' = \frac{1}{4}Px^2 - \frac{1}{2}P\left\langle x - \frac{1}{2}L\right\rangle^2 + C_1$$

$$EIv = \frac{1}{12}Px^3 - \frac{1}{6}P\left\langle x - \frac{1}{2}L\right\rangle^3 + C_1x + C_2$$

$At\ x = 0,\ v = 0$, therefore, $C_2 = 0$

$At\ x = L,\ v = 0$

$$0 = \frac{1}{12}PL^3 - \frac{1}{6}P\left\langle L - \frac{1}{2}L\right\rangle^3 + C_1L$$

$$0 = \frac{1}{12}PL^3 - \frac{1}{48}PL^3 + C_1L$$

$$C_1 = -\frac{1}{16}PL^2$$

Thus,

$$EIv = \frac{1}{12}Px^3 - \frac{1}{6}P\left\langle x - \frac{1}{2}L\right\rangle^3 - \frac{1}{16}PL^2x$$

Maximum deflection will occur at $x = ½\,L\ (midspan)$

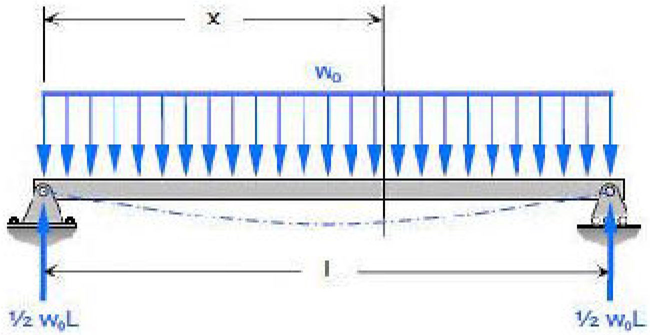
$$EIv_{max} = \frac{1}{12}P\left(\frac{1}{2}L\right)^3 - \frac{1}{6}P\left(\frac{1}{2}L - \frac{1}{2}L\right)^3 - \frac{1}{16}PL^2\left(\frac{1}{2}L\right)$$

$$EIv_{max} = \frac{1}{96}PL^3 - 0 - \frac{1}{32}PL^3$$

$$v_{max} = -\frac{PL^3}{48EI}\quad Ans.$$

EXAMPLE 5.3.5

Determine the maximum deflection v in a simply supported beam of length L carrying a uniformly distributed load of intensity w_o applied over its entire length.



Solution

$$EIv'' = \frac{1}{2}w_o Lx - w_o x\left(\frac{1}{2}x\right)$$

$$EIv'' = \frac{1}{2}w_o Lx - \frac{1}{2}w_o x^2$$

$$EIv' = \frac{1}{4}w_o Lx^2 - \frac{1}{6}w_o x^3 + C_1$$

$$EIv = \frac{1}{12}w_o Lx^3 - \frac{1}{24}w_o x^4 + C_1 x + C_2$$

$At\ x = 0,\ v = 0$, therefore $C_2 = 0$

$At\ x = L,\ v = 0$

$$0 = \frac{1}{12}w_o L^4 - \frac{1}{24}w_o L^4 + C_1 L$$

$$C_1 = -\frac{1}{24}w_o L^3$$

Therefore,

$$EIv = \frac{1}{12}w_o Lx^3 - \frac{1}{24}w_o x^4 - \frac{1}{24}w_o L^3 x$$

Maximum deflection will occur at $x = ½\ L\ (midspan)$

$$EIv_{max} = \frac{1}{12}w_o L\left(\frac{1}{2}L\right)^3 - \frac{1}{24}w_o\left(\frac{1}{2}L\right)^4 - \frac{1}{24}w_o L^3\left(\frac{1}{2}L\right)$$

$$EIv_{max} = \frac{1}{96}w_o L^4 - \frac{1}{384}w_o L^4 - \frac{1}{48}w_o L^4$$

$$EIv_{max} = -\frac{5}{384}w_o L^4$$

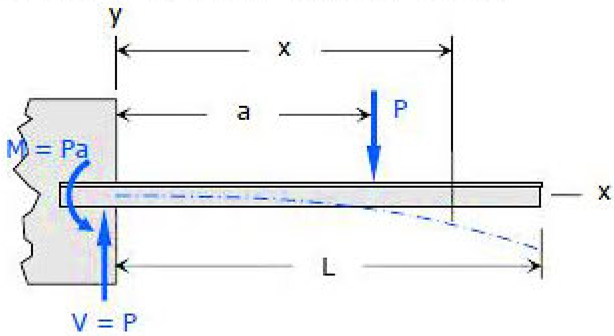
$$v_{max} = -\frac{5w_o L^4}{384EI} \qquad Ans.$$

From **Example 5.3.1** $\quad W_0 = 4$ **kN/m** and $L = 10$ **m**

$$v_{max} = -\frac{5w_o L^4}{384EI} = -\frac{5(4)(10)^4}{384EI} = \frac{520.8333}{EI} \cong -\frac{521}{EI}$$

EXAMPLE 5.3.6

Determine the maximum deflection v for the cantilever beam loaded as shown in the Figure.



Solution

$$EIv'' = -Pa + Px - P\langle x - a\rangle$$

$$EIv' = -Pax + \frac{1}{2}Px^2 - \frac{1}{2}P\langle x - a\rangle^2 + C_1$$

$$EIv = -\frac{1}{2}Pax^2 + \frac{1}{6}Px^3 - \frac{1}{6}P\langle x - a\rangle^3 + C_1x + C_2$$

$At\ x = 0,\ v' = 0$, therefore $C_1 = 0$

$At\ x = 0,\ v = 0$, therefore $C_2 = 0$

Therefore,

$$EIv = -\frac{1}{2}Pax^2 + \frac{1}{6}Px^3 - \frac{1}{6}P\langle x - a\rangle^3$$

The maximum value of EIv is at $x = L\ (free\ end)$

$$EIv_{\max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}P\left(L - a\right)^3$$

$$EIv_{\max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}P\left(L^3 - 3L^2a + 3La^2 - a^3\right)$$

$$EIv_{\max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}PL^3 + \frac{1}{2}PL^2a - \frac{1}{2}PLa^2 + \frac{1}{6}Pa^3$$

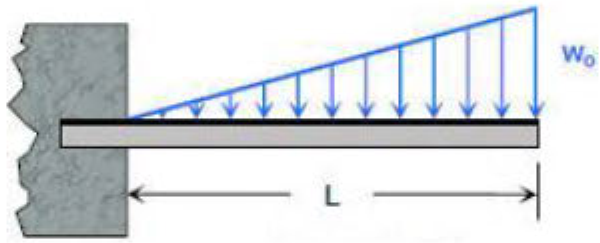
$$EIv_{\max} = -\frac{1}{2}PLa^2 + \frac{1}{6}Pa^3$$

$$EIv_{\max} = -\frac{1}{6}Pa^2(3L - a)$$

$$v_{\max} = -\frac{Pa^2}{6EI}(3L - a)\quad Ans.$$

EXAMPLE 5.3.7

Find the equation of the elastic curve for the cantilever beam shown in the Figure; it carries a load that varies from zero at the wall to w_o at the free end. Take the origin at the wall.



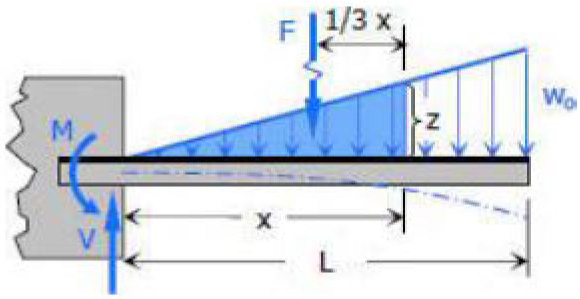
Solution

$$V = \frac{1}{2} w_o L$$

$$M = \frac{1}{2} w_o L \left(\frac{2}{3} L\right)$$

$$M = \frac{1}{3} w_o L^2$$

By ratio and proportion



$$\frac{z}{x} = \frac{w_o}{L}$$

$$z = \frac{w_o}{L} x$$

$$F = \frac{1}{2} x z$$

$$F = \frac{1}{2} x \left(\frac{w_o}{L} x\right)$$

$$F = \frac{w_o}{2L} x^2$$

$$EIv'' = -M + Vx - F\left(\frac{1}{3} x\right)$$

$$EIv'' = -\frac{1}{3} w_o L^2 + \frac{1}{2} w_o L x - \frac{1}{3} x \left(\frac{w_o}{2L} x^2\right)$$

$$EIv'' = -\frac{w_o L^2}{3} + \frac{w_o L}{2} x - \frac{w_o}{6L} x^3$$

$$EIv' = -\frac{w_o L^2}{3} x + \frac{w_o L}{4} x^2 - \frac{w_o}{24L} x^4 + C_1$$

$$EIv = -\frac{w_o L^2}{6} x^2 + \frac{w_o L}{12} x^3 - \frac{w_o}{120L} x^5 + C_1 x + C_2$$

At $x = 0$, $v' = 0$, therefore $C_1 = 0$

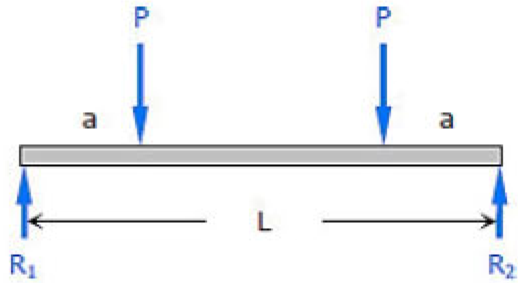
At $x = 0$, $v = 0$, therefore $C_2 = 0$

Therefore, the equation of the elastic curve is

$$EIv = -\frac{w_o L^2}{6} x^2 + \frac{w_o L}{12} x^3 - \frac{w_o}{120L} x^5 \qquad Ans.$$

EXAMPLE 5.3.8

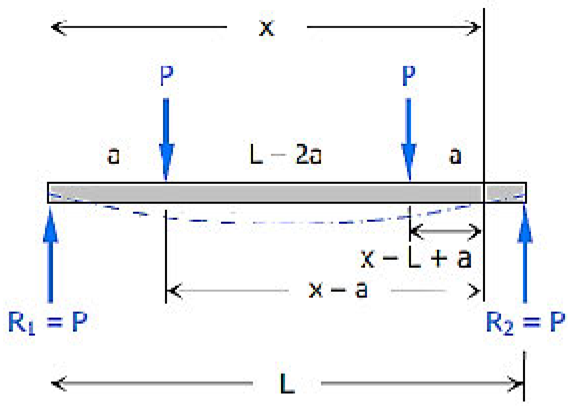
As shown in the figure, a simply supported beam carries two symmetrically placed concentrated loads. Compute the maximum deflection v.



Solution

By symmetry

$R_1 = R_2 = P$



$$EIv'' = Px - P\langle x-a\rangle - P\langle x-L+a\rangle$$

$$EIv' = \frac{1}{2}Px^2 - \frac{1}{2}P\langle x-a\rangle^2 - \frac{1}{2}P\langle x-L+a\rangle^2 + C_1$$

$$EIv = \frac{1}{6}Px^3 - \frac{1}{6}P\langle x-a\rangle^3 - \frac{1}{6}P\langle x-L+a\rangle^3 + C_1x + C_2$$

At $x = 0$, $v = 0$, therefore $C_2 = 0$

At $x = L$, $v = 0$

$$0 = \frac{1}{6}PL^3 - \frac{1}{6}P\langle x-a\rangle^3 + C_1L$$

$$0 = PL^3 - P\left(L^3 - 3L^2a + 3La^2 - a^3\right) - Pa^3 + 6C_1L$$

$$0 = PL^3 - PL^3 + 3PL^2a - 3PLa^2 + Pa^3 - Pa^3 + 6C_1L$$

$$0 = 3PL^2a - 3PLa^2 + 6C_1L \quad \Rightarrow 0 = 3PLa(L-a) + 6C_1L \quad \Rightarrow C_1 = -\frac{1}{2}Pa(L-a)$$

Therefore,

$$EIv = \frac{1}{6}Px^3 - \frac{1}{6}P\langle x-a\rangle^3 - \frac{1}{6}P\langle x-L+a\rangle^3 - \frac{1}{2}Pa(L-a)x$$

Maximum deflection will occur at $x = ½ L$ (midspan)

$$EIv_{max} = \frac{1}{6}P\left(\frac{1}{2}L\right)^3 - \frac{1}{6}P\left(\frac{1}{2}L-a\right)^3 - \frac{1}{2}Pa(L-a)\left(\frac{1}{2}L\right)$$

$$EIv_{max} = \frac{1}{48}PL^3 - \frac{1}{6}P\left[\frac{1}{2}(L-2a)\right]^3 - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

$$EIv_{max} = \frac{1}{48}PL^3 - \frac{1}{48}P\left[L^3 - 3L^2(2a) + 3L(2a)^2 - (2a)^3\right] - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

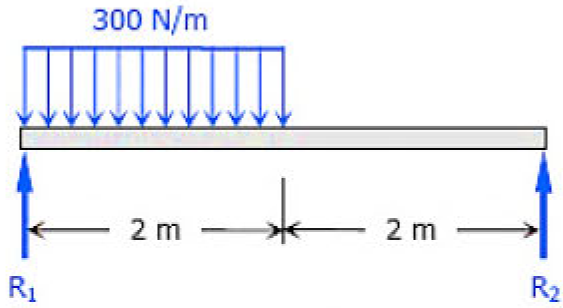
$$EIv_{max} = \frac{1}{48}PL^3 - \frac{1}{48}PL^3 + \frac{1}{8}PL^2a - \frac{1}{4}PLa^2 + \frac{1}{6}Pa^3 - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

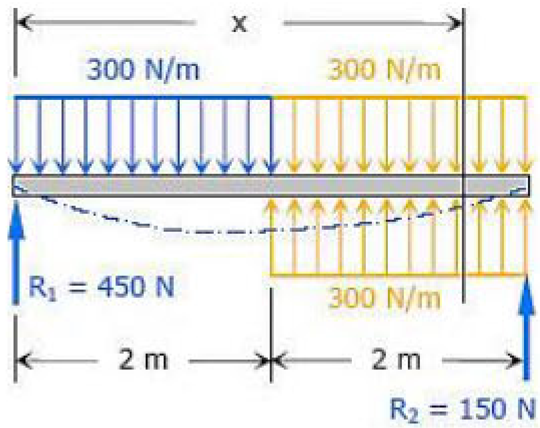
$$EIv_{max} = -\frac{1}{8}PL^2a + \frac{1}{6}Pa^3 \quad \Rightarrow EIv_{max} = -\frac{1}{24}Pa\left(3L^2 - 4a^2\right)$$

$$v_{max} = -\frac{Pa}{24EI}\left(3L^2 - 4a^2\right) \qquad Ans.$$

EXAMPLE 5.3.9

Compute the value of ***EI v*** at midspan for the beam loaded as shown in the figure.





$\Sigma M_{R2} = 0$

$4R_1 = 300(2)(3)$

$R1 = 450\text{N}$

$\Sigma M_{R1} = 0$

$4R_2 = 300(2)(1)$

$R_2 = 150\text{N}$

$$EIv'' = 450x - \frac{1}{2}(300)x^2 + \frac{1}{2}(300)\langle x-2\rangle^2$$

$$EIv'' = 450x - 150x^2 + 150\langle x-2\rangle^2$$

$$EIv' = 225x^2 - 50x^3 + 50\langle x-2\rangle^3 + C_1$$

$$EIv = 75x^3 - 12.5x^4 + 12.5\langle x-2\rangle + C_1x + C_2$$

$At\ x = 0, \quad v = 0$, therefore $C_2 = 0$

$At\ x = 4\,m, \quad v = 0$

$$0 = 75(4^3) - 12.5(4^4) + 12.5(4-2)^4 + 4C_1 \Rightarrow C_1 = -450\ \text{N.m}^2$$

Therefore,

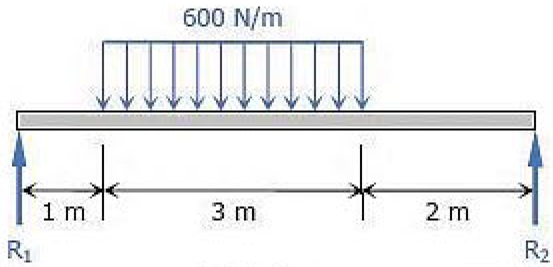
$$EIy = 75x^3 - 12.5x^4 + 12.5\langle x-2\rangle^4 - 450x$$

$$At\ x = 2\,m\ (midspan) \quad \Rightarrow EIy_{midspan} = 75(2^3) - 12.5(2^4) + 12.5(2-2)^4 - 450(2)$$

$$\Rightarrow EIv_{midspan} = -500\ \text{N.m}^3$$

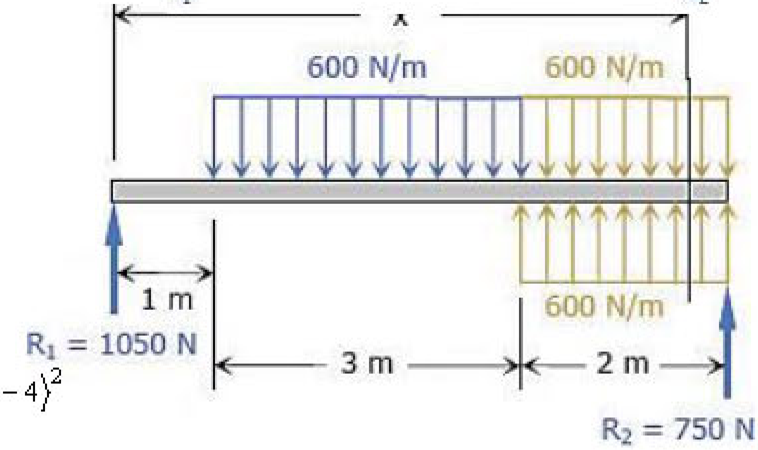
EXAMPLE 5.3.10

Compute the midspan value of **EI δ** for the beam loaded as shown in the figure.



Solution

$$\Sigma M_{R2} = 0$$
$$6R_1 - 600(3)(3.5) = 0$$
$$R_1 = 1050 \text{ N}$$
$$\Sigma M_{R1} = 0$$
$$-6R_2 + 600(3)(2.5) = 0$$
$$R_2 = 750\text{N}$$



$$EIv'' = 1050x - \frac{1}{2}(600)\langle x-1\rangle^2 + \frac{1}{2}(600)\langle x-4\rangle^2$$
$$EIv'' = 1050x - 300\langle x-1\rangle^2 + 300\langle x-4\rangle^2$$
$$EIv' = 525x^2 - 100\langle x-1\rangle^3 + 100\langle x-4\rangle^3 + C_1$$
$$EIv = 175x^3 - 25\langle x-1\rangle^4 + 25\langle x-4\rangle^4 + C_1x + C_2$$

At $x = 0$, $v = 0$, therefore $C_2 = 0$

At $x = 6\ m$, $v = 0$

$$0 = 175(6^3) - 25(6-1)^4 + 25(6-4)^4 + 6C_1 \quad \Rightarrow C1 = -3762.5 \text{ N.m}^2$$

Therefore,

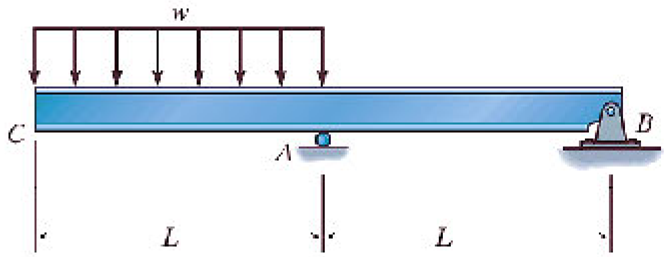
$$EIv = 175x^3 - 25(x-1)^4 + 25(x-4)^4 - 3762.5x$$

At midspan, $x = 3$ m

$$EIv_{midspan} = 175(3^3) - 25(3-1)^4 - 3762.5(3)$$
$$EIv_{midspan} = -6962.5 \text{ N.m}^3$$

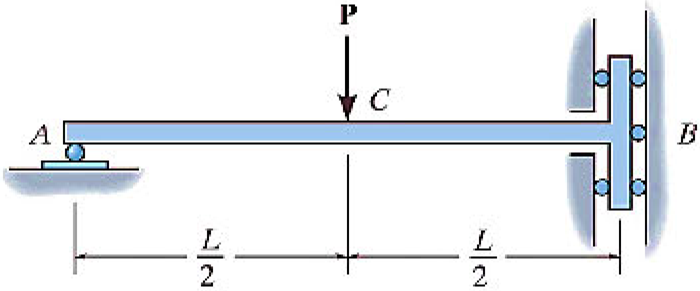
Hw.14

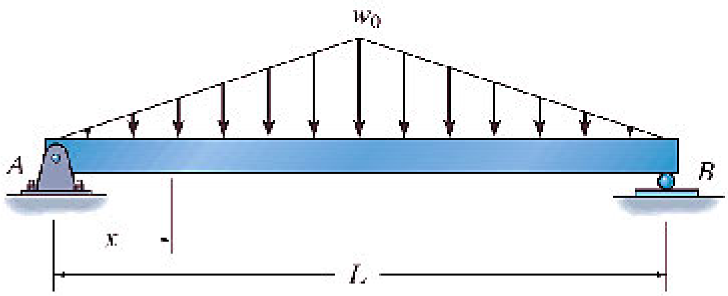
Determine the maximum deflection between the supports *A* and *B*. *EI* is constant. Use the method of integration.



Hw.15

The bar is supported by a roller constraint at *B*, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at *A* and the deflection at *C*. *EI* is constant.



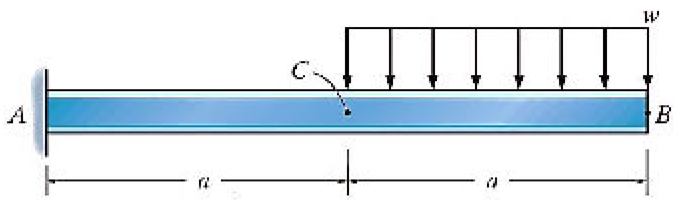


Hw.16

Determine the elastic curve for the simply supported beam using the *x* coordinate $0 \le x \le L/2$
Also, determine the slope at *A* and the maximum deflection of the beam. *EI* is constant.

Hw.17

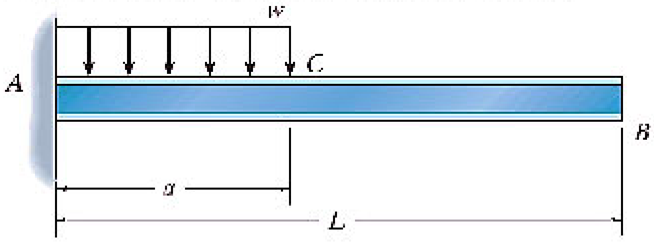
Determine the equations of the elastic curve and specify the slope and deflection at *B* and *C*. *EI* is constant.



.

Hw.18

Determine the equations of the elastic curve, and specify the slope and deflection at point ***B*** and ***C***. ***EI*** is constant.



6 DEFLECTIONS USING ENERGY METHODS

6.1 Method of Virtual Work: Beams and Frames

The method of virtual work can be applied to deflection problems involving beams and frames.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in **Fig.6-1*b***. Here the displacement Δ of point ***A*** is to be determined.

To compute Δ a virtual unit load acting in the direction

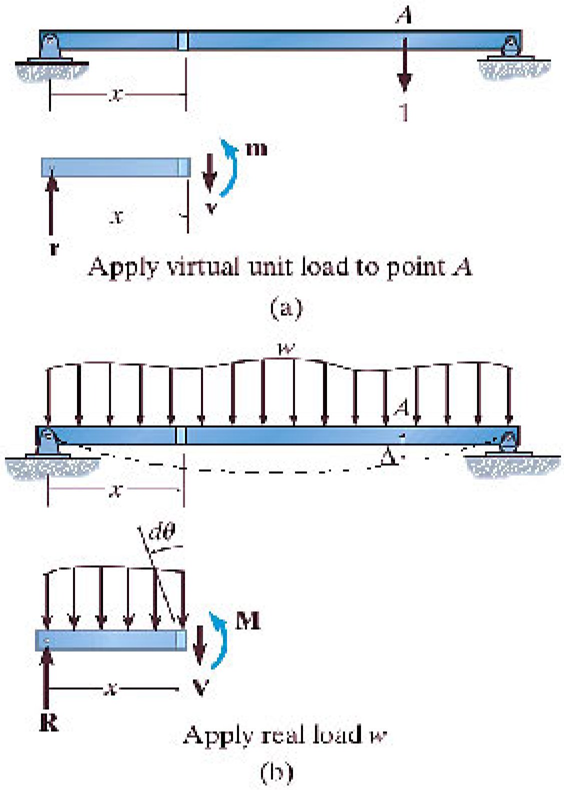


Fig. 6-1

of Δ is placed on the beam at ***A***, and the ***internal virtual moment*** **m** is determined by the method of sections at an arbitrary location ***x*** from the left support, **Fig. 6-1*a***. When the real loads act on the beam, **Fig. 6–1*b***, point ***A*** is displaced Δ.

Provided these loads cause ***linear elastic material response,*** then from the equation below, the element ***dx*** deforms or rotates

$$d\theta = \left(\frac{M}{EI}\right) dx$$

Here ***M*** is the internal moment at ***x*** caused by the real loads. Consequently, the ***external virtual work*** done by the unit load is **1 .** Δ, and the ***internal virtual work*** done by the moment **m** is

$$m d\theta = m\left(\frac{M}{EI}\right) dx$$

Summing the effects on all the elements ***dx*** along the beam requires an integration,

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI}\, dx \qquad ...(6\text{-}1)$$

where

1 = external virtual unit load acting on the beam or frame in the direction of Δ

m = internal virtual moment in the beam or frame, expressed as a function of ***x*** and caused by the external virtual unit load.

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

M = internal moment in the beam or frame, expressed as a function of ***x*** and caused by the real loads.

E = modulus of elasticity of the material.

I = moment of inertia of cross-sectional area, computed about the neutral axis.

In a similar manner, if the tangent rotation or slope angle at a point *A* on the beam's elastic curve is to be determined, **Fig. 6-2**, a ***unit couple moment*** is first applied at the point, and the corresponding internal moments m_θ have to be determined. Since the work of the unit couple is $1 \cdot \theta$, then

$$1 \cdot \theta = \int_0^L \frac{m_\theta\, M}{EI}\, dx \qquad ...(6\text{-}2)$$

When applying **Eqs. 6-1** and **6-2**, it is important to realize that the definite integrals on the right side actually represent the amount of virtual strain energy that is ***stored*** in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length. Instead, separate *x* coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each ***x*** have the ***same origin***; however, the ***x*** selected for determining the real moment ***M*** in a particular region must be the ***same x*** as that selected for determining the virtual moment ***m*** **or** m_θ within the same region.

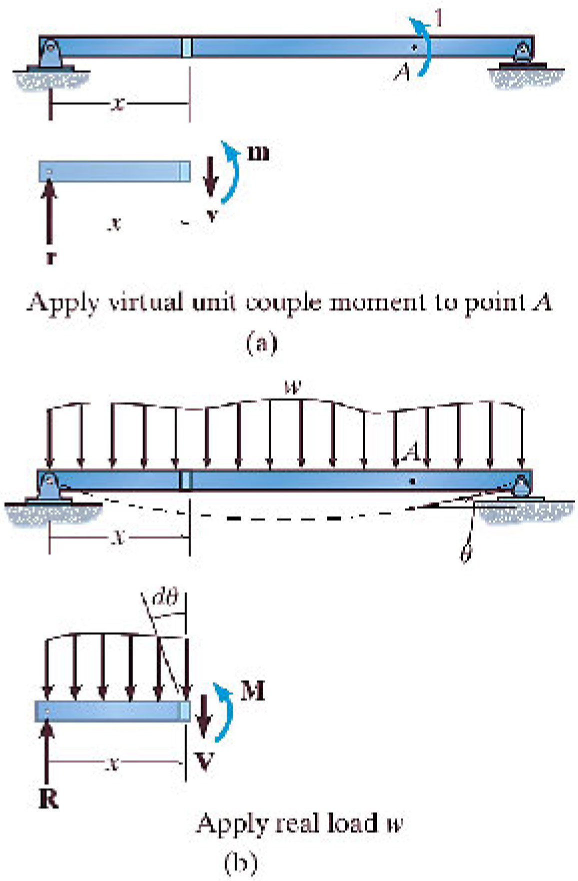
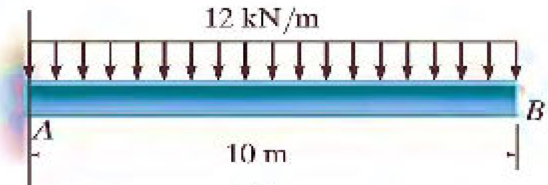


Fig. 6-2

EXAMPLE 6.1.1

Determine the displacement of point B of the steel beam shown in **Fig. *a*.** Take **E = 200 GPa, I = 500(10⁶) mm⁴.**

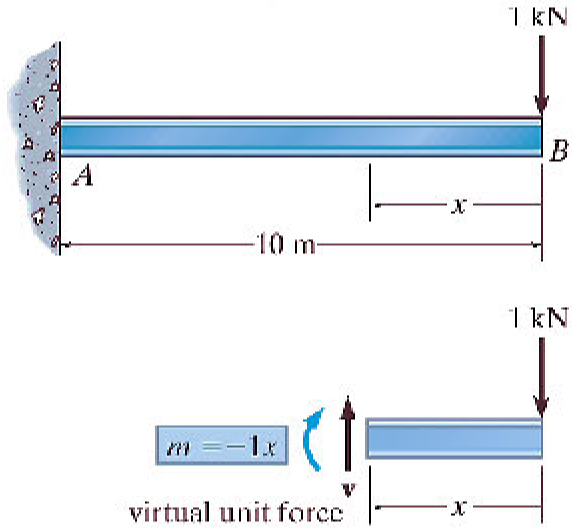


(a)

Solution

Virtual Moment m.

The vertical displacement of point B is obtained by placing a virtual unit load of **1 kN** at B, **Fig.*b***. By inspection there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a ***single*** x coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at B, since then the reactions at A do not have to be determined in order to find the internal moments m and M. Using the method of sections, the internal moment m is formulated as shown in **Fig.*b***.



(b)

Real Moment M. Using the ***same*** x coordinate, the internal moment M is formulated as shown in **Fig.*c***.

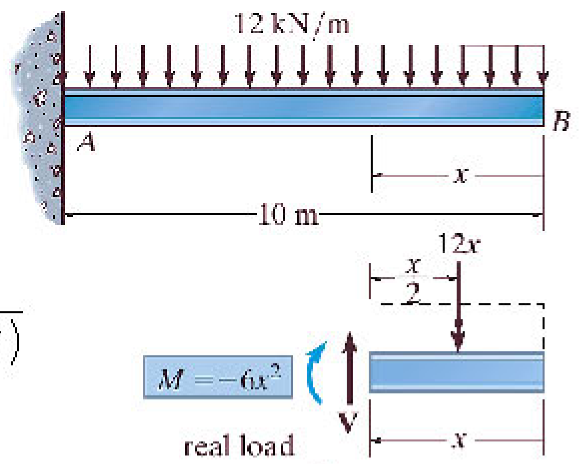
Virtual-Work Equation. The vertical displacement of B is thus,

$$1\text{kN}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x)(-6x^2)}{EI}dx$$

$$1\text{kN}.\Delta_B = \frac{15(10^3)\text{kN}^2.m^3}{EI}$$

$$\Delta_B = \frac{15(10^3)\text{kN.m}^3}{200(10^6)\text{kN/m}^2\left(500(10^6)\text{mm}^4\right)\left(10^{-12}\text{m}^4/\text{mm}^4\right)}$$

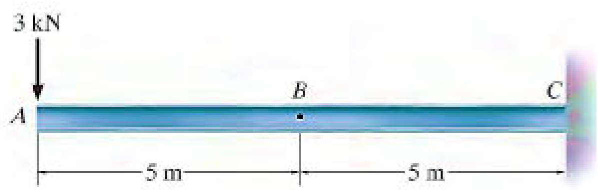
$$= 0.150\text{ m} = 150\text{ mm}$$



(c)

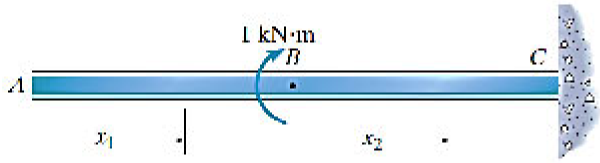
EXAMPLE 6.1.2

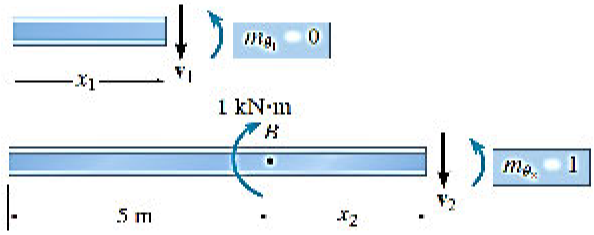
Determine the slope θ at point B of the steel beam shown in **Fig. *a***. Take **E = 200 GPa, I = 60(10⁶) mm⁴**



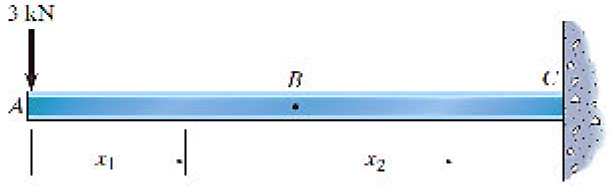
(a)

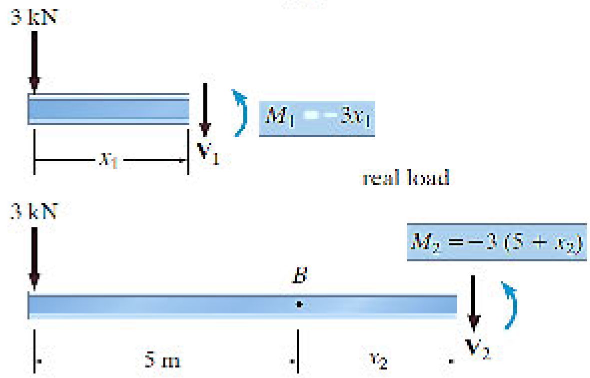
Solution





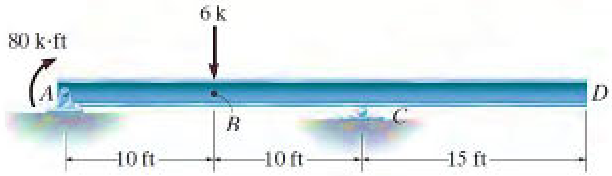
virtual unit couple
(b)





(c)

Note: The *negative sign* indicates is θ_B ***opposite*** to the direction of the virtual couple moment shown in **Fig. b**



(a)

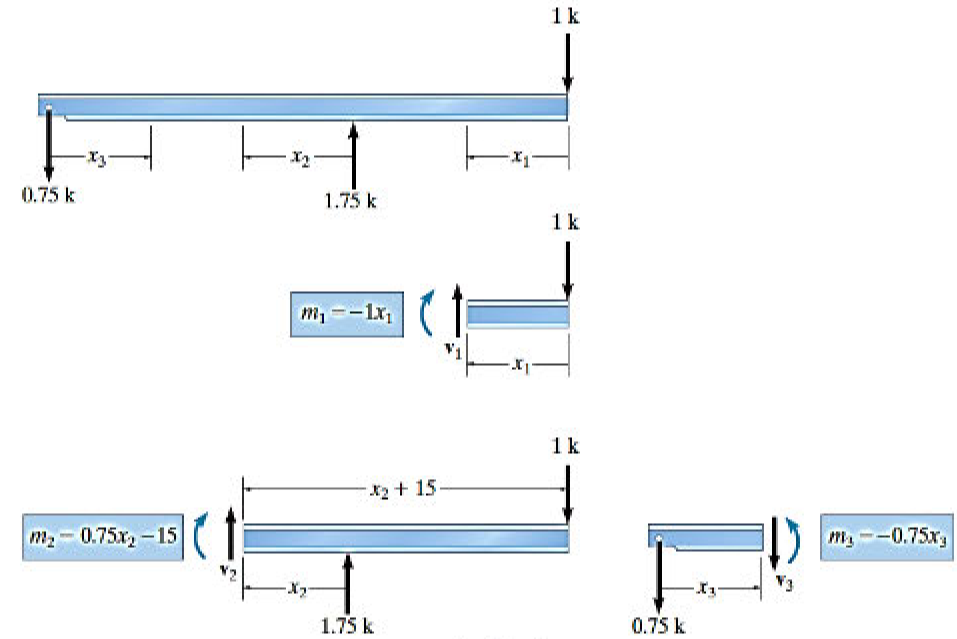
$$(1\ \text{kN.m}).\theta_B = \int_0^L \frac{m_\theta M}{EI} dx = \int_0^5 \frac{(0)(-3x_1)}{EI} dx_1 + \int_0^5 \frac{(1)[-3(5+x_2)]}{EI} dx_2$$

$$\theta_B = \frac{-112.5\text{kN.m}^2}{EI} = -0.00938\ \text{rad}$$

EXAMPLE 6.1.3

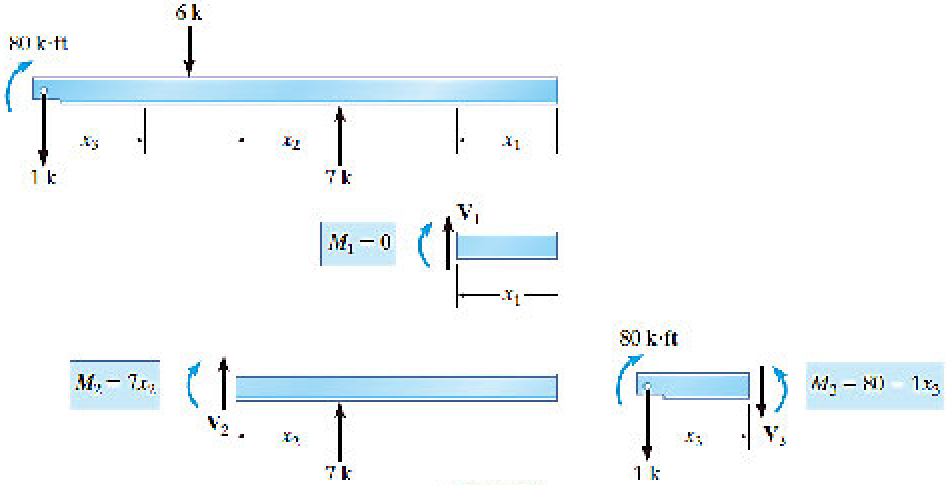
Determine the displacement at ***D*** of the steel beam in **Fig.*a***. Take **E = 29(10³) ksi, I = 800 in⁴**.

Solution



virtual loads

(b)



real loads

(c)

$$1\text{kN}.\Delta_D = \int_0^L \frac{mM}{EI}dx = \int_0^{15} \frac{(-1x_1)(0)}{EI}dx_1 + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)}{EI}dx_2 + \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)}{EI}dx_3$$

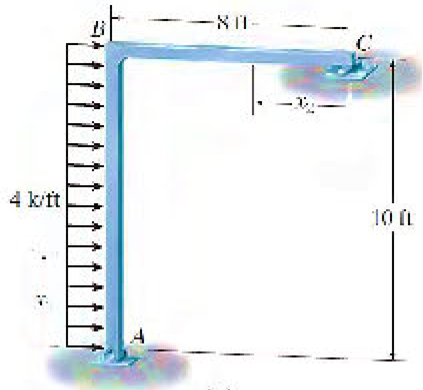
$$\Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250\ \text{k.ft}^3}{EI}$$

$$\Delta_D = \frac{-6250\ \text{k.ft}^3 (12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2 (800\ \text{in}^4)} = -0.466\ \text{in}$$

Note: The negative sign indicates the displacement is upward, opposite to the downward unit load, **Fig. *b***. Also note that m_1 did not actually have to be calculated since $M_1 = 0$.

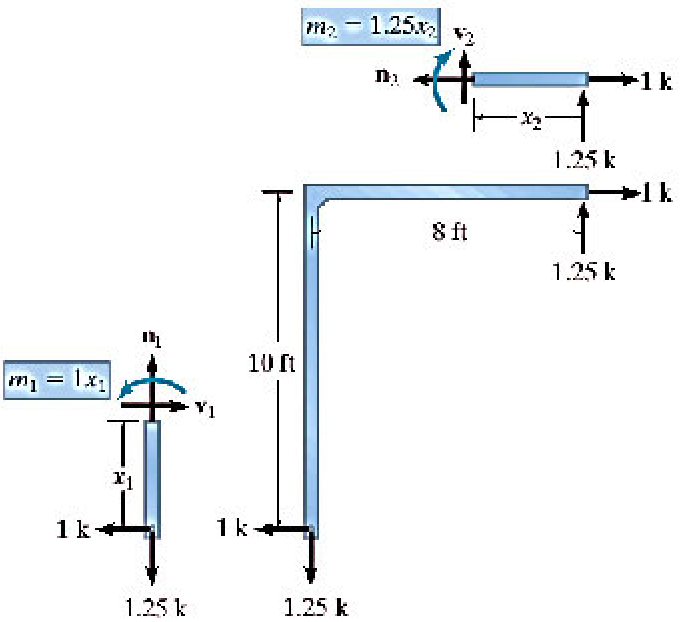
EXAMPLE 6.1.4

Determine the horizontal displacement of point C on the frame shown in **Fig. *a***. Take $\mathbf{E = 29(10^3)\ ksi, I = 600\ in^4}$ for both members.



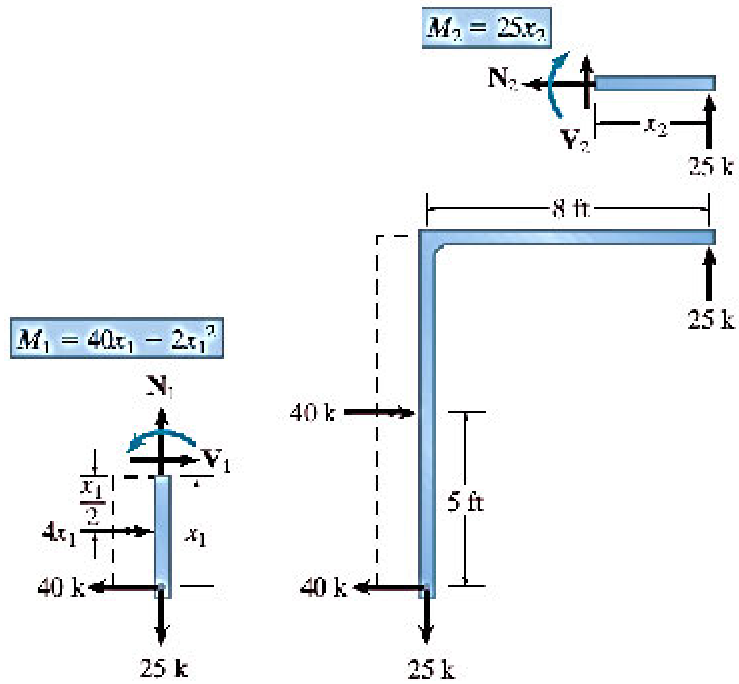
(a)

Solution



virtual loadings

(b)



real loadings

(c)

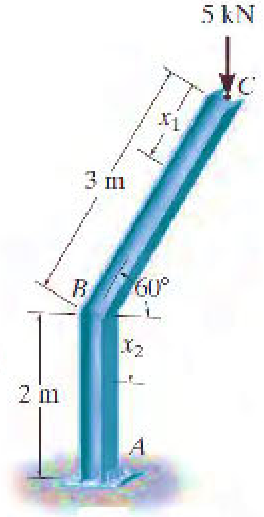
$$1\text{kN}.\Delta_{C_h} = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x_1)(40x_1 - 2x_1^2)}{EI}dx_1 + \int_0^8 \frac{(1.25x_2)(25x_2)}{EI}dx_2$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13666.7\ \text{k.ft}^3}{EI}$$

$$\Delta_{C_h} = \frac{13666.7\ \text{k.ft}^3(12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2(600\ \text{in}^4)} = 1.357\ \text{in}$$

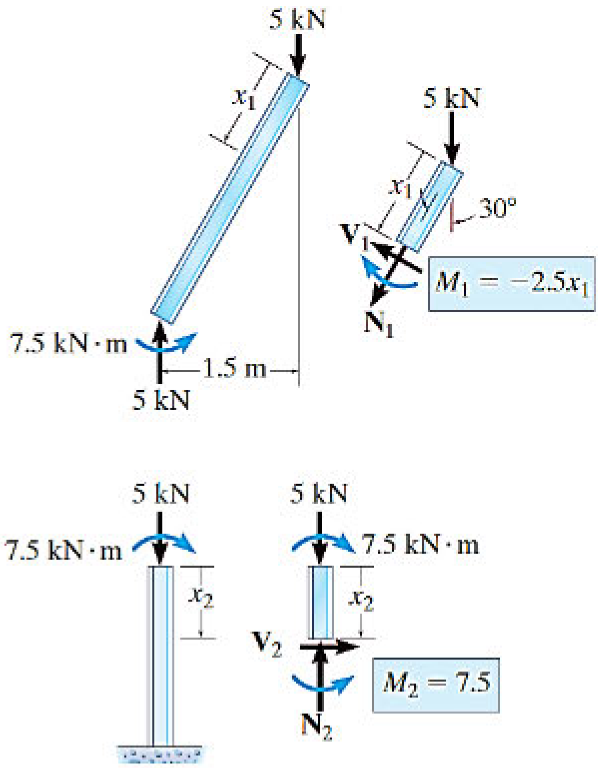
EXAMPLE 6.1.5

Determine the tangential rotation at point C of the frame shown in **Fig.** a. Take E = 200 GPa, $I = (15)10^6$ mm^4.

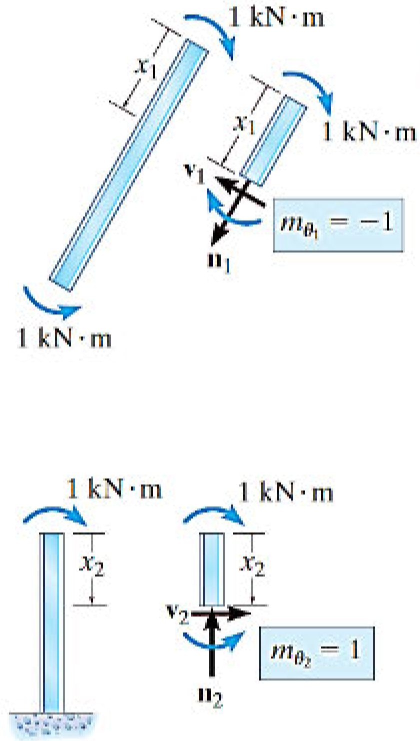


(a)

Solution



real loads
(b)



virtual loads
(c)

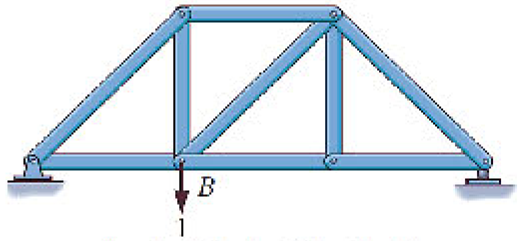
$$(1\ \text{kN.m}).\theta_C = \int_0^L \frac{m_\theta M}{EI} dx = \int_0^3 \frac{(-1)(-2.5x_1)}{EI} dx_1 + \int_0^2 \frac{(1)(7.5)}{EI} dx_2$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25\,\text{kN.m}^2}{EI}$$

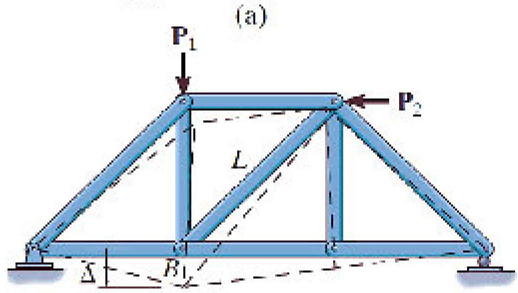
$$\theta_C = \frac{26.25\,\text{kN.m}^2}{200(10^6)\,\text{kN/m}^2\ \left[15\left(10^6\right)\,\text{mm}^4\right]\left(10^{-12}\ \text{m}^4/\text{mm}^4\right)} = 0.00875\ \text{rad}$$

6.2 Method of Virtual Work: Trusses

The method of virtual work can be used to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.



Apply virtual unit load to B
(a)



Apply real loads P_1, P_2
(b)

External Loading.

For the purpose of explanation let us consider the vertical displacement Δ of joint ***B*** of the truss in **Fig.*a***. Here a typical element of the truss would be one of its ***members*** having a length ***L***, **Fig.*b***. If the applied loadings $\mathbf{P_1}$ and $\mathbf{P_2}$ cause a ***linear elastic material response,*** then this element deforms an amount,

$$\Delta L = \frac{NL}{AE}$$

where ***N*** is the normal or axial force in the member, caused by the loads. The virtual-work equation for the truss is therefore

$$1.\,\Delta = \sum \frac{nNL}{AE} \qquad \ldots(6\text{-}3)$$

where

1	= external virtual unit load acting on the truss joint in the stated direction of Δ.
n	= internal virtual normal force in a truss member caused by the external virtual unit load.
Δ	= external joint displacement caused by the real loads on the truss.
N	= internal normal force in a truss member caused by the real loads.
E	= modulus of elasticity of a member.
A	= cross-sectional area of a member.
L	= length of a member.

Temperature.

In some cases, truss members may change their length due to temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in its temperature, the change in length of a member is $\Delta L = \alpha\,\Delta T\;L$

Hence, we can determine the displacement of a selected truss joint due to this temperature change from.

$$1.\,\Delta = \sum n\;\alpha\,\Delta T\;L \qquad \ldots(6\text{-}4)$$

where

1	= external virtual unit load acting on the truss joint in the stated direction of Δ.
n	= internal virtual normal force in a truss member caused by the external virtual unit load.
Δ	= external joint displacement caused by the temperature change.

α = coefficient of thermal expansion of member.
ΔT = change in temperature of member.
L = length of a member.

Note: If any of the members undergoes an ***increase in temperature,*** ΔT will be ***positive,*** whereas a ***decrease in temperature*** results in a ***negative*** value for ΔT.

Fabrication Errors and Camber.

Occasionally, errors in fabricating the lengths of the members of a truss may occur. Also, in some cases truss members must be made slightly longer or shorter in order to give the truss a camber. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from ,

$$1.\,\Delta = \sum n\;\Delta L \qquad \text{...(6 - 5)}$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.
n = internal virtual normal force in a truss member caused by the external virtual unit load.
Δ = external joint displacement caused by the fabrication errors.
ΔL = difference in length of the member from its intended size as caused by a fabrication error.

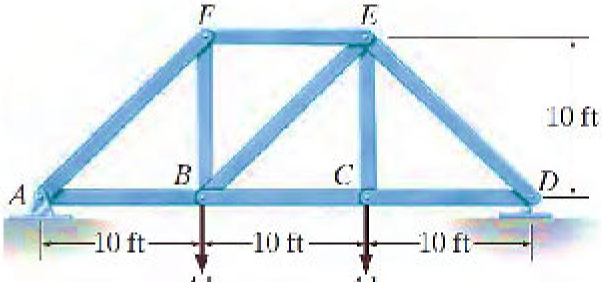
Note: When a fabrication error ***increases the length*** of a member, ΔL **is** ***positive,*** whereas a ***decrease in length*** **is** ***negative***.

A combination of the right sides of **Eqs. 6–3** through **6–5** will be necessary if both external loads act on the truss and some of the members undergo a thermal change or have been fabricated with the wrong dimensions.

$$1.\,\Delta = \sum \frac{nNL}{AE} + \sum n\;\alpha\,\Delta T\;L + \sum n\;\Delta L$$

EXAMPLE 6.2.1

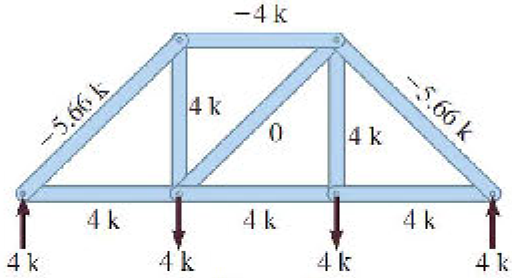
Determine the vertical displacement of joint C of the steel truss shown in **Fig. *a***. The cross-sectional area of each member is $A = 0.5\ \text{in}^2$ and $E = 29(10^3)$ ksi.



(a)

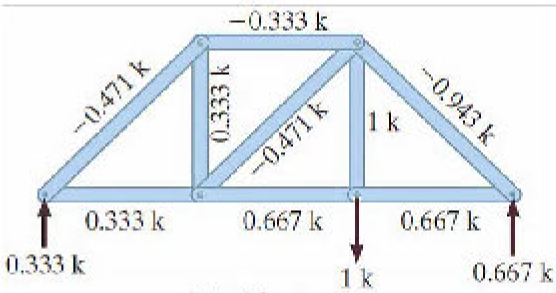
Solution

Real Forces N. The real forces in the members are calculated using the method of joints. The results are shown in **Fig. *b***.



real forces **N**
(b)

Virtual Forces n. Only a vertical **1-k** load is placed at joint C, and the force in each member is calculated using the method of joints. The results are shown in **Fig. *c***. Positive numbers indicate tensile forces and negative numbers indicate compressive forces.



virtual forces **n**
(c)

Virtual-Work Equation. Arranging the data in tabular form, we have

Member	n (k)	N (k)	L (ft)	nNL (k^2. ft)
AB	0.333	4	10	13.320
BC	0.667	4	10	26.680
CD	0.667	4	10	26.680
DE	-0.943	-5.66	14.14	75.471
FE	-0.333	-4	10	13.320
EB	-0.471	0	14.14	0.000
BF	0.333	4	10	13.320
AF	-0.471	-5.66	14.14	37.695
CE	1	4	10	40.000
				$\sum$ **246.486**

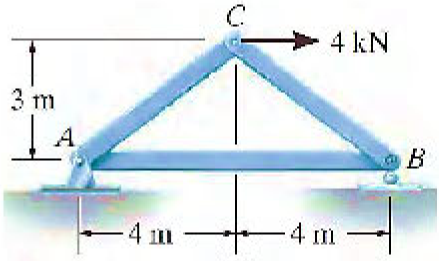
Thus

$$1\ \text{k}.\ \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.486\ \text{k}^2.\text{ft}}{AE}$$

$$1\ \text{k}.\ \Delta_{C_v} = \frac{\left(246.486\ \text{k}^2.\text{ft}\right)\left(12\ \text{in/ft}\right)}{\left(0.5\ \text{in}^2\right)\left(29\left(10^3\right)\ \text{k/in}^2\right)} = 0.204\ \text{in.}$$

EXAMPLE 6.2.2

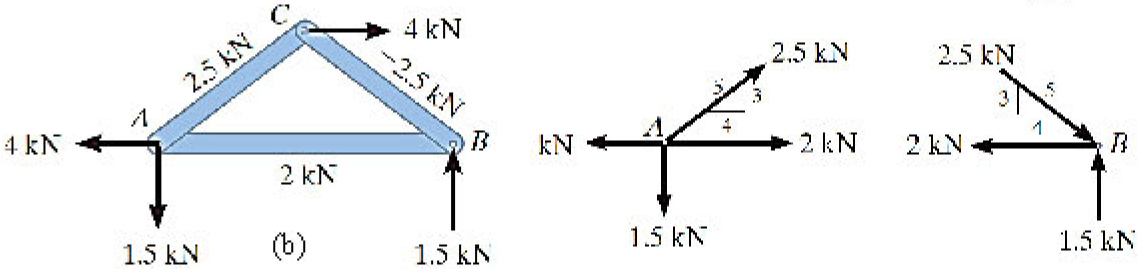
The cross-sectional area of each member of the truss shown in **Fig.*a*** is A = **400 mm²** and E = **200 GPa.** **(a)** Determine the vertical displacement of joint C if a **4-kN** force is applied to the truss at C. **(b)** If no loads act on the truss, what would be the vertical displacement of joint C if member AB were **5 mm** too short?



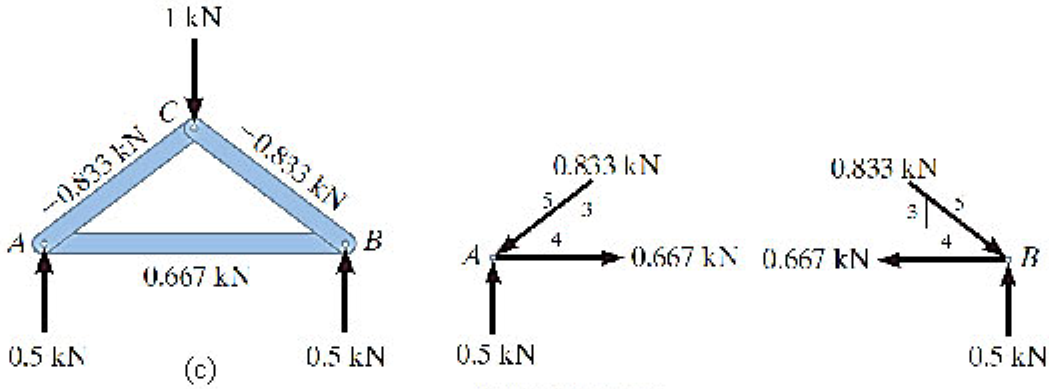
(a)

Solution

(a)



real forces **N**



virtual forces **n**

Member	n (kN)	N (kN)	L (m)	nNL (kN². m)
AB	0.667	2	8	10.672
AC	-0.833	2.5	5	-10.413
CB	-0.833	-2.5	5	10.413
				$\sum$ **10.672**

$$1\ \text{kN}.\ \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{10.672\ \text{kN}^2.\text{m}}{AE}$$

$$1\ \text{kN}.\ \Delta_{C_v} = \frac{10.672\ \text{kN}^2.\text{m}}{400(10^{-6})\ \text{m}^2\,(200(10^{-6})\ \text{kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133\ \text{m} = 0.133\ \text{mm}$$

(b)

Since the vertical displacement of C is to be determined, we can use the results of **Fig. *c***. Only member ***AB*** undergoes a change in length, namely, of $\Delta L = 0.005$ **m**.

Thus,

$$1.\ \Delta = \sum n\ \Delta L$$

$$1\ \text{kN}.\ \Delta_{C_v} = (0.667\ \text{kN})(-0.005\ \text{m})$$

$$\Delta_{C_v} = -0.00333\ \text{m} = -3.33\ \text{mm}$$

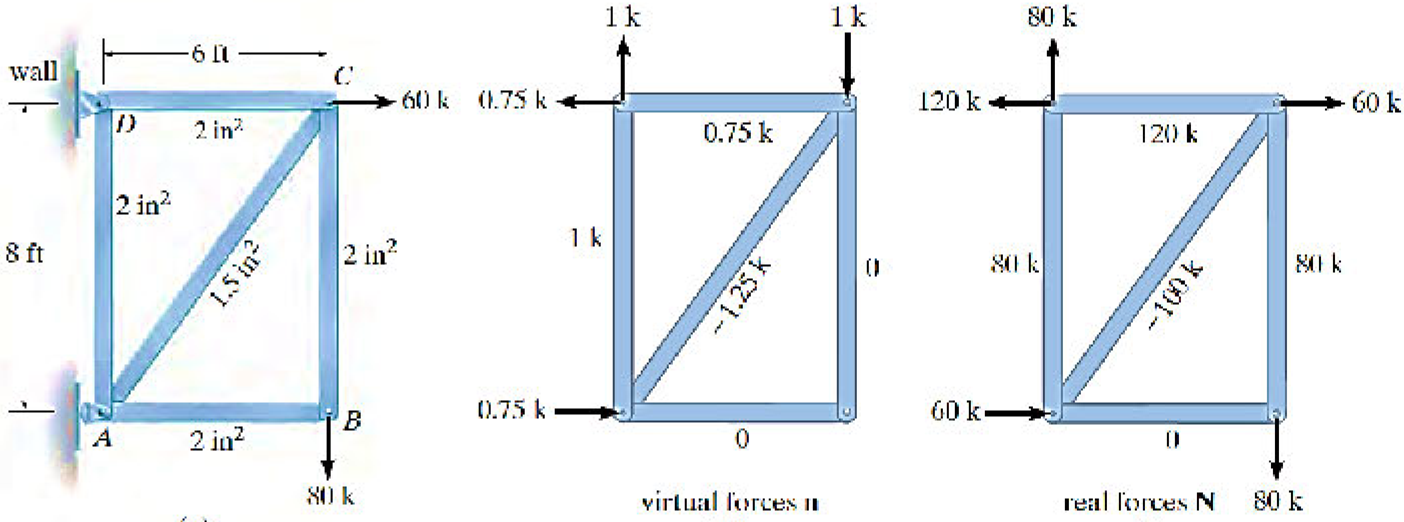
The negative sign indicates joint ***C*** is displaced ***upward,*** opposite to the **1-kN** vertical load.

Note: If the **4-kN** load and fabrication error are both accounted for, the resultant displacement is then

$$\Delta_{C_v} = 0.133 - 3.33 = -3.20\ \text{mm (upward)}.$$

EXAMPLE 6.2.3

Determine the vertical displacement of joint ***C*** of the steel truss shown in **Fig. *a***. Due to radiant heating from the wall, member ***AD*** is subjected to an ***increase*** in temperature of $\Delta T = +120°F$. Take $\alpha = 0.6(10^{-5})/°F$ and $E = 29(10^3)$ **kis.** The cross-sectional area of each member is indicated in the figure.



(a) (b) (c)

Solution

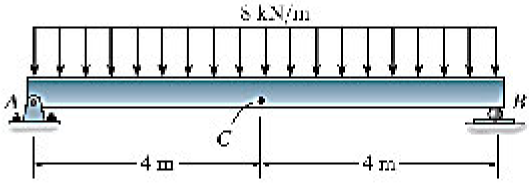
$$1.\ \Delta_{C_v} = \sum \frac{nNL}{AE} + \sum n\ \alpha\ \Delta T\ \ L$$

$$= \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]}$$

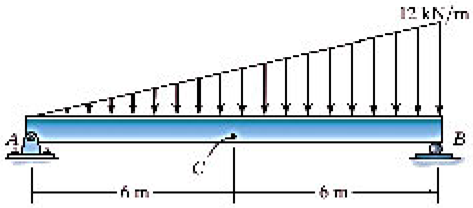
$$+ \frac{(-1.25)(-100)(10)(12)}{1.5[29(10^3)]} + (1)[0.6(10^{-5})](8)(12)$$

$$\Delta_{C_v} = 0.658\,\text{in}$$

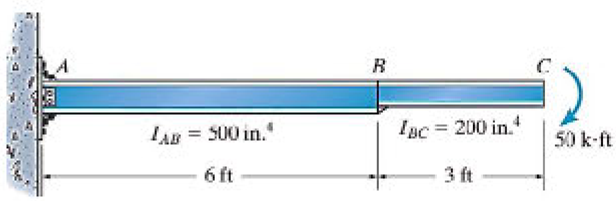
Hw.15
Determine the slope at ***A*** and displacement at point ***C***. ***EI*** is constant. Use the principle of virtual work.



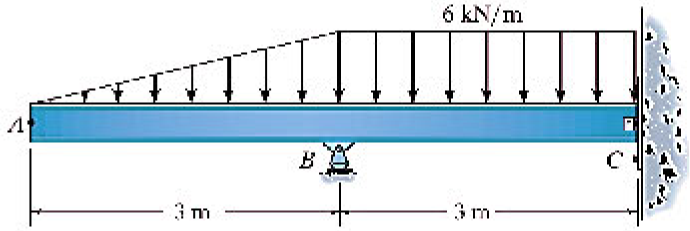
Hw.16
Determine the displacement at point ***C***. ***EI*** is constant. Use the principle of virtual work.



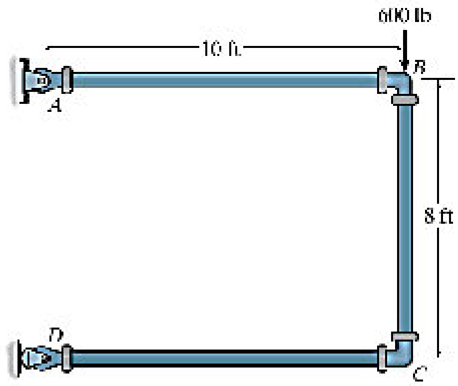
Hw.17
Determine the displacement and slope at point ***C*** of the cantilever beam. The moment of inertia of each segment is indicated in the figure. Take $\mathbf{E = 29(10^3)\ ksi}$. Use the principle of virtual work.



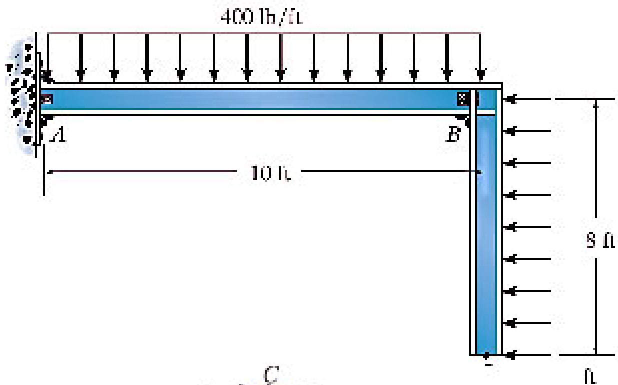
Hw.18
Determine the slope and displacement at point A. Assume *C* is pinned. Use the principle of virtual work. ***EI*** is constant.



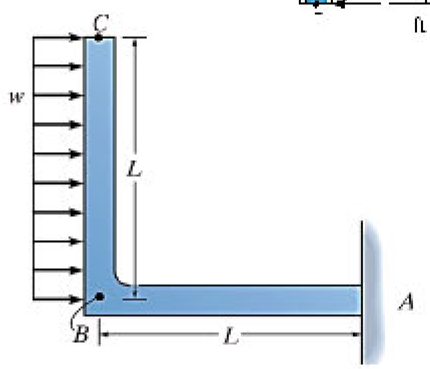
Hw.19
Use the method of virtual work and determine the vertical deflection at the rocker support ***D***. ***EI*** is constant



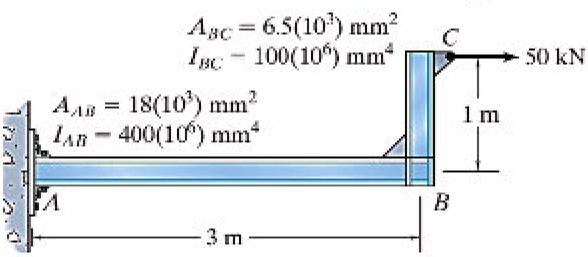
Hw.20
Determine the horizontal displacement of point ***C***. ***EI*** is constant. Use the method of virtual work.



Hw.21
The **L-shaped** frame is made from two segments, each of length ***L*** and flexural stiffness ***EI***. If it is subjected to the uniform distributed load, determine the horizontal displacement of the end ***C***, and the vertical displacement of point ***B***. Use the method of virtual work.

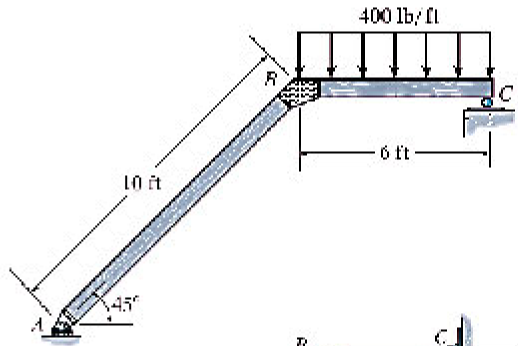


Hw.22
Determine the vertical deflection at *C*. The cross-sectional area and moment of inertia of each segment is shown in the figure. Take ***E*** **= 200 GPa** Assume ***A*** is a fixed support. Use the method of virtual work.



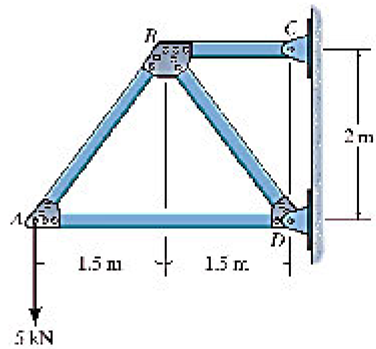
Hw.23

Use the method of virtual work and determine the horizontal deflection at *C*. ***EI*** is constant. There is a pin at ***A***. Assume ***C*** is a roller and ***B*** is a fixed joint.



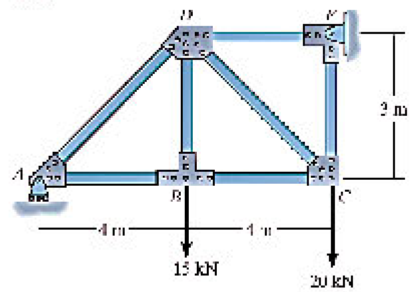
Hw.24

Determine the vertical displacement of joint ***A***. Each bar is made of steel and has a cross-sectional area of **600 mm2**. Take ***E*** = **200 GPa**. Use the method of virtual work.



Hw.25

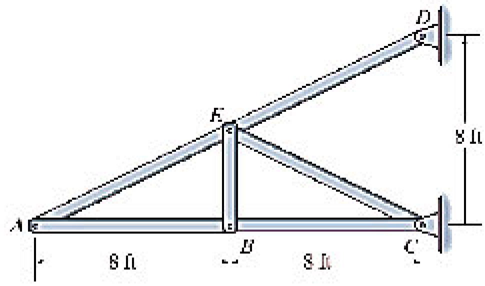
Determine the vertical displacement of joint ***D***. Use the method of virtual work. ***AE*** is constant. Assume the members are pin connected at their ends.



Hw.26

(A) Determine the vertical displacement of joint ***A*** if members ***AB*** and ***BC*** experience a temperature increase of ΔT = **200°F**. Take ***A*** = **2 in^2** and ***E*** = **29(10^3) ksi.** Also, **α= 6.60 (10-6)/°F.**

(B) Determine the vertical displacement of joint ***A*** if member ***AE*** is fabricated **0.5 in**. too short.



DEFORMATION OF STRUCTURES

Moment Diagrams by Parts

Basic Principles

1. The bending moment caused by all forces to the left or to the right of ***any section*** is equal to the respective algebraic sum of the bending moments at that section caused by each load acting separately.

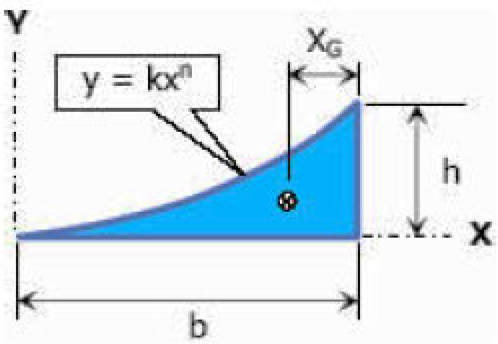
$$M = (\Sigma M)_L = (\Sigma M)_R$$

2. The moment of a load about a specified axis is always defined by the equation of a spandrel.

$$y = k\,x^n$$

where n is the degree of power of x.

The graph of the above equation is as shown below



Area and centroid of moment diagram (spandrel)

and the area and location of centroid are defined as follows.

$$A = \frac{1}{n+1}bh$$

$$X_G = \frac{1}{n+2}b$$

Cantilever Loadings

A = area of moment diagram
M_x = moment about a section of distance x
$\bar{x}$ = location of centoid
Degree = degree power of the moment diagram

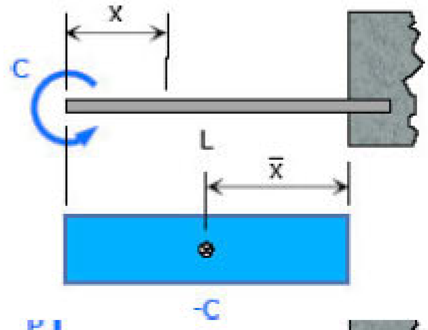
Couple or Moment Load

$$A = -CL$$

$$M_x = -C$$

$$\bar{x} = \frac{1}{2}L$$

Degree : zero



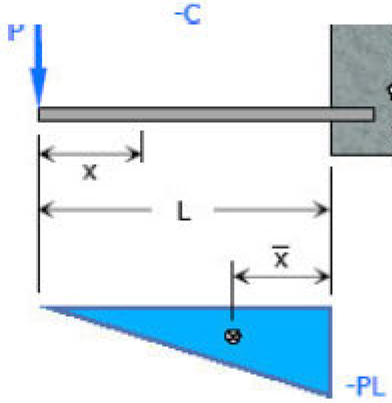
Concentrated Load

$$A = -\frac{1}{2}PL^2$$

$$M_x = -Px$$

$$\bar{x} = \frac{1}{3}L$$

Degree : first



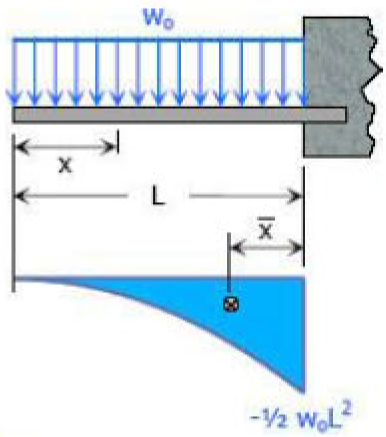
Uniformly Distributed Load

$$A = -\frac{1}{6}w_o L^3$$

$$Mx = -\frac{1}{2}w_o x^2$$

$$\bar{x} = \frac{1}{4}L$$

Degree : second



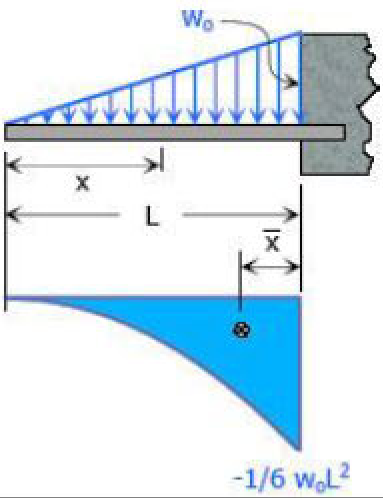
Uniformly Varying Load

$$A = -\frac{1}{24}w_o L^3$$

$$M_x = -\frac{w_o}{6L}x^3$$

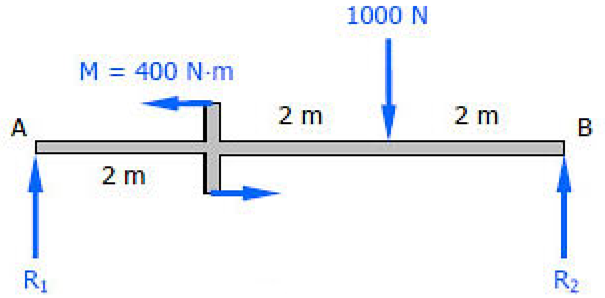
$$x = \frac{1}{5}L$$

Degree : third



Example 1

For the beam loaded as shown in the figure , compute the moment of area of the ***M*** diagrams between the reactions about both the left and the right reaction.

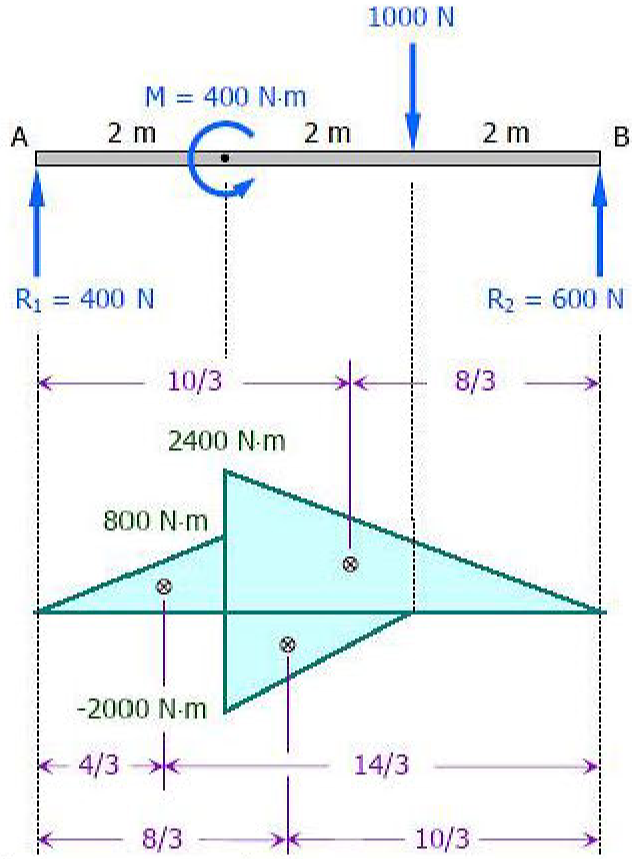


Solution

$$\Sigma M_{R2} = 0$$
$$6R_1 = 400 + 1000(2) \quad \Rightarrow \quad R_1 = 400N$$
$$\Sigma M_{R1} = 0$$
$$6R_2 + 400 = 1000(2) \quad \Rightarrow \quad R_2 = 600N$$

Moment diagram by parts can be drawn in different ways;

<u>1st Solution</u>



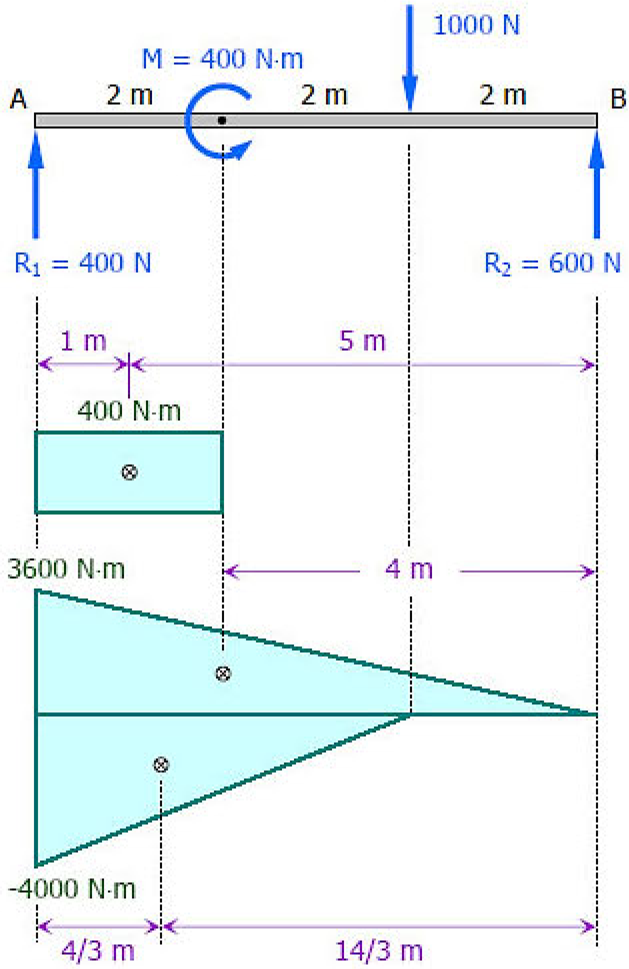
$$(Area_{AB})\bar{X}_A = \frac{1}{2}(2)(800)\left(\frac{4}{3}\right) + \frac{1}{2}(4)(2400)\left(\frac{10}{3}\right) - \frac{1}{2}(2)(2000)\left(\frac{8}{3}\right)$$

$$(Area_{AB})\bar{X}_A = 11733.33 \text{ N.m}^3$$

$$(Area_{AB})\bar{X}_B = \frac{1}{2}(2)(800)\left(\frac{14}{3}\right) + \frac{1}{2}(4)(2400)\left(\frac{8}{3}\right) - \frac{1}{2}(2)(2000)\left(\frac{10}{3}\right)$$

$$(Area_{AB})\bar{X}_B = 9866.67 \text{ N.m}^3$$

2nd Solution



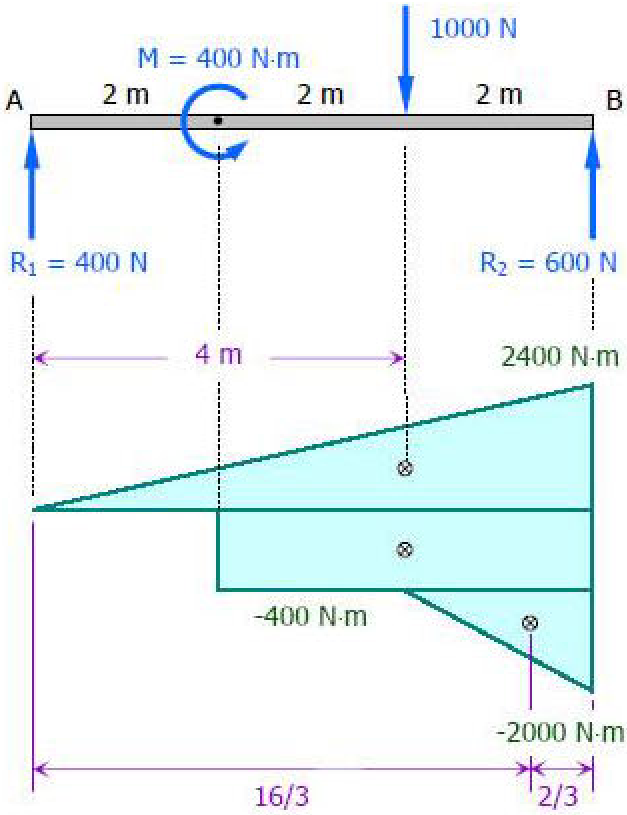
$$(Area_{AB})\bar{X}_A = 400(2)(1)+\frac{1}{2}(6)(3600)(2)-\frac{1}{2}(4)(4000)\left(\frac{4}{3}\right)$$

$$(Area_{AB})\bar{X}_A = 11733.33\,\text{N.m}^3$$

$$(Area_{AB})\bar{X}_B = 400(2)(5)+\frac{1}{2}(6)(3600)(4)-\frac{1}{2}(4)(4000)\left(\frac{14}{3}\right)$$

$$(Area_{AB})\bar{X}_B = 9866.67\,\text{N.m}^3$$

3rd Solution



$$(Area_{AB})\bar{X}_A = \frac{1}{2}(6)(2400)(4) - 400(4)(4) - \frac{1}{2}(2)(2000)\left(\frac{16}{3}\right)$$

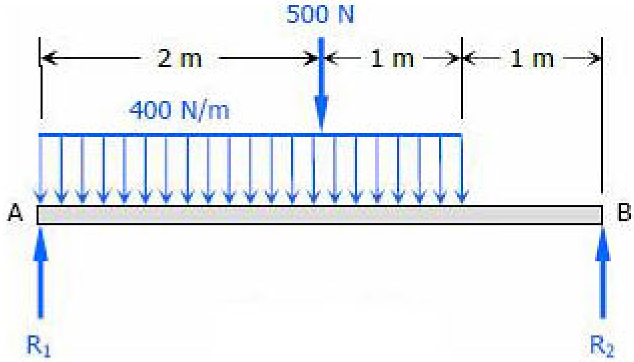
$$(Area_{AB})\bar{X}_A = 11733.33 \text{ N.m}^3$$

$$(Area_{AB})\bar{X}_B = \frac{1}{2}(6)(2400)(2) - 400(4)(2) - \frac{1}{2}(2)(2000)\left(\frac{2}{3}\right)$$

$$(Area_{AB})\bar{X}_B = 9866.67 \text{ N.m}^3$$

Example 2

For the beam loaded as shown in the figure, compute the moment of area of the ***M*** diagrams between the reactions about both the left and the right reaction.



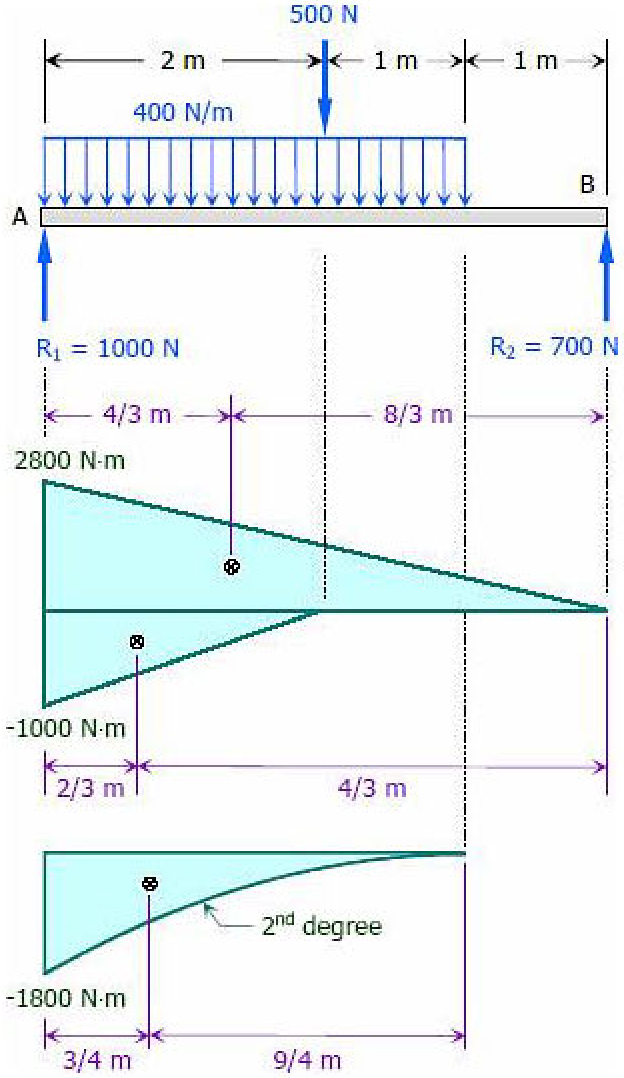
Solution

$$\Sigma M_{R2} = 0$$

$$4R_1 = 400(3)(2.5) + 500(2) \quad \Rightarrow R_1 = 1000\text{N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 = 400(3)(1.5) + 500(2) \quad \Rightarrow R_2 = 700\text{N}$$



$$(Area_{AB})\bar{X}_A = \frac{1}{2}(4)(2800)\left(\frac{4}{3}\right) - \frac{1}{2}(2)(1000)\left(\frac{2}{3}\right) - \frac{1}{3}(3)(1800)\left(\frac{3}{4}\right)$$

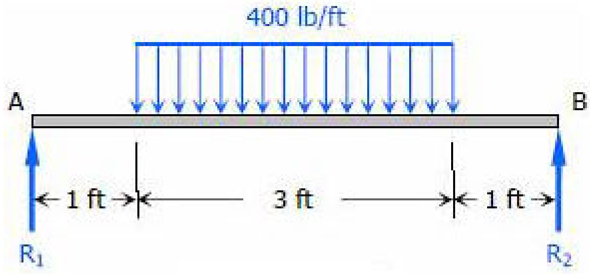
$$(Area_{AB})\bar{X}_A = 5450 \ \ \text{N.m}^3 \qquad Ans.$$

$$(Area_{AB})\bar{X}_B = \frac{1}{2}(4)(2800)\left(\frac{8}{3}\right) - \frac{1}{2}(2)(1000)\left(\frac{4}{3}\right) - \frac{1}{3}(3)(1800)\left(\frac{9}{4}+1\right)$$

$$(Area_{AB})\bar{X}_B = 7750 \ \ \text{N.m}^3 \qquad Ans.$$

Example 3

For the eam loaded as shown in the figure, compute the moment of area of the ***M*** diagrams between the reactions about both the left and the right reaction.



Solution

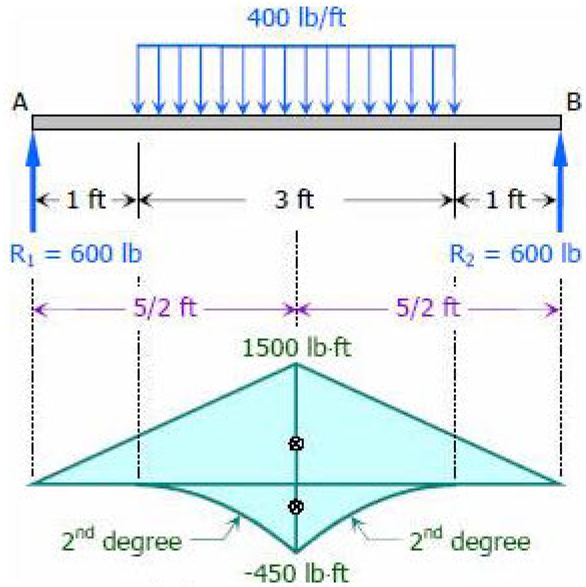
By symmetry

$$R_1 = R_2 = \frac{1}{2}(400)(3)$$

$$R_1 = R_2 = 600 \text{ lb}$$

and

$$(Area_{AB})\bar{X}_A = (Area_{AB})\bar{X}_B$$



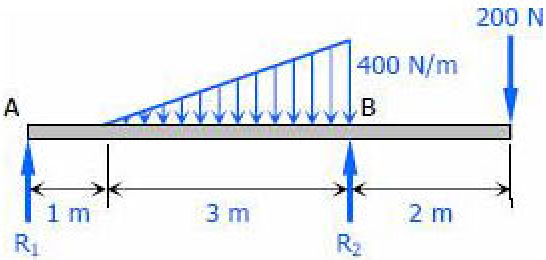
$$(Area_{AB})\bar{X}_A = \frac{1}{2}(5)(1500)\left(\frac{5}{2}\right) - \frac{1}{3}(3)(450)\left(\frac{5}{2}\right)$$

$$(Area_{AB})\bar{X}_A = 8250 \quad \text{lb.ft}^3 \quad \text{Ans.}$$

Thus,

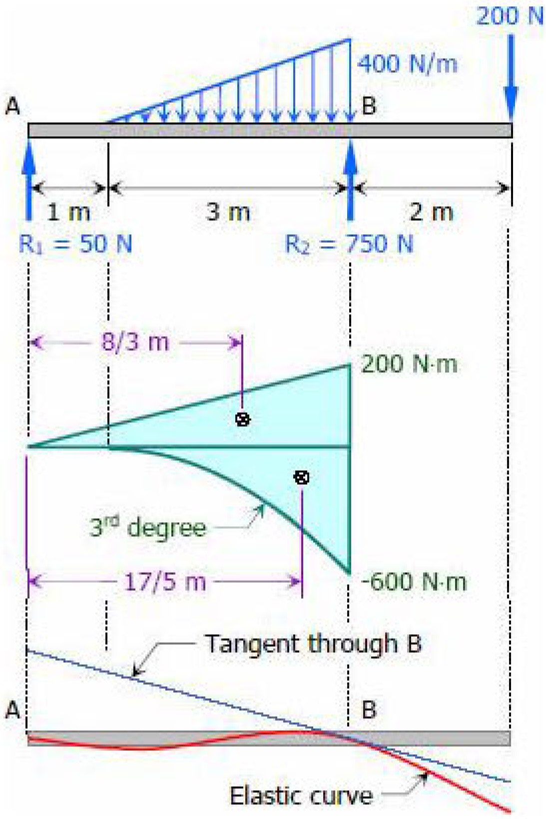
$$(Area_{AB})\bar{X}_B = 8250 \quad \text{lb.ft}^3 \quad \text{Ans.}$$

Example 4



For the beam loaded as shown in the figure, compute the value of $(Area_{AB})(\bar{X})_A$. From this result, is the tangent drawn to the elastic curve at ***B*** directed up or down to the right?

Solution



$$\Sigma M_{R2} = 0$$

$$4R_1 + 200(2) = \frac{1}{2}(3)(400)(1) \quad \Rightarrow R_1 = 50 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 = 200(6) + \frac{1}{2}(3)(400)(3) \quad \Rightarrow R_2 = 750 \text{ N}$$

$$(Area_{AB})\bar{X}_A = \frac{1}{2}(4)(200)\left(\frac{8}{3}\right) - \frac{1}{4}(3)(600)\left(\frac{17}{5}\right)$$

$$(Area_{AB})\bar{X}_A = -463.33 \quad \text{N.m}^3 \qquad \text{Ans.}$$

The value of $(Area_{AB})(\bar{X})_A$ is negative; therefore point ***A*** is below the tangent through ***B***, thus **the tangent through B slopes downward to the right**.

DEFORMATION OF STRUCTURES

Conjugate Beam Method

Properties of Conjugate Beam

1. The length of a conjugate beam is always equal to the length of the actual beam.
2. The load on the conjugate beam is the M/EI diagram of the loads on the actual beam.
3. A simple support for the real beam remains simple support for the conjugate beam.
4. A fixed end for the real beam becomes free end for the conjugate beam.
5. The point of zero shear for the conjugate beam corresponds to a point of zero slope for the real beam.
6. The point of maximum moment for the conjugate beam corresponds to a point of maximum deflection for the real beam.

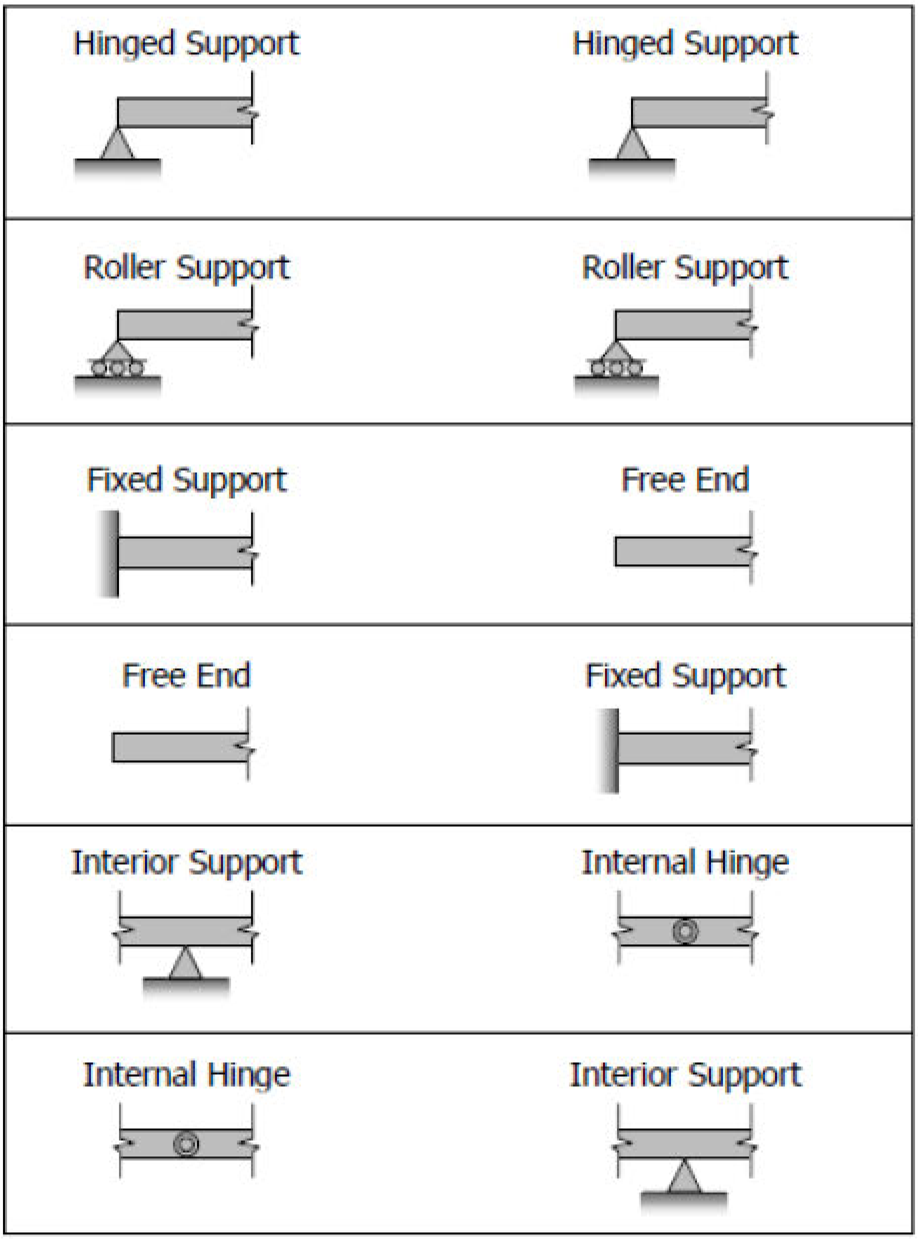
Slope on real beam = Shear on conjugate beam

Deflection on real beam = Moment on conjugate beam

Supports of Conjugate Beam

Knowing that the slope on the real beam is equal to the shear on conjugate beam and the deflection on real beam is equal to the moment on conjugate beam, the shear and bending moment at any point on the conjugate beam must be consistent with the slope and deflection at that point of the real beam. Take for example a real beam with fixed support; at the point of fixed support there is neither slope nor deflection, thus, the shear and moment of the corresponding conjugate beam at that point must be zero. Therefore, the conjugate of fixed support is free end.

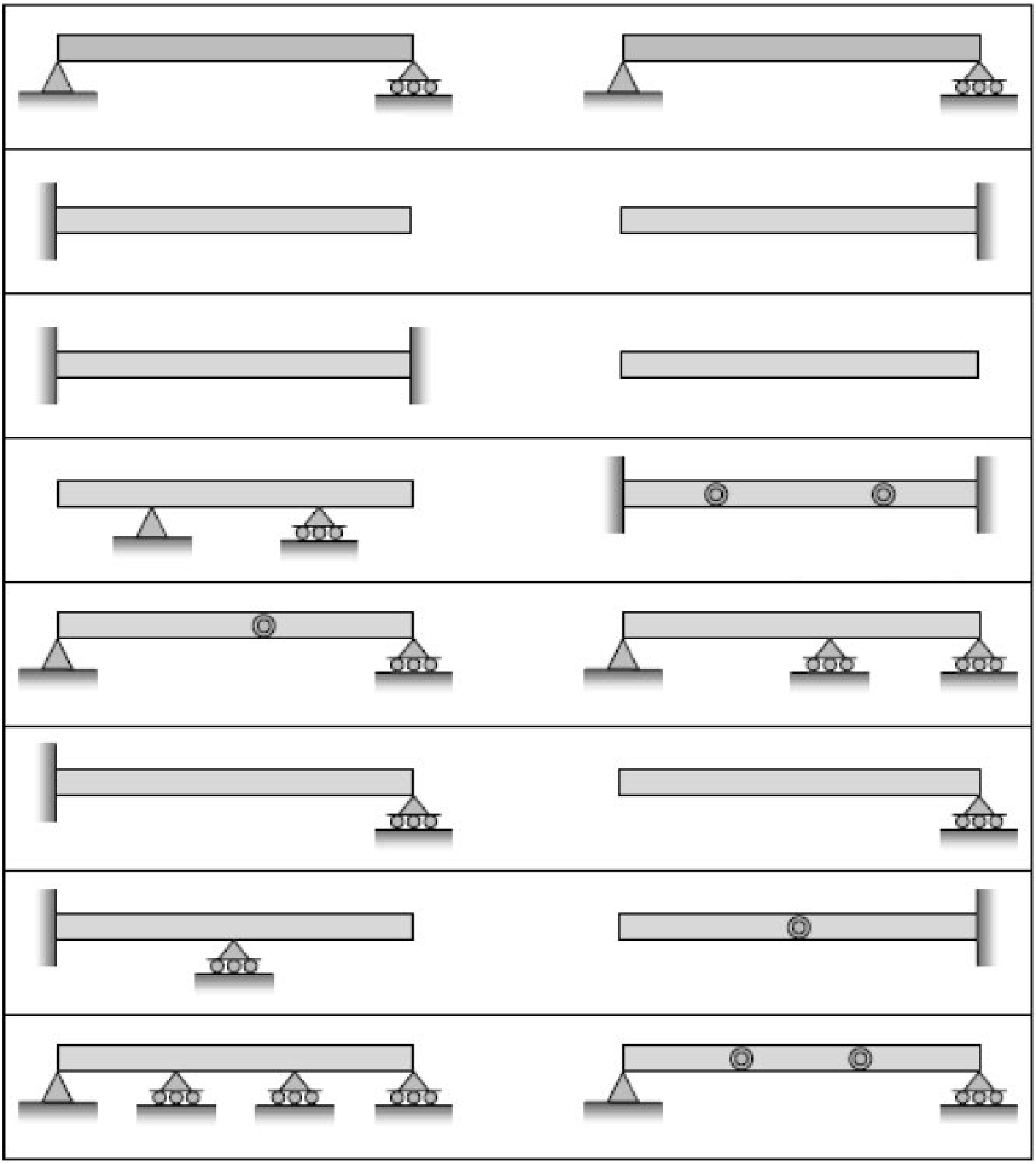
Real Beam Support **Conjugate Beam Support**



Examples of Beam and its Conjugate

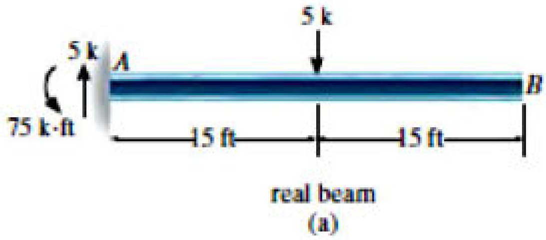
The following are some examples of beams and its conjugate. Loadings are omitted.

Real Beam **Conjugate Beam**



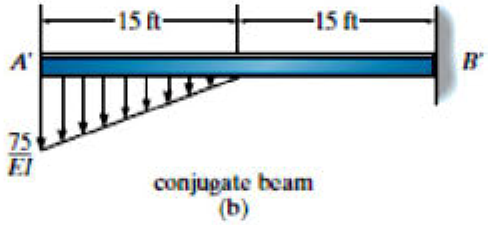
Example 1

Determine the slope and deflection at point B of the steel beam shown in **Fig. a**. The reactions have been computed. $E = 29(10^3)$ **ksi**, $I = 800$ **in**^{4}.

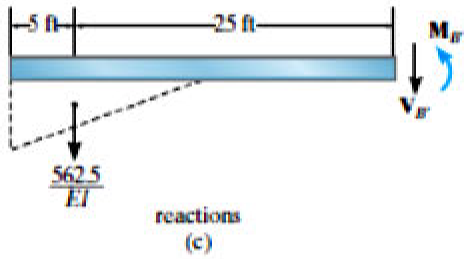


Solution

Conjugate Beam. The conjugate beam is shown in **Fig. b**. The supports at A' and B' correspond to supports A and B on the real beam. The M/EI diagram is *negative*, so the distributed load acts *downward*, i.e., away from the beam.



Equilibrium. Since θ_B and Δ_B are to be determined, we must compute $V_{B'}$ and $M_{B'}$ in the conjugate beam, **Fig. c**.



$$+\uparrow \sum F_y = 0; \quad -\frac{562.5 \text{ k.ft}^2}{EI} - V_{B'} = 0$$

$$\theta_B = V_{B'} = -\frac{562.5 \text{ k.ft}^2}{EI}$$

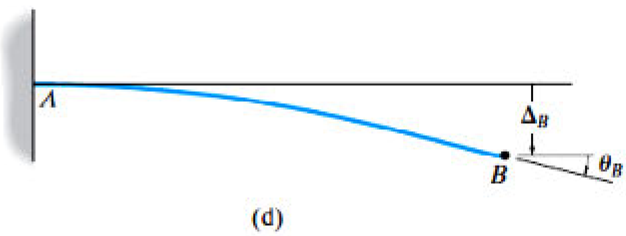
$$= \frac{-562.5 \text{ k.ft}^2}{29(10^3)\text{k/in}^2(144 \text{ in}^2/\text{ft}^2)\ 800 \text{ in}^4(1 \text{ ft}^4/(12)^4 \text{ in}^4)} = -0.00349 \text{ rad}$$

$\circlearrowleft +$

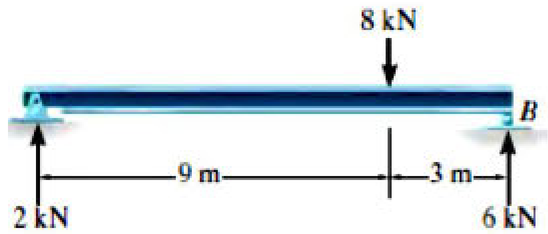
$$\sum M_{B'} = 0; \qquad \frac{562.5 \text{ k.ft}^2}{EI}(25 \text{ ft}) + M_{B'} = 0$$

$$\Delta_B = M_{B'} = \frac{14062.5 \text{ k.ft}^3}{EI} = -0.0873 \text{ ft} = -1.05 \text{ in}$$

The negative signs indicate the slope of the beam is measured clockwise and the displacement is downward, **Fig. d**.



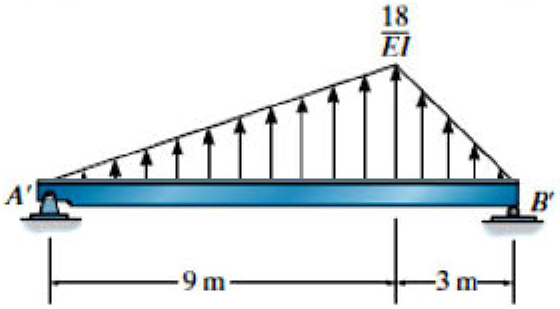
Example 2



Determine the maximum deflection of the steel beam shown in **Fig. a**. The reactions have been computed. $E = 200$ **GPa**, $I = 60(10^6)$ **mm**^{4}.

Solution

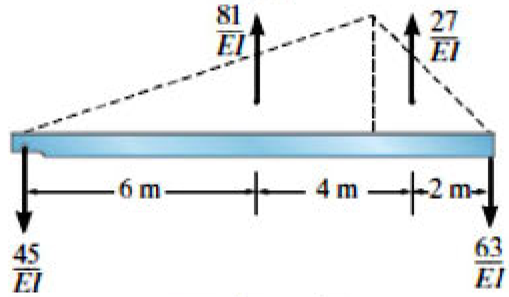
Conjugate Beam. The conjugate beam loaded with the ***M/EI*** diagram is shown in **Fig. b**. Since the ***M/EI*** diagram is positive, the distributed load acts upward (away from the beam).



conjugate beam

(b)

Equilibrium. The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram in Fig. c. ***Maximum deflection*** of the real beam occurs at the point where the ***slope*** of the beam is ***zero***. This corresponds to the same point in the conjugate beam where the ***shear*** is ***zero***.



external reactions

(c)

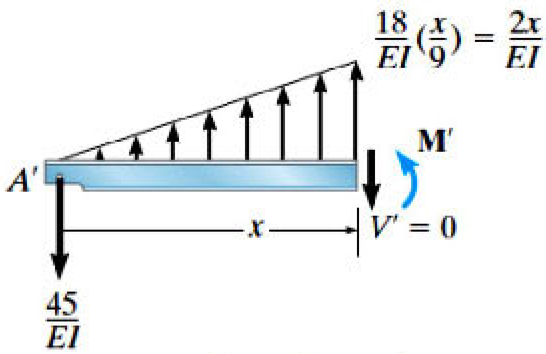
Assuming this point acts within the region **0** $\leq x \geq 9$ **m** from ***A′***. we can isolate the section shown in Fig. d. Note that the peak of the distributed loading was determined from proportional triangles, that is,

$$w/x = (18/EI)/9$$

We require $V' = 0$ so that,

$$+\uparrow \sum F_y = 0; \quad -\frac{45}{EI} + \frac{1}{2}\left(\frac{2x}{EI}\right)x = 0$$

$$x = 6.71 \text{ m} \qquad (0 \leq x \geq 9 \text{ m}) \text{ OK}$$



internal reactions

(d)

Using this value for *x*, the maximum deflection in the real beam corresponds to the moment ***M′***. Hence,

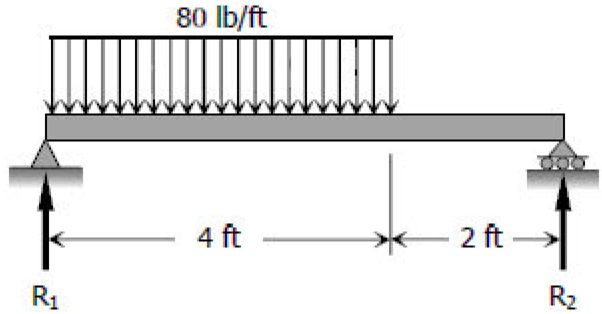
$$+\circlearrowleft \sum M = 0; \quad \frac{45}{EI}(6.71) - \left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71) + M'$$

$$\Delta_{max} = M' = -\frac{201.2 \text{ kN.m}^3}{EI} = -0.0168 \text{ m} = -16.8 \text{ mm}$$

The negative sign indicates the deflection is downward.

Example 3

For the beam in the figure find the value of ***EIδ*** at ***2 ft*** from ***R₂***



Solution

Solving for reactions

$\Sigma M_{R2} = 0$

$6R_1 = 80(4)(4)$

$R1 = 213.33\,\text{lb}$

$\Sigma M_{R1} = 0$

$6R_2 = 80(4)(2)$

$R_2 = 106.67\,\text{lb}$

From the conjugate beam

$\Sigma M_A = 0$

$$6F_2 + \frac{1}{3}(4)(640)\left[\frac{3}{4}(4)\right] =$$

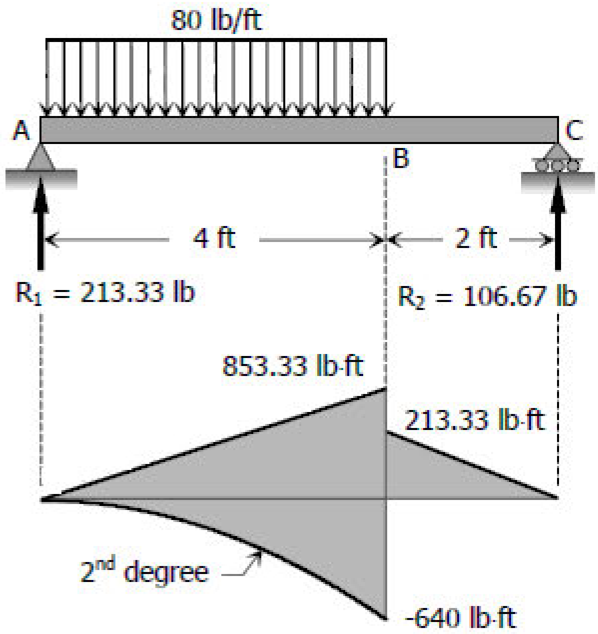
$$\frac{1}{2}(4)(853.33)\left[23(4)\right] + \frac{1}{2}(2)(213.33)\left[4 + \frac{1}{3}(2)\right]$$

$F_2 = 497.77\,\text{lb.ft}^2$

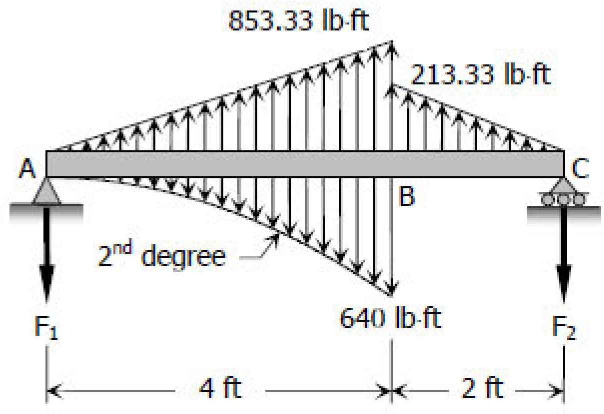
$$M_B = \frac{1}{2}(2)(213.33\left[\frac{1}{3}(2)\right] \text{-} 2F_2$$

$$M_B = \frac{1}{2}(2)(213.33\left[\frac{1}{3}(2)\right] \text{-} 2(497.77)$$

$M_B = \text{-}853.32\,\text{lb.ft}^3$



Moment Diagram by Parts



Conjugate Beam Loaded with M/EI Diagram

Thus, the deflection at B **is**

$EI\,\delta_B = M_B$

$EI\,\delta_B = \text{-}853.32\ \text{lb.ft}^3$

$EI\,\delta_B = 853.32\ \text{lb.ft}^3$ **downward**