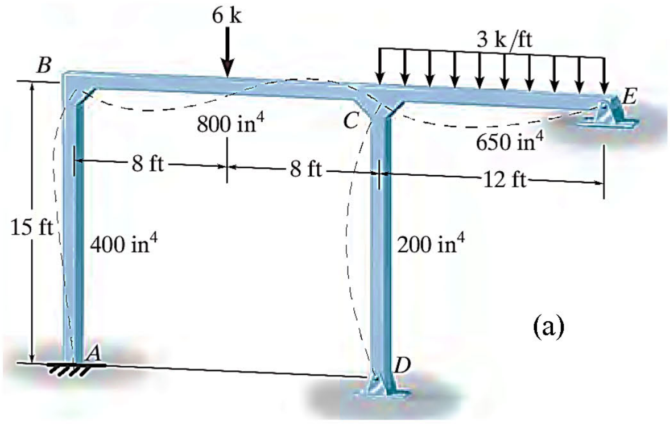
EXAMPLE 8.3.2

Determine the internal moments at each joint of the frame shown in **Fig. *a***. The moment of inertia ***I*** for each member is given in the figure. Take $E = 29(10^3)$ **ksi**.



(a)

Solution

$\theta_A = 0$

$\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$, since no sidesway will occur.

The member stiffness's,

$$k_{AB} = \frac{400}{15(12)^4} = 0.001286 \text{ ft}^3 \quad , \quad k_{BC} = \frac{800}{16(12)^4} = 0.002411 \text{ ft}^3$$

$$k_{CD} = \frac{200}{15(12)^4} = 0.000643 \text{ ft}^3 \quad , \quad k_{CE} = \frac{650}{12(12)^4} = 0.002612 \text{ ft}^3$$

The *FEMs* due to the loadings are

$$(FEM)_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12 \text{ k.ft} \qquad (FEM)_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12 \text{ k.ft}$$

$$(FEM)_{CE} = -\frac{wL^2}{12} = -\frac{3(12)^2}{12} = -36 \text{ k.ft} \qquad (FEM)_{EC} = \frac{wL^2}{12} = \frac{3(12)^2}{12} = 36 \text{ k.ft}$$

For member AB

$$M_{AB} = 2[29(10^3)(12)^2](0.001286)[2(0) + \theta_B - 3(0)] + 0 = 10740.70\ \theta_B \qquad ...(1)$$

$$M_{BA} = 2[29(10^3)(12)^2](0.001286)[2(\theta_B) + 0 - 3(0)] + 0 = 12481.50\ \theta_B \qquad ...(2)$$

For member BC

$$M_{BC} = 2[29(10^3)(12)^2](0.002411)[2(\theta_B) + \theta_C - 3(0)] - 12 = 40277.8\ \theta_B + 20138.9\ \theta_C - 12 \qquad ...(3)$$

$$M_{CB} = 2[29(10^3)(12)^2](0.002411)[2(\theta_C) + \theta_B - 3(0)] - 12 = 20138.9\ \theta_B + 40277.8\ \theta_C + 12 \qquad ...(4)$$

For member CD

$$M_{CD} = 2[29(10^3)(12)^2](0.000643)[2(\theta_C) + \theta_D - 3(0)] + 0 = 10740.74\ \theta_C + 5370.37\ \theta_D \qquad ...(5)$$

$$M_{DC} = 2[29(10^3)(12)^2](0.000643)[2(\theta_D) + \theta_C - 3(0)] + 0 = 5370.37\ \theta_C + 10740.74\ \theta_D \qquad ...(6)$$

For member CE

$$M_{CE} = 2[29(10^3)(12)^2](0.02612)[2(\theta_C) + \theta_E - 3(0)] - 36 = 43634.26\ \theta_C + 21817.13\ \theta_E - 36 \qquad ...(7)$$

$$M_{EC} = 2[29(10^3)(12)^2](0.02612)[2(\theta_E) + \theta_C - 3(0)] - 36 = 21817.13\theta_C + 43634.26\theta_E + 36 \qquad ...(8)$$

$\because M_{DC} = 0,$

From **Eqs (5)**, and **(6)** ***eliminates*** the unknown and θ_D

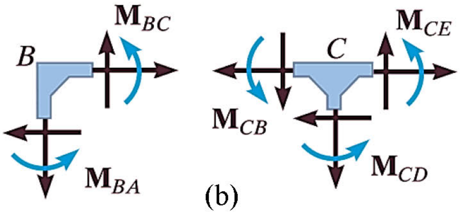
$$M_{CD} = 8055.6\ \theta_D \qquad ...(9)$$

$\because M_{EC} = 0,$

From **Eqs (7)**, and **(8)** ***eliminates*** the unknown and θ_E

$$M_{CE} = 32725.7\ \theta_C - 54 \qquad ...(10)$$

Equations of Equilibrium. These ***six*** equations contain ***eight*** unknowns. ***Two moment equilibrium*** equations can be written for joints ***B*** and ***C***, **Fig. *b***. We have



(b)

$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(11)$$

$$+\circlearrowleft \sum M_C = 0 \qquad M_{CB} + M_{CD} + M_{CE} = 0 \qquad ...(12)$$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (11)**, and **Eqs. (4),(9)** and **(10)** into **Eq. (12)**.This gives

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$

$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

Solving these equations simultaneously yields

$$\boldsymbol{\theta_B = 2.758(10^{-5})\ \text{rad} \qquad \theta_C = 5.113(10^{-4})\ \text{rad}}$$

These values, being ***clockwise***, tend to distort the frame as shown in **Fig. *a***. Substituting these values into **Eqs. (1), (2), (3), (4), (9),** and **(10)** and solving,

$$\boldsymbol{M_{AB} = 0.296\ \text{k.ft}}$$

$$\boldsymbol{M_{BA} = 0.592\ \text{k.ft}}$$

$$\boldsymbol{M_{BC} = -0.592\ \text{k.ft}}$$

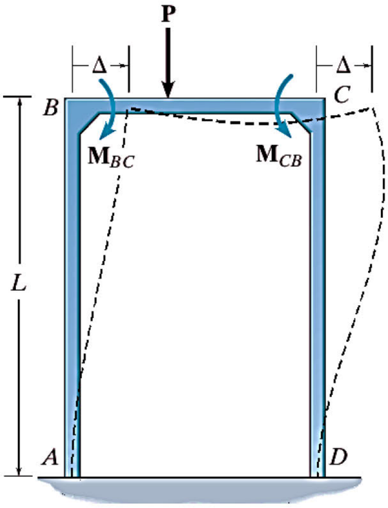
$$\boldsymbol{M_{CB} = 33.1\ \text{k.ft}}$$

$$\boldsymbol{M_{CD} = 4.12\ \text{k.ft}}$$

$$\boldsymbol{M_{CE} = -37.3\ \text{k.ft}}$$

8.4 Analysis of Frames: Sidesway

A frame will sidesway, or be displaced to the side, when it or the loading acting on it is nonsymmetric. When applying the slope-deflection equation to each column of this frame, we must therefore consider the column rotation ψ (since $\psi = \Delta/L$) as unknown in the equation. As a result an extra equilibrium equation must be included for the solution. In the previous sections it was shown that unknown *angular displacements* $\boldsymbol{\theta}$ were related by joint *moment equilibrium equations*. In a similar manner, when unknown joint *linear displacements* Δ (or span rotations ψ) occur, we must write *force equilibrium equations* in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal *moments* acting at the ends of the columns, since the slope-deflection equations involve these moments.

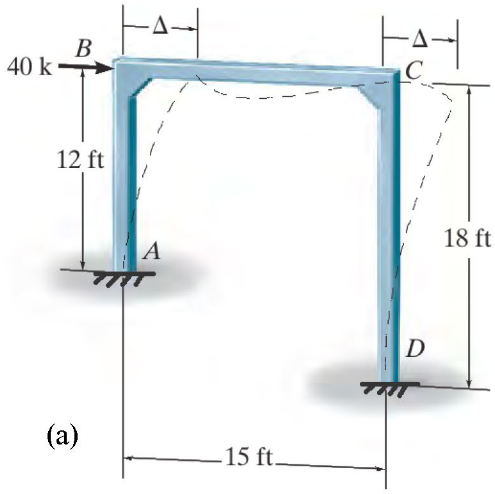


EXAMPLE 8.4.1

Determine the moments at each joint of the frame shown in **Fig. *a***. ***EI*** is constant.

Solution

Both joints ***B*** and ***C*** are assumed to be displaced an ***equal amount*** **△.** Consequently,



(a)

$\psi_{AB} = \Delta/12$ and $\psi_{DC} = \Delta/18$

Both terms are ***positive*** since the cords of members ***AB*** and ***CD*** "***rotate***" ***clockwise.***

Relating ψ_{AB} to ψ_{DC}

$$\psi_{AB} = (18/12)\,\psi_{DC}$$

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0)+\theta_B-3\left(\frac{18}{12}\psi_{DC}\right)\right]+0 = EI\left(0.1667\theta_B-0.75\psi_{DC}\right) \quad ...(1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2(\theta_B)+0-3\left(\frac{18}{12}\psi_{DC}\right)\right]+0 = EI\left(0.333\theta_B-0.75\psi_{DC}\right) \quad ...(2)$$

$$M_{BC} = 2E\left(\frac{I}{15}\right)\left[2(\theta_B)+\theta_C-3(0)\right]+0 = EI\left(0.267\theta_B+0.133\theta_C\right) \quad ...(3)$$

$$M_{CB} = 2E\left(\frac{I}{15}\right)\left[2(\theta_C)+\theta_B-3(0)\right]+0 = EI\left(0.267\theta_C+0.133\theta_B\right) \quad ...(4)$$

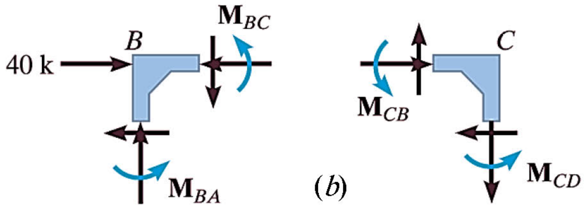
$$M_{CD} = 2E\left(\frac{I}{18}\right)\left[2(\theta_C)+0-3\psi_{DC}\right]+0 = EI\left(0.222\theta_C-0.333\psi_{DC}\right) \quad ...(5)$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)\left[2(0)+\theta_C-3\psi_{DC}\right]+0 = EI\left(0.111\theta_C-0.333\psi_{DC}\right) \quad ...(6)$$

Equations of Equilibrium. The ***six*** equations contain ***nine*** unknowns. ***Two*** moment equilibrium equations for joints ***B*** and ***C***, **Fig.*b***, can be written, namely,

$$M_{BA} + M_{BC} = 0 \qquad ...(7)$$

$$M_{CB} + M_{CD} = 0 \qquad ...(8)$$



(*b*)

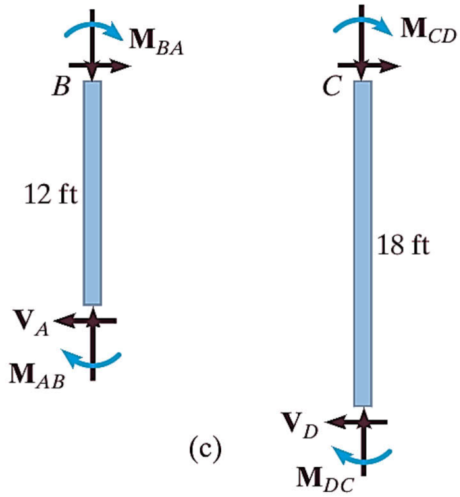
Since a horizontal displacement △ occurs, we will consider summing forces on the ***entire frame*** in the ***x*** direction. This yields

$$+\rightarrow \sum F_x = 0; \qquad 40 - V_A - V_D = 0$$

The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, **Fig. *c***. We have

$$\sum M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\sum M_C = 0; \qquad V_A = -\frac{M_{DC} + M_{CD}}{18}$$



(c)

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0 \qquad ...(9)$$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (7), Eqs. (4)** and **(5)** into **Eq. (8)**, and **Eqs. (1)**, **(2), (5), (6)** into **Eq. (9)**. This yields

$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$

$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$

$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

Solving simultaneously, we have

$$EI\theta_B = 438.81 \qquad EI\theta_C = 136.18 \qquad EI\psi_{DC} = 375.26$$

Finally, using these results and solving **Eqs. (1)–(6)** yields

$M_{AB} = \text{-208 k.ft}$

$M_{BA} = \text{-135 k.ft}$

$M_{BC} = \text{135 k.ft}$

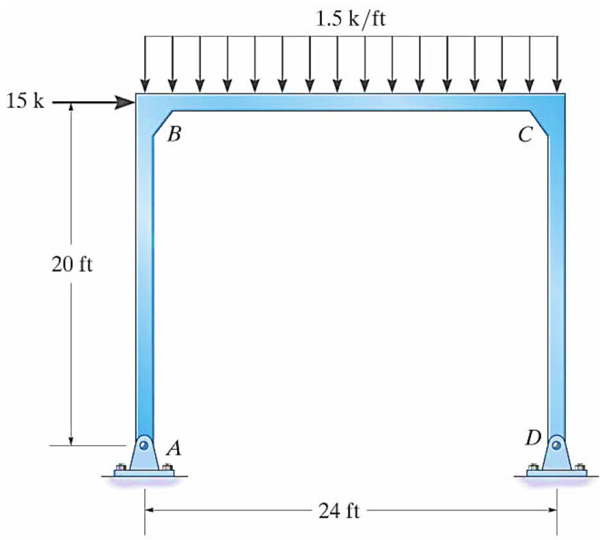
$M_{CB} = \text{94.8 k.ft}$

$M_{CD} = \text{-94.8 k.ft}$

$M_{DC} = \text{-110 k.ft}$

EXAMPLE 8.4.2

Determine the rotation and the horizontal displacement at joints ***B*** and ***C*** of the frame shown in **Fig. *a***. ***EI*** is constant.



Solution

For member AB

$$M_{AB} = 0$$

$$M_{BA} = \frac{3EI}{20}\left[\theta_B - \frac{3\Delta}{20}\right] + 0 \qquad ...(1)$$

For member BC

$$M_{BC} = \frac{2EI}{24}\left[2\theta_B + \theta_C - 0\right] - \frac{wL^2}{12} = \frac{EI}{12}\cdot(2\theta_B + \theta_C) - 72 \qquad ...(2)$$

$$M_{CB} = \frac{2EI}{24}\left[\theta_B + 2\theta_C - 0\right] + \frac{wL^2}{12} = \frac{EI}{12}\cdot(\theta_B + 2\theta_C) + 72 \qquad ...(3)$$

For member CD

$$M_{CD} = \frac{3EI}{20}\left[\theta_C - \left(\frac{3\Delta}{20}\right)\right] + 0 \qquad ...(4)$$

$$M_{DC} = 0$$

Equations of Equilibrium. These ***four*** equations contain ***seven*** unknowns. ***Two moment equilibrium*** equations can be written for joints ***B*** and ***C***,

$$\because M_{BA} + M_{BC} = 0$$

$$\frac{3EI}{20}\left[\theta_B - \frac{3\Delta}{20}\right] + \frac{EI}{12}\cdot(2\theta_B + \theta_C) - 72 = 0$$

$$0.3166\theta_B + 0.0833\theta_C - 0.0225\Delta = \frac{72}{EI} \qquad ...(5)$$

$$\because M_{CB} + M_{CD} = 0$$

$$\frac{EI}{12}\cdot(\theta_B + 2\theta_C) + 72 + \frac{3EI}{20}\left[\theta_C - \left(\frac{3\Delta}{20}\right)\right] = 0$$

$$0.0833\theta_B + 0.3166\theta_C - 0.0225\Delta = -\frac{72}{EI} \qquad ...(6)$$

$$\because 15 - V_A - V_D = 0$$

$$\sum M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\sum M_C = 0; \qquad V_A = -\frac{M_{DC} + M_{CD}}{18}$$

Thus,

$$15 + \frac{M_{AB} + M_{BA}}{20} + \frac{M_{DC} + M_{CD}}{20} = 0$$

$$0.15\theta_B + 0.15\theta_C - 0.045 = -\frac{18}{EI} \qquad ...(7)$$

Solve **Eq. (5), (6),** and **(7)**

$$\boldsymbol{EI\,\theta_B = 338.63 \qquad EI\,\theta_C = -278.60 \qquad EI\,\Delta = 533.41}$$