8 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

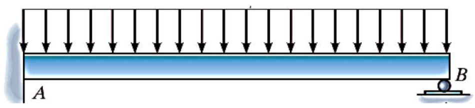
8.1 Statically Indeterminate Structures

A structure of any type is classified as *statically indeterminate* when the number of unknown reactions or internal forces exceeds the number of equilibrium equations available for its analysis. The most of the structures designed today are statically indeterminate. This indeterminacy may arise as a result of added supports or members, or by the general form of the structure. For example, reinforced concrete buildings are almost always statically indeterminate since the columns and beams are poured as continuous members through the joints and over supports.

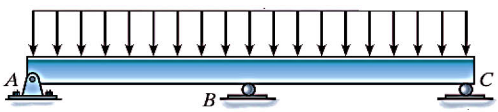
8.2 Slope-Deflection Method (Displacement Method)

All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety.

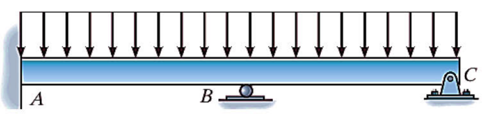
- ✓ This method considers the deflection as the primary unknowns.
- ✓ In this method, if the slopes at the ends and the relative displacement of the ends are known, the end moment can be found in terms of slopes, deflection, stiffness and length of the members.
- ✓ In the slope-deflection method, the relationship is established between moments at the ends of the members and the corresponding rotations and displacements.
- ✓ The *basic assumption* used in the slope-deflection method is that a typical member can flex but the shear and axial deformation are negligible. It is no different from that used with the force method.
- ✓ *Kinematically indeterminate* structures versus statically indeterminate structures:



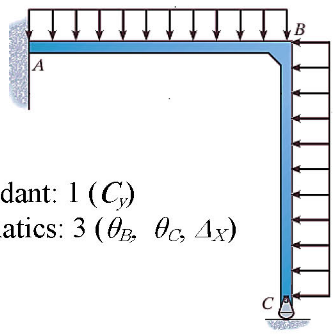
Redundant: 1 (B_y)
Kinematics: 1 (θ_B)



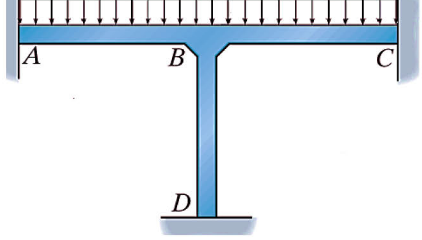
Redundant: 1 (B_y)
Kinematics: 3 (θ_A, θ_B, θ_C)



Redundant: 2 (B_y, C_y)
Kinematics: 2 (θ_B, θ_C)



Redundant: 1 (C_y)
Kinematics: 3 (θ_B, θ_C, Δ_X)



Redundant: 6 (A_x, A_y, M_A, B_x, B_y, M_B)
Kinematics: 1 (θ_B)

Sign convention

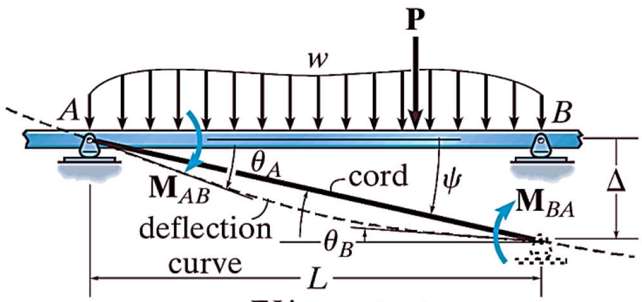
All **clockwise internal moments** and **end rotation** are ***positive***.

Basic Idea of Slope Deflection Method

The basic idea of the slope deflection method is to write the equilibrium equations for ***each node*** in terms of the ***deflections and rotations***. Solve for the generalized displacements. Using moment-displacement relations, moments are then known. The structure is thus reduced to a determinate structure.

8.2.1 Fundamental Slope-Deflection Equations:

The slope-deflection method is so named since it relates the unknown slopes and deflections to the applied load on a structure. In order to develop the general form of the slope-deflection equations, we will consider the typical span ***AB*** of a continuous beam as shown in the figure, which is subjected to the arbitrary loading and has a constant ***EI***. We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom, namely, its angular displacements θ_A and θ_B and linear displacement Δ which could be caused by a relative settlement between the supports. Since we will be developing a formula, ***moments*** and ***angular displacements*** will be considered ***positive*** when they act ***clockwise on the span***, as shown in the Figure Furthermore, the ***linear displacement*** Δ is considered ***positive*** as shown, since this displacement causes the cord of the span and the span's cord angle ψ to rotate ***clockwise***.



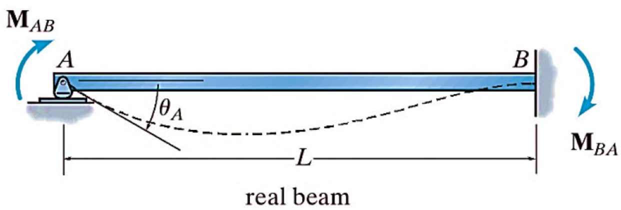
EI is constant
positive sign convention

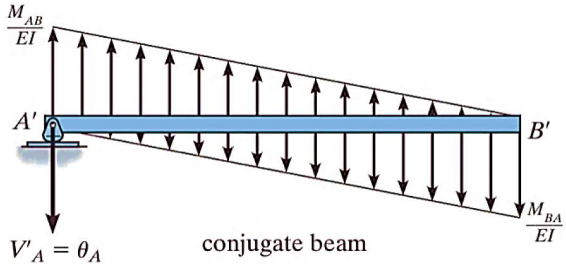
The slope-deflection equations can be obtained by using the principle of superposition by considering *separately* the moments developed at each support due to each of the displacements θ_A, θ_B, and Δ and then the loads

Case A*: Rotation at A*

θ_A, Unknown

$\theta_B = 0, \quad \Delta_{AB} = 0, \quad P = 0$





$$\theta_A = \frac{M_{AB}}{2EI}L - \frac{M_{BA}}{2EI}L$$

$$\Delta = 0 = \frac{M_{AB}L}{2EI}\cdot\frac{L}{3} - \frac{M_{BA}L}{2EI}\cdot\frac{2L}{3}$$

$$M_{AB} = 2M_{BA}$$

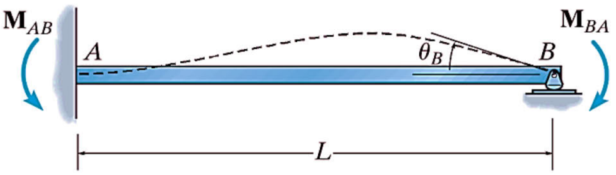
$$\theta_A = \frac{M_{AB}L}{4EI}$$

$$M_{AB} = \frac{4EI\cdot\theta_A}{L}, \qquad M_{BA} = \frac{2EI\cdot\theta_A}{L}$$

Case B: ***Rotation at B***

θ_B, Unknown

$\theta_A = 0, \quad \Delta_{BA} = 0, \quad P = 0$



$$M_{AB} = \frac{2EI \cdot \theta_B}{L}, \qquad M_{BA} = \frac{4EI \cdot \theta_B}{L}$$

Case C: ***Displacement of End B Related to End A*** **(Relative Linear Displacement, Δ)**

Δ_{AB}, Unknown

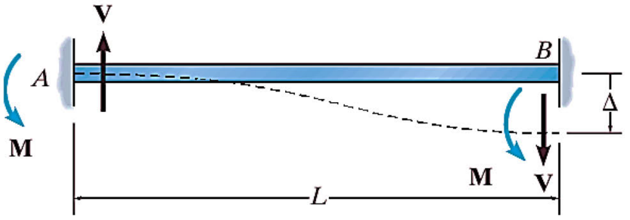
$[\theta_A = \theta_B = 0, \quad P = 0]$

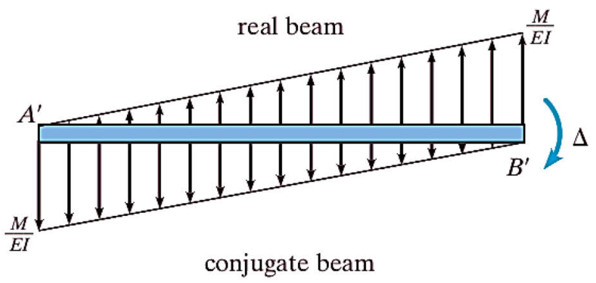
$$\theta = 0 = \frac{-M_{AB}}{EI}\frac{L}{2} + \frac{M_{BA}}{EI}\frac{L}{2}$$

$$\Delta_{AB} = \frac{-M_{AB}L}{2EI} \cdot \frac{2L}{3} + \frac{M_{BA}L}{2EI} \cdot \frac{L}{3}$$

$$\Delta_{AB} = \frac{-M_{AB}L^2}{6EI}$$

$$M_{AB} = M_{BA} = \frac{-6EI\,\Delta_{AB}}{L^2}$$



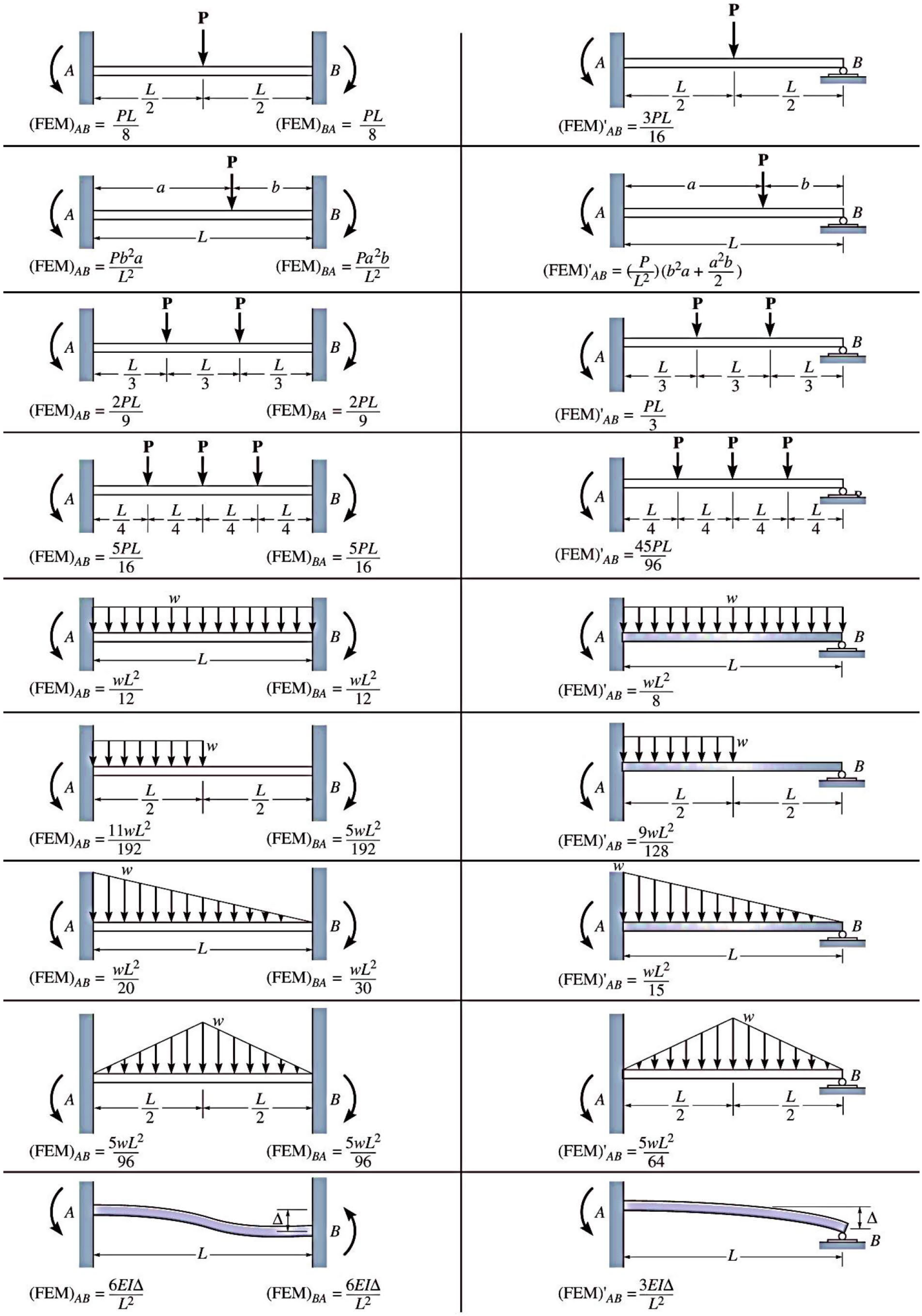


Case D: ***Fixed-End Moments*** **(FEM)**

In order to develop the slope-deflection equations, we must transform these ***span loadings*** into equivalent moments acting at the nodes and then use the load-displacement relationships just derived. This is done simply by finding the reaction moment that each load develops at the nodes.

This moment is called a ***fixed-end moment*** **(FEM)**. Note that according to our sign convention, it is ***negative*** at ***node A (counterclockwise)*** and ***positive*** at ***node B (clockwise).***

For convenience in solving problems, fixed-end moments have been calculated for many loadings and are tabulated.



Both ends fixed		One end fixed, one end pinned
$(FEM)_{AB} = \frac{PL}{8}$	$(FEM)_{BA} = \frac{PL}{8}$	$(FEM)'_{AB} = \frac{3PL}{16}$
$(FEM)_{AB} = \frac{Pb^2a}{L^2}$	$(FEM)_{BA} = \frac{Pa^2b}{L^2}$	$(FEM)'_{AB} = (\frac{P}{L^2})(b^2a + \frac{a^2b}{2})$
$(FEM)_{AB} = \frac{2PL}{9}$	$(FEM)_{BA} = \frac{2PL}{9}$	$(FEM)'_{AB} = \frac{PL}{3}$
$(FEM)_{AB} = \frac{5PL}{16}$	$(FEM)_{BA} = \frac{5PL}{16}$	$(FEM)'_{AB} = \frac{45PL}{96}$
$(FEM)_{AB} = \frac{wL^2}{12}$	$(FEM)_{BA} = \frac{wL^2}{12}$	$(FEM)'_{AB} = \frac{wL^2}{8}$
$(FEM)_{AB} = \frac{11wL^2}{192}$	$(FEM)_{BA} = \frac{5wL^2}{192}$	$(FEM)'_{AB} = \frac{9wL^2}{128}$
$(FEM)_{AB} = \frac{wL^2}{20}$	$(FEM)_{BA} = \frac{wL^2}{30}$	$(FEM)'_{AB} = \frac{wL^2}{15}$
$(FEM)_{AB} = \frac{5wL^2}{96}$	$(FEM)_{BA} = \frac{5wL^2}{96}$	$(FEM)'_{AB} = \frac{5wL^2}{64}$
$(FEM)_{AB} = \frac{6EI\Delta}{L^2}$	$(FEM)_{BA} = \frac{6EI\Delta}{L^2}$	$(FEM)'_{AB} = \frac{3EI\Delta}{L^2}$

Table (8-1): Fixed-End Moments (FEM)

8.2.2 Slope-Deflection Equation.

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + (FEM)_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + (FEM)_{BA}$$

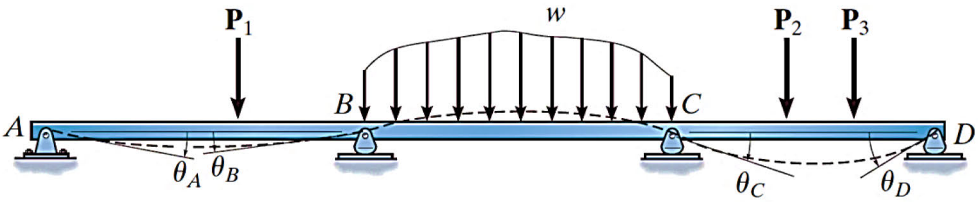
or

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + (FEM)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + (FEM)_F$$

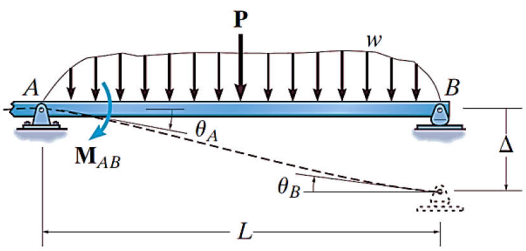
where

- M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.
- M_F = internal moment in the far end of the span; this moment is *positive clockwise* when acting on the span.
- E , k = modulus of elasticity of material and span stiffness $k = I/L$
- θ_N , θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*
- ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.* (**ψ 'psi'**)
- $(FEM)_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table for various loading conditions.

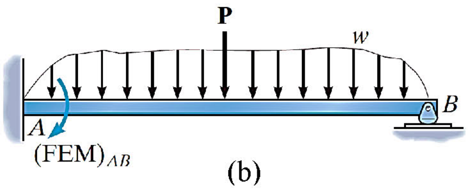


Pin-Supported End Span.

Occasionally an end span of a beam or frame is supported by a pin or roller at its ***far end,*** **Fig.*a***. When this occurs, the moment at the roller or pin must be **zero**; and **provided** the angular displacement θ_B at this support does not have to be determined, we can modify the general slope-deflection equation so that it has to be applied ***only once*** to the span rather than twice.



(a)



(b)

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + (FEM)_N$$

$$0 = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + (FEM)_F$$

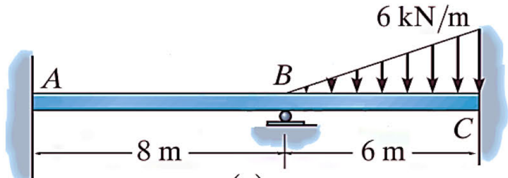
Multiplying the first equation by **2** and subtracting the second equation from it yields

$$M_N = 3Ek\left[\theta_N - \psi\right] + (FEM)'_N$$

Only for End Span with Far End Pinned or Roller Supported

EXAMPLE 8.2.1

Draw the shear and moment diagrams for the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

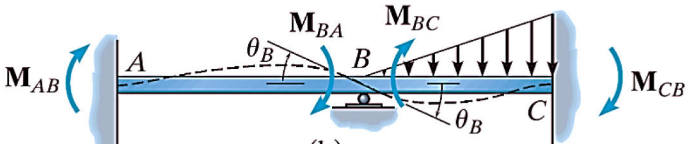
$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

For member AB

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{BA}$$



(b)

(*FEM*)$_{AB}$ = (*FEM*)$_{BA}$ = 0 (There is no load on span ***AB***.)

$$M_{AB} = \frac{2EI}{8}\left[2(0) + \theta_B - 3(0)\right] + 0 = \frac{EI}{4}\cdot\theta_B \qquad ...(1)$$

$$M_{BA} = \frac{2EI}{8}\left[0 + 2\theta_B - 3(0)\right] + 0 = \frac{EI}{2}\cdot\theta_B \qquad ...(2)$$

$$\Rightarrow 2M_{AB} = M_{BA}$$

For member BC

$$\left(FEM\right)_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2 \text{ kN.m}$$

> **Note: (*FEM*)$_{BC}$** is negative since it acts counterclockwise *on the beam* at ***B***.

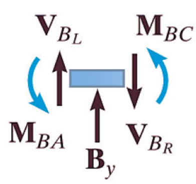
$$\left(FEM\right)_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{30} = 10.8 \text{ kN.m}$$

$$M_{BC} = \frac{2EI}{6}\left[2\theta_B + 0 - 3(0)\right] - 7.2 = \frac{2EI}{3}\cdot\theta_B - 7.2 \qquad ...(3)$$

$$M_{CB} = \frac{2EI}{6}\left[\theta_B + 2(0) - 3(0)\right] + 10.8 = \frac{EI}{3}\cdot\theta_B + 10.8 \qquad ...(4)$$

Equilibrium Equations. The above **four** equations contain **five** unknowns. The necessary fifth equation comes from the condition of ***moment equilibrium*** at support ***B***.

$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(5)$$



(c)

To solve, substitute **Eqs. (2)** and **(3)** into **Eq. (5)**, which yields

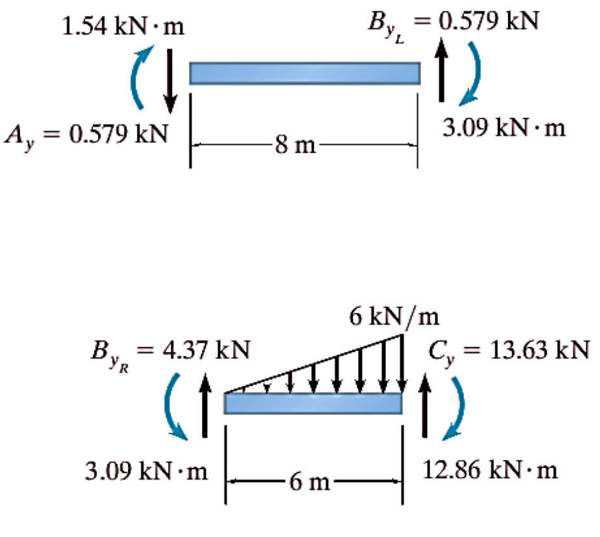
$$\frac{EI}{2}\cdot\theta_B + \frac{2EI}{3}\cdot\theta_B - 7.2 = 0 \qquad \Rightarrow \qquad \boldsymbol{\theta_B = \frac{6.17}{EI}}$$

Substituting this value into **Eqs. (1)–(4)** yields

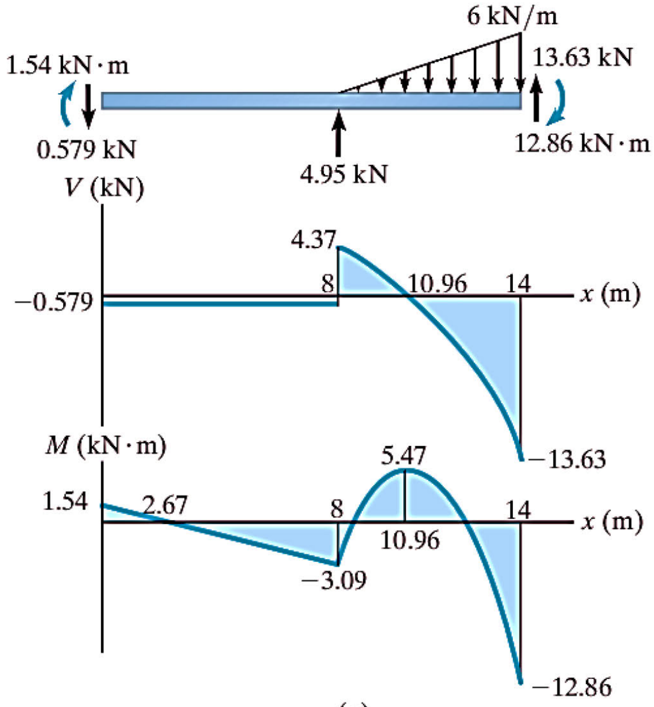
$$M_{AB} = 1.54 \text{ kN.}m \qquad M_{BA} = 3.09 \text{ kN.}m$$

$$M_{BC} = -3.09 \text{ kN.}m \qquad M_{CB} = 12.86 \text{ kN.}m$$

Using these results, the shears at the end spans are determined from the equilibrium equations, **Fig. *d***. The free-body diagram of the entire beam and the shear and moment diagrams are shown in **Fig. *e***.



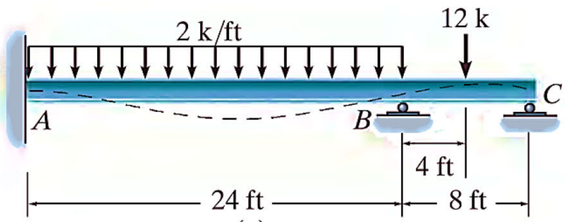
(d)



(e)

EXAMPLE 8.2.2

Draw the shear and moment diagrams for the beam shown in **Fig. *a***. ***EI*** is constant.



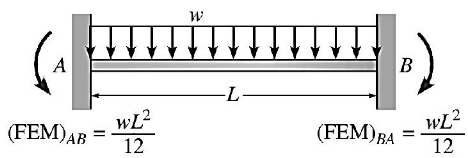
(a)

Solution

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

For member AB



$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{BA}$$

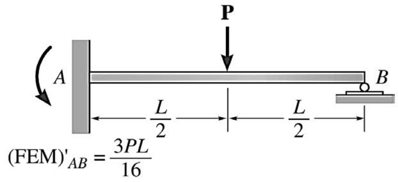
$$M_{AB} = \frac{2EI}{24}\left[0 + \theta_B - 0\right] - \frac{2(24)^2}{12} = \frac{EI}{12}\cdot\theta_B - 96 \qquad ...(1)$$

$$M_{BA} = \frac{2EI}{24}\left[0 + 2\theta_B - 0\right] + \frac{2(24)^2}{12} = \frac{EI}{6}\cdot\theta_B + 96 \qquad ...(2)$$

For member BC

$$M_N = 3Ek\left[\theta_N - \psi\right] + \left(FEM\right)'_N$$

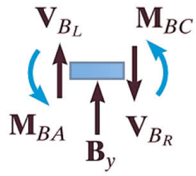
$$\therefore \quad M_{BC} = \frac{3EI}{L}\left[\theta_B - \frac{\Delta}{L}\right] + \left(FEM\right)'_{BC}$$



$$M_{BC} = \frac{3EI}{8}\left[\theta_B - 0\right] - \frac{3(12)(8)}{16} = \frac{3EI}{8}\cdot(\theta_B) - 18 \qquad ...(3)$$

$$M_{CB} = 0$$

Equilibrium Equations. The above **three** equations contain **four** unknowns. The necessary **fourth** equation comes from the condition of ***moment equilibrium*** at support ***B***.



$$+\circlearrowleft\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(4)$$

To solve, substitute **Eqs. (2)** and **(3)** into **Eq. (4)**, which yields

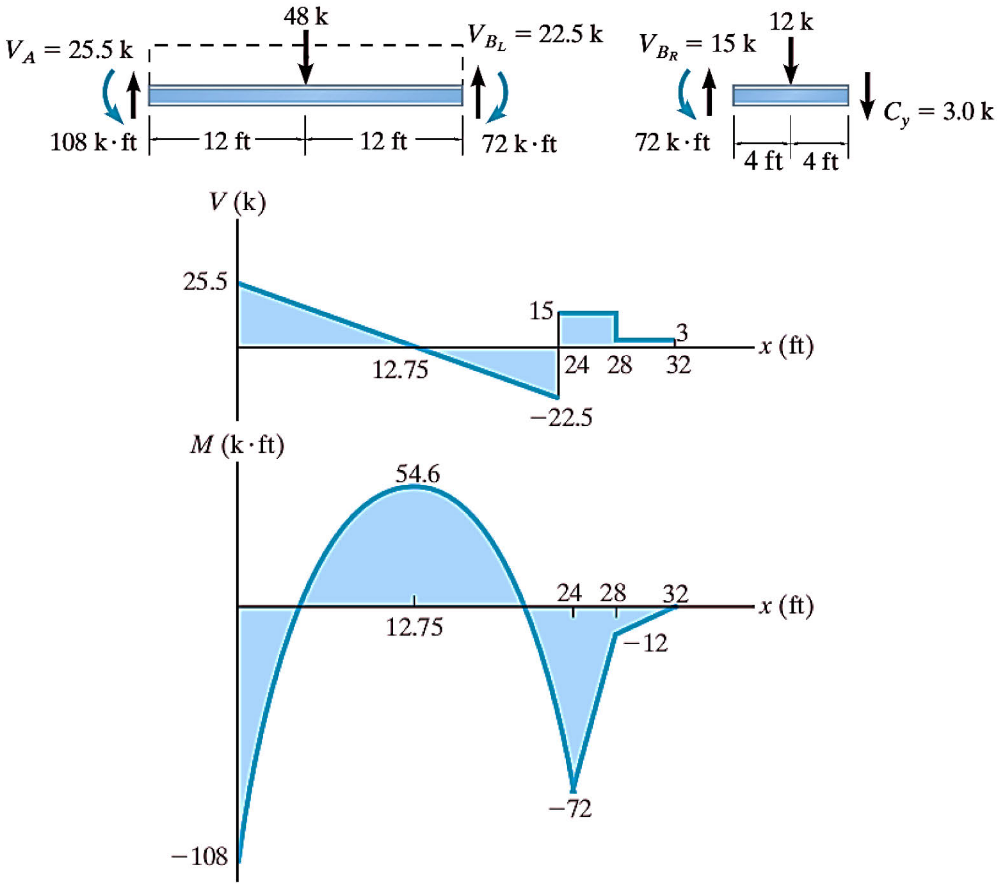
$$\left[\frac{EI}{6}\cdot\theta_B + 96\right] + \left[\frac{3EI}{8}\cdot(\theta_B) - 18\right] = 0$$

$$\frac{26EI}{48}\cdot(\theta_B) + 78 = 0 \quad \Rightarrow \quad \boldsymbol{\theta_B = \frac{-144}{EI}}$$

Note: Since $\boldsymbol{\theta_B}$ is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in **Fig. *a***. Substituting $\boldsymbol{\theta_B}$ into **Eqs. (1)–(3),** we get

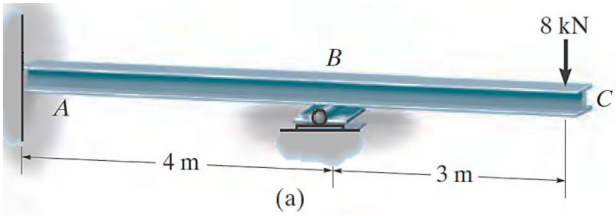
$M_{AB} = -108.0 \text{ k.ft}$ $\qquad M_{BA} = -72.0 \text{ k.ft}$

$M_{BC} = 72.0 \text{ kN.ft}$ $\qquad M_{CB} = 0$



EXAMPLE 8.2.3

Determine the moment at *A* and *B* for the beam shown in **Fig. *a***. The support at ***B*** is displaced (settles) **80 mm**. Take **E = 200 GPa , I = 5(10^6) mm^4.**

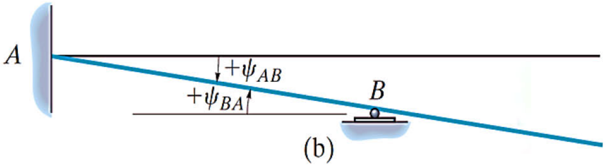


(a)

Solution

Only one span (***AB***) must be considered in this problem since the moment M_{BC} due to the overhang can be calculated from statics. Since there is no loading on span ***AB***, the **FEMs** are ***zero***.

As shown in **Fig.*b***, the downward displacement (settlement) of ***B*** causes the cord for span ***AB*** to rotate ***clockwise***. Thus,



(b)

$$\psi_{AB} = \psi_{BA} = \frac{0.08 \text{ m}}{4 \text{ m}} = 0.02 \text{ rad}$$

The stiffness for AB is

$$k = \frac{I}{L} = \frac{5(10^6)\text{ mm}^4(10^{-12})\text{ m}^4/\text{mm}^4}{4 \text{ m}} = 1.25(10^{-6})\text{ m}^3$$

Applying the slope-deflection equation, to span ***AB***, with $\theta_A = 0$ we have,

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

$$M_{AB} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2(0) + \theta_B - 3(0.02)] + 0 \qquad ...(1)$$

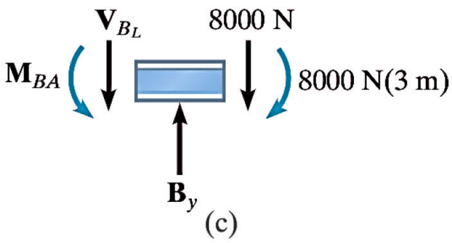
$$M_{BA} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2\theta_B + (0) - 3(0.02)] + 0 \qquad ...(2)$$

The free-body diagram of the beam at support ***B*** is shown in **Fig.*c***. Moment equilibrium requires

$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} - 8000 \text{ N}(3 \text{ m}) = 0 \qquad ...(3)$$

$$\Rightarrow M_{BA} = 24000 \text{ N. m}$$

Substituting in **Eq. (2)** yields



(c)

$$1(10^6)\theta_B - 30(10^3) = 24(10^3)$$

$$\Rightarrow \theta_B = 0.054 \text{ rad}$$

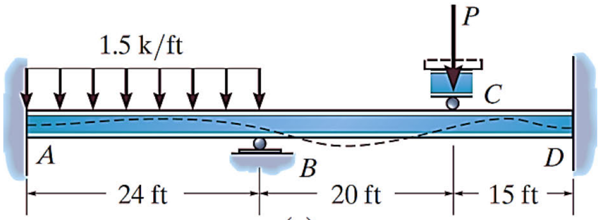
Thus, from **Eqs. (1)** and **(2),**

$$M_{AB} = -3.00 \text{ kN.m}$$

$$M_{BA} = 24.0 \text{ kN.m}$$

EXAMPLE 8.2.4

Determine the internal moments at the supports of the beam shown in **Fig.*a***. The roller support at *C* is pushed downward **0.1 ft** by the force **P**. Take ***E* = 29(10^3) ksi, *I* = 1500 in^4.**



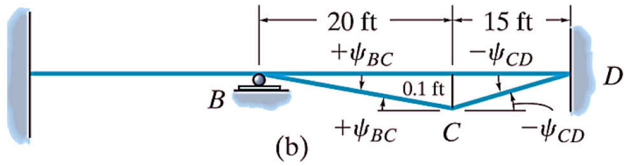
(a)

Solution

For member BC

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{1(1.5)(24)^2}{12} = -72.0 \text{ k.ft}$$

$$(FEM)_{BA} = -\frac{wL^2}{12} = \frac{1(1.5)(24)^2}{12} = 72.0 \text{ k.ft}$$



(b)

As shown in **Fig. *b***, the displacement (or settlement) of the support *C* causes ψ_{BC} to be ***positive***, since the cord for span ***BC*** rotates ***clockwise***, and ψ_{CD} to be ***negative***, since the cord for span ***CD*** rotates ***counterclockwise***. Hence,

$$\psi_{BC} = \frac{0.1 \text{ ft}}{20 \text{ ft}} = 0.005 \text{ rad} \quad , \quad \psi_{CD} = -\frac{0.1 \text{ ft}}{15 \text{ ft}} = -0.00667 \text{ rad}$$

Also, expressing the units for the stiffness in feet, we have

$$k_{AB} = \frac{1500}{24(12)^4} = 0.003014 \text{ ft}^3, \quad k_{BC} = \frac{1500}{20(12)^4} = 0.003617 \text{ ft}^3, \quad k_{CD} = \frac{1500}{15(12)^4} = 0.004823 \text{ ft}^3$$

Noting that $\theta_A = \theta_D = 0$ since *A* and *D* are fixed supports, and applying the slope-deflection twice to each span, we have

For span *AB*:

$$M_{AB} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2(0) + \theta_B - 3(0)\right] - 72 = 25173.6\theta_B - 72 \quad ...(1)$$

$$M_{BA} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2\theta_B + 0 - 3(0)\right] + 72 = 50347.2\theta_B + 72 \quad ...(2)$$

For span *BC*:

$$M_{BC} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2(\theta_B) + \theta_C - 3(0.005)\right] + 0$$

$$M_{BC} = 60416.7\theta_B + 30208.3\theta_C - 453.1 \quad ...(3)$$

$$M_{CB} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2(\theta_C) + \theta_B - 3(0.005)\right] + 0$$

$$M_{CB} = 60416.7\theta_C + 30208.3\theta_B - 453.1 \quad ...(4)$$

For span *CD*:

$$M_{CD} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2(\theta_C) + 0 - 3(-0.00667)\right] + 0 = 80555.6\theta_C + 805.6 \quad ...(5)$$

$$M_{DC} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2(0) + \theta_C - 3(-0.00667)\right] + 0 = 40277.8\theta_C + 805.6 \quad ...(6)$$

Equilibrium Equations. These six equations contain eight unknowns. Writing the moment equilibrium equations for the supports at ***B*** and ***C***, **Fig. *c***, we have

$+\circlearrowleft \sum M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad ...(7)$

$+\circlearrowleft \sum M_C = 0; \qquad M_{CB} + M_{CD} = 0 \qquad ...(8)$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (7)**, and **Eqs. (4)** and **(5)** into **Eq. (8)**.This yields

$\theta_C + 3.667\theta_B = 0.01262 \qquad ...(9)$

$-\theta_C - 0.214\theta_B = 0.00250 \qquad ...(10)$

solve, **Eqs. (9)** and **(10)** yields,

$$\boldsymbol{\theta_B = 0.00438\ \text{rad} \quad , \quad \theta_C = -0.00344\ \text{rad}}$$

The negative value for θ_C indicates ***counterclockwise*** rotation of the tangent at ***C***, **Fig. *a***. Substituting these values into **Eqs. (1)–(6)** yields

$\boldsymbol{M_{AB} = 38.2}$ **k.ft**

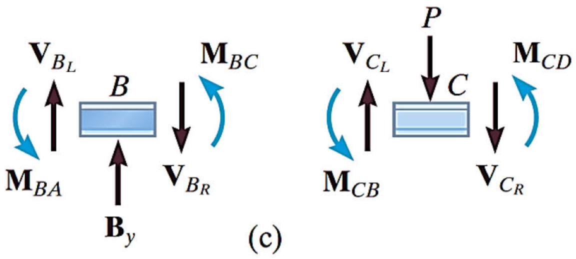
$\boldsymbol{M_{BA} = 292}$ **k.ft**

$\boldsymbol{M_{BC} = -292}$ **k.ft**

$\boldsymbol{M_{CB} = -529}$ **k.ft**

$\boldsymbol{M_{CD} = 529}$ **k.ft**

$\boldsymbol{M_{DC} = 667}$ **k.ft**



(c)

Apply these end moments to spans ***BC*** and ***CD*** and show that

V_{CL} **= 41.05 k**

V_{CR} **= -79.73 k**

and the force on the roller is

P **= 121 k .**