# 6 DEFLECTIONS USING ENERGY METHODS

## 6.1 Method of Virtual Work: Beams and Frames

The method of virtual work can be applied to deflection problems involving beams and frames.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in **Fig.6-1*b***. Here the displacement Δ of point *A* is to be determined.

To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at *A*, and the ***internal virtual moment*** **m** is determined by the method of sections at an arbitrary location ***x*** from the left support, **Fig. 6-1*a***.When the real loads act on the beam, **Fig. 6–1*b***, point *A* is displaced **Δ.**

Provided these loads cause ***linear elastic material response,*** then from the equation below , the element ***dx*** deforms or rotates

$$d\theta = \left(\frac{M}{EI}\right) dx$$

Here ***M*** is the internal moment at ***x*** caused by the real loads. Consequently, the ***external virtual work*** done by the unit load is **1 .** Δ, and the ***internal virtual work*** done by the moment **m** is

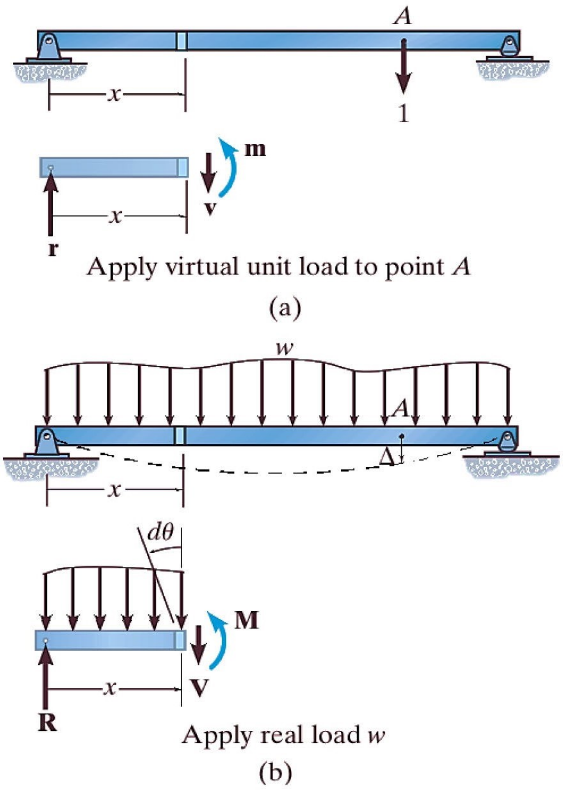



**Fig. 6-1**

$$md\theta = m\left(\frac{M}{EI}\right) dx$$

Summing the effects on all the elements ***dx*** along the beam requires an integration,

$$1 \,.\, \Delta = \int_0^L \frac{mM}{EI}\, dx \qquad \text{...(6-1)}$$

where

- **1** = external virtual unit load acting on the beam or frame in the direction of Δ
- ***m*** = internal virtual moment in the beam or frame, expressed as a function of ***x*** and caused by the external virtual unit load.
- Δ = external displacement of the point caused by the real loads acting on the beam or frame.
- ***M*** = internal moment in the beam or frame, expressed as a function of ***x*** and caused by the real loads.
- ***E*** = modulus of elasticity of the material.
- ***I*** = moment of inertia of cross-sectional area, computed about the neutral axis.

In a similar manner, if the tangent rotation or slope angle at a point $A$ on the beam's elastic curve is to be determined, **Fig. 6-2,** a ***unit couple moment*** is first applied at the point, and the corresponding internal moments $m_\theta$ have to be determined. Since the work of the unit couple is $1 \cdot \theta$, then

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx \qquad ...(6\text{-}2)$$

When applying **Eqs. 6-1** and **6-2**, it is important to realize that the definite integrals on the right side actually represent the amount of virtual strain energy that is ***stored*** in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length. Instead, separate $x$ coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each $x$ have the ***same origin***; however, the $x$ selected for determining the real moment $M$ in a particular region must be the ***same x*** as that selected for determining the virtual moment $m$ **or** $m_\theta$ within the same region.

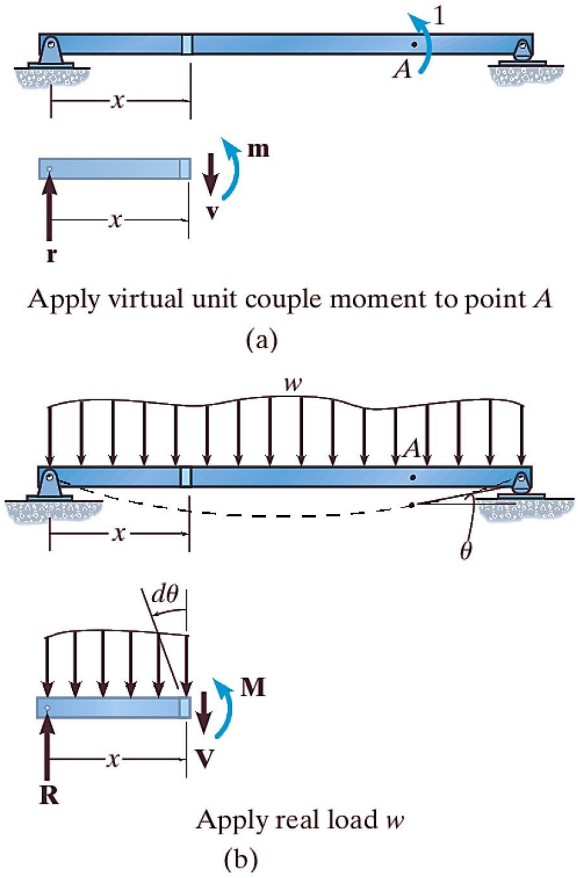
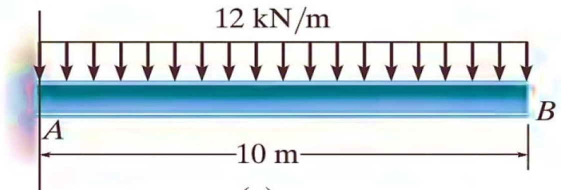



**Fig. 6-2**

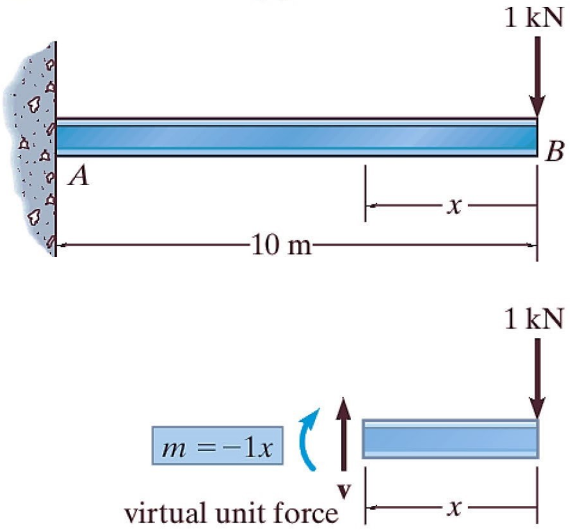
**EXAMPLE 6.1.1**
Determine the displacement of point $B$ of the steel beam shown in **Fig. *a*.** Take **E = 200 GPa, I = 500($10^6$) mm$^4$.**




(a)

**Solution**
**Virtual Moment m.**
The vertical displacement of point $B$ is obtained by placing a virtual unit load of **1 kN** at ***B*, Fig.*b***. By inspection there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a ***single x*** coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at $B$, since then the reactions at $A$ do not have to be determined in order to find the internal moments $m$ and $M$. Using the method of sections, the internal moment $m$ is formulated as shown in **Fig.*b*.**

**Real Moment M**. Using the ***same x*** coordinate, the internal moment $M$ is formulated as shown in **Fig.*c*.**

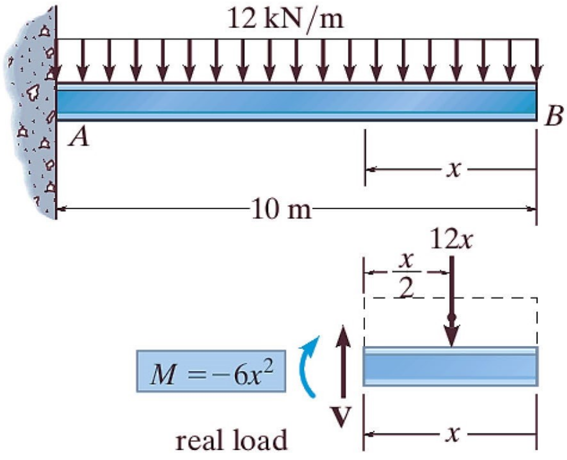



(b)

**Virtual-Work Equation**. The vertical displacement of ***B*** is thus,

$$1\text{kN}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x)(-6x^2)}{EI}dx$$

$$1\text{kN}.\Delta_B = \frac{15(10^3)\text{kN}^2.m^3}{EI}$$



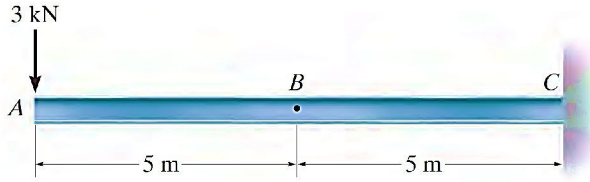
(c)

$$\Delta_B = \frac{15(10^3)\text{kN.m}^3}{200(10^6)\text{kN/m}^2(500(10^6)\text{mm}^4)(10^{-12}\text{m}^4/\text{mm}^4}$$

$= 0.150$ m $= 150$ mm

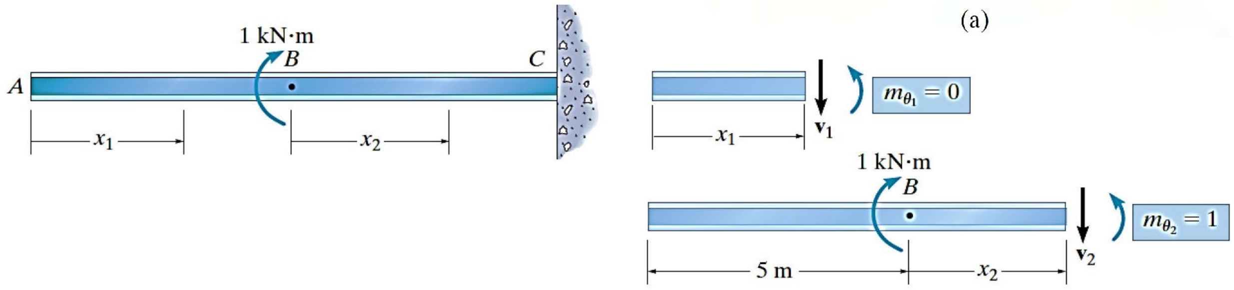
**EXAMPLE 6.1.2**

Determine the slope ***θ*** at point ***B*** of the steel beam shown in **Fig. *a***. Take **E = 200 GPa, I = 60($10^6$) mm$^4$**

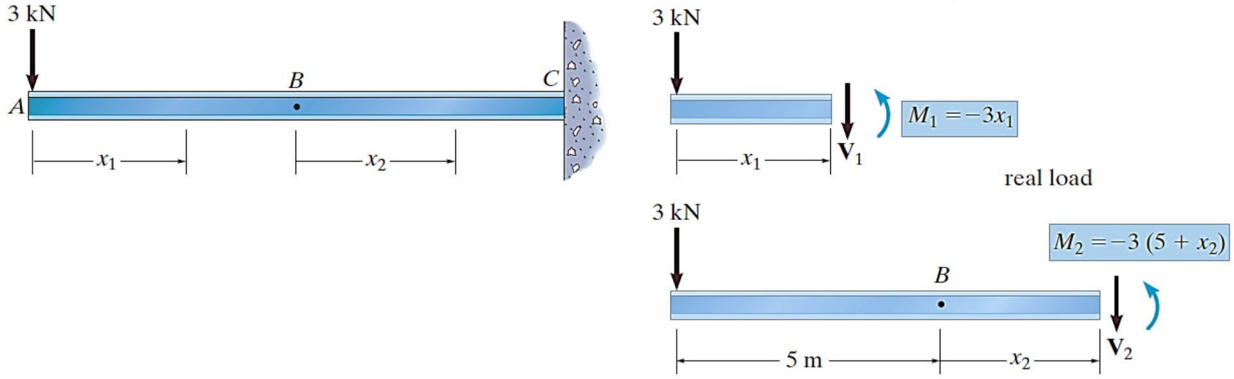


(a)

**Solution**



virtual unit couple
(b)



(c)

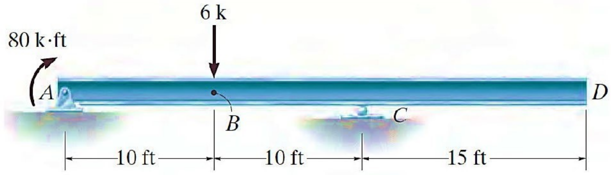
$$(1\ \text{kN.m}).\theta_B = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^5 \frac{(0)(-3x_1)}{EI}dx_1 + \int_0^5 \frac{(1)[-3(5+x_2)]}{EI}dx_2$$

$$\theta_B = \frac{-112.5\text{kN.m}^2}{EI} = -0.00938 \text{ rad}$$

**Note:** The *negative sign* indicates is ***θ_B opposite*** to the direction of the virtual couple moment shown in **Fig. *b***

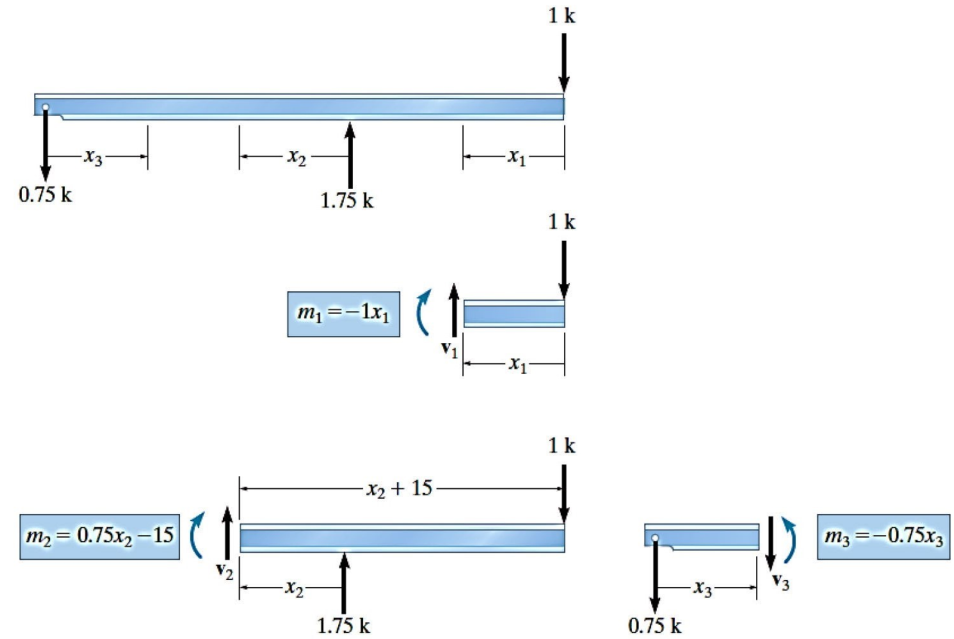
## EXAMPLE 6.1.3

Determine the displacement at ***D*** of the steel beam in **Fig.*a***. Take **E = 29($10^3$) ksi, I = 800 in$^4$**.



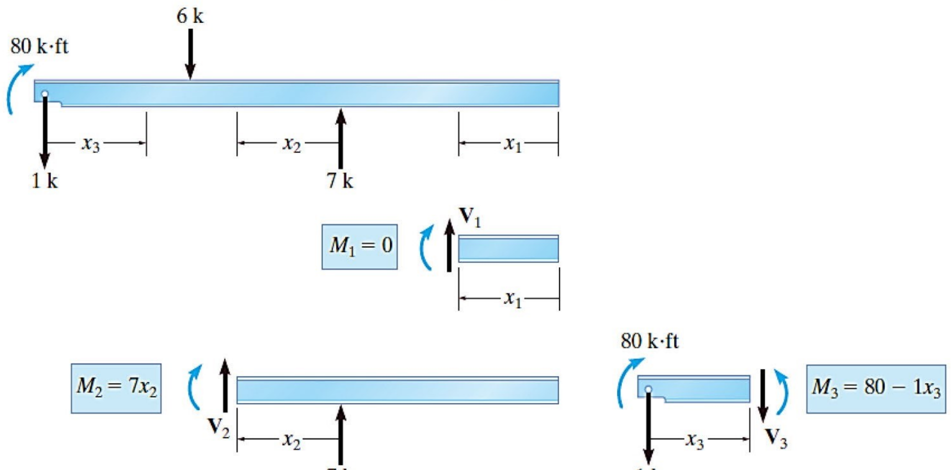

(a)

**Solution**




virtual loads

(b)




real loads

(c)

$$1\text{kN}.\Delta_D = \int_0^L \frac{mM}{EI}dx = \int_0^{15} \frac{(-1x_1)(0)}{EI}dx_1 + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)}{EI}dx_2 + \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)}{EI}dx_3$$

$$\Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250\ \text{k.ft}^3}{EI}$$

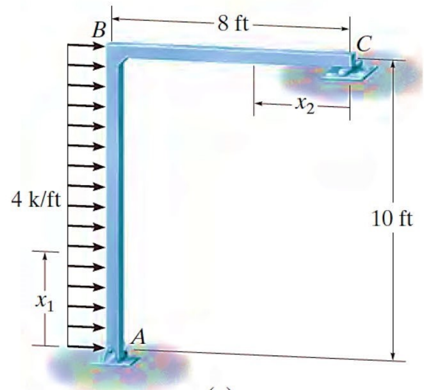
$$\Delta_D = \frac{-6250\ \text{k.ft}^3(12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2(800\ \text{in}^4)} = -0.466\ \text{in}$$

**Note:** The negative sign indicates the displacement is upward, opposite to the downward unit load, **Fig. *b***. Also note that $m_1$ did not actually have to be calculated since $M_1 = 0$.

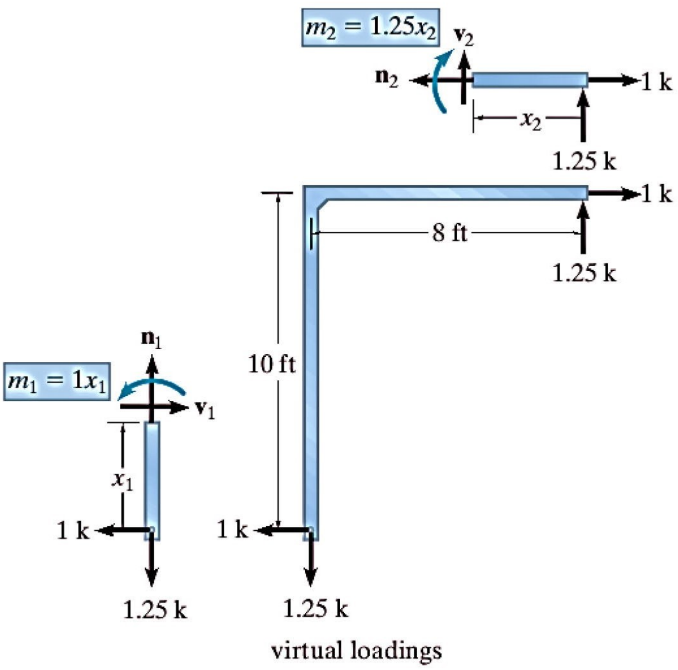
**EXAMPLE 6.1.4**

Determine the horizontal displacement of point $C$ on the frame shown in **Fig.** ***a***. Take $\mathbf{E = 29(10^3)}$ **ksi**, $\mathbf{I = 600}$ $\mathbf{in^4}$ for both members.

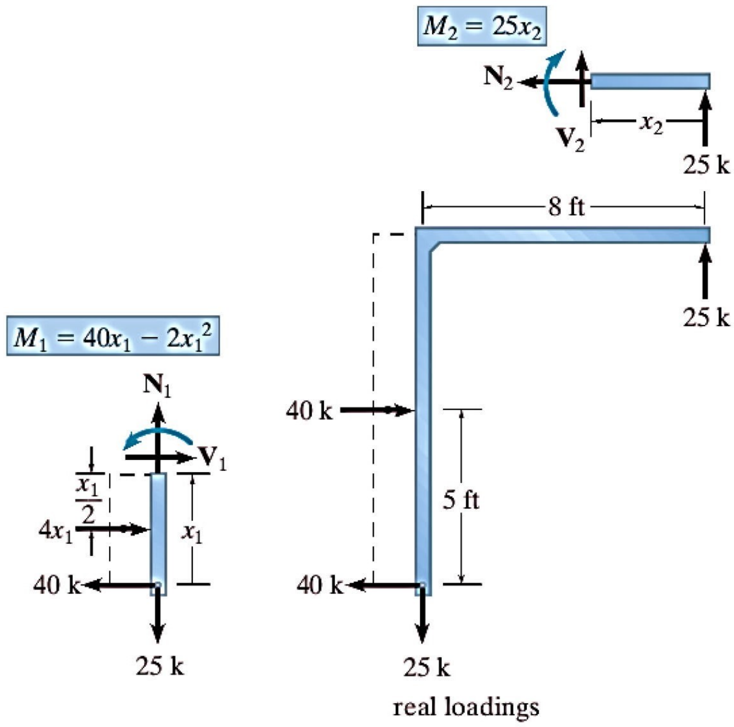
**Solution**




(a)




(b)




(c)

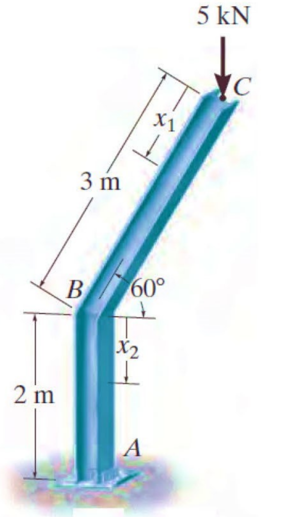
$$1\text{kN}.\Delta_{C_h} = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x_1)(40x_1 - 2x_1^2)}{EI}dx_1 + \int_0^8 \frac{(1.25x_2)(25x_2)}{EI}dx_2$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13666.7\ \text{k.ft}^3}{EI}$$

$$\Delta_{C_h} = \frac{13666.7\ \text{k.ft}^3(12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2(600\ \text{in}^4)} = 1.357\ \text{in}$$

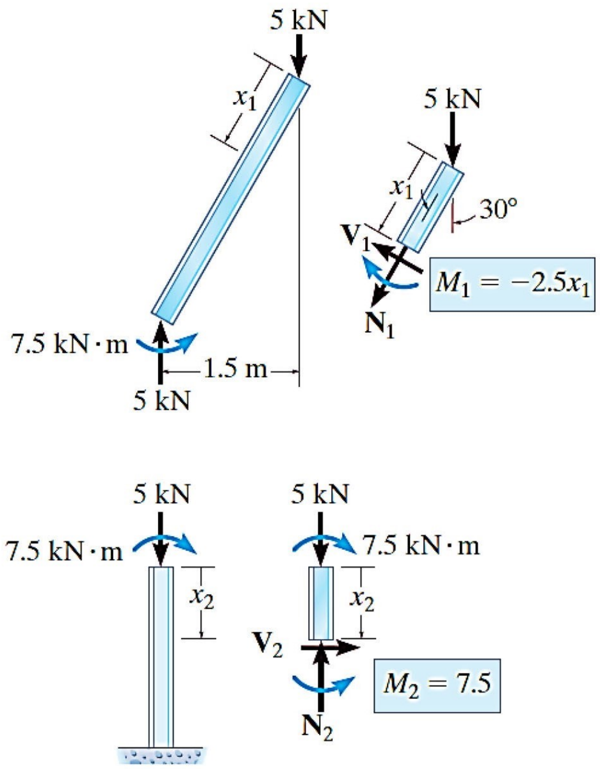
**EXAMPLE 6.1.5**

Determine the tangential rotation at point $C$ of the frame shown in **Fig. *a***. Take $E$ **= 200 GPa,** $I$ **= (15)10⁶ mm⁴**.



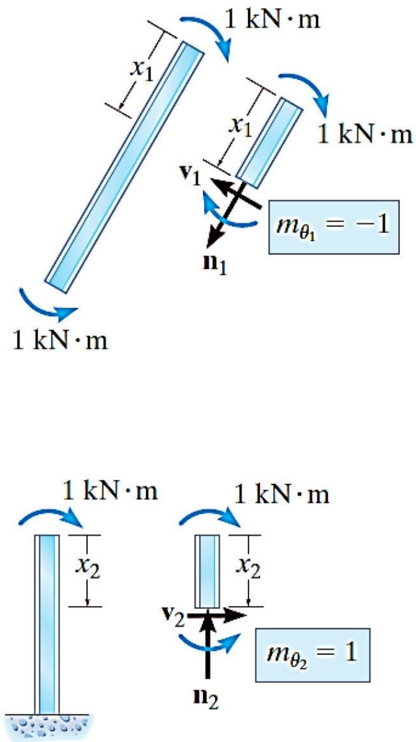

(a)

**Solution**




real loads

(b)




virtual loads

(c)

$$(1\ \text{kN.m}).\theta_C = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^3 \frac{(-1)(-2.5x_1)}{EI}\,dx_1 + \int_0^2 \frac{(1)(7.5)}{EI}\,dx_2$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25\,\text{kN.m}^2}{EI}$$

$$\theta_C = \frac{26.25\,\text{kN.m}^2}{200(10^6)\text{kN/m}^2\left[15\left(10^6\right)\ \text{mm}^4\right]\left(10^{-12}\ \text{m}^4/\text{mm}^4\right)} = 0.00875\ \text{rad}$$