Third Isomorphism Theorem

Lecture 11

In the following lecture we study the third important theorem of fundamental isomorphism theorem.

Theorem (1): (Third Isomorphism Theorem)

If H and K are normal subgroups of a group G with $K \leq H$, then H/K is normal in G/K and

$(G/K)/(H/K) \cong G/H$.

Proof:

Define f: G/K□ G/H by f(a K) = a H. Note that f is a (well- defined function, for if a G = b K, then a-1b□ K But K□ H, thus a-1b□ H, and so aH= b H , and we are done.

It is easy to see that f is an epimorphism.

Now ker f = H/K. Also clearly H/K is a normal subgroup of G/K. Since f is monomorphism, so by the first isomorphism theorem we have: (G/K)/(H/K) □ G/H

The third isomorphism theorem is easy to remember: the K’s in the fraction (G/K)/(H/K) can be canceled. One can better appreciate the first isomorphism theorem after having proved the third one. The quotient group (G/K)/(H/K) consists of cosets (of H/K) whose representatives are themselves cosets (of G/K).

Here is another construction of a new group from two given groups.

Definition (2): If H and K are groups, then their **direct product**, denoted by HxK, is the set of all ordered pairs (h, k) equipped with the following operation:

$(h, k)(h1, k1) = (hh1, kk1)$

It is routine to check that HXK is a group [the identity element is $(e, e1)$ and $(h, k)-1 = (h-1, k-1)$.

Remark (3): let G and h be groups.Then HxK is abelian if and only if both H and K are abelian.

We end the tenth lecture by the following example.

Example: $Z \times 2Z$ is the direct product between $(Z, +)$ and $(2Z, +)$ groups.

The identity element is $(0, 0)$, and the inverse element of (a, b) is $(-a, -b)$.