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College of Education for Pure Sciences

Department of Mathematics

Lecture Note On Mathematical Statistics 1

B.Sc. in Mathematics

Fourth Stage

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Transformation Methods of Two Dimensional

المحاضرة الرابعة

الكورس الاول

When two random variables are involved, many interesting problems can result. In the case of a single-valued inverse, the rule is about the same as that in the one-variable case, with the derivative being replaced by the Jacobian. That is, if X_1 and X_2 are two continuous-type random variables with joint pdf $f(x_1, x_2)$, and if $Y_1 = u_1(X_1, X_2)$, $Y_2 = u_2(X_1, X_2)$ has the single-valued inverse $X_1 = v_1(Y_1, Y_2)$, $X_2 = v_2(Y_1, Y_2)$, then the joint pdf of Y_1 and Y_2 is

$$g(y_1, y_2) = |J| f\left[v_1(y_1, y_2), v_2(y_1, y_2)\right], \qquad (y_1, y_2) \in S_Y,$$

where the Jacobian J is the determinant

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}.$$

Of course, we find the support S_Y of Y_1, Y_2 by considering the mapping of the support S_X of X_1, X_2 under the transformation $y_1 = u_1(x_1, x_2)$, $y_2 = u_2(x_1, x_2)$. This method of finding the distribution of Y_1 and Y_2 is called the **change-of-variables technique.**

It is often the mapping of the support S_X of X_1, X_2 into that (say, S_Y) of Y_1, Y_2 which causes the biggest challenge. That is, in most cases, it is easy to solve for x_1 and x_2 in terms of y_1 and y_2, say,

$$x_1 = v_1(y_1, y_2), \qquad x_2 = v_2(y_1, y_2),$$

and then to compute the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial v_1(y_1, y_2)}{\partial y_1} & \dfrac{\partial v_1(y_1, y_2)}{\partial y_2} \\ \dfrac{\partial v_2(y_1, y_2)}{\partial y_1} & \dfrac{\partial v_2(y_1, y_2)}{\partial y_2} \end{vmatrix}.$$

However, the mapping of $(x_1, x_2) \in S_X$ into $(y_1, y_2) \in S_Y$ can be more difficult. Let us consider two simple examples.

Let X_1, X_2 have the joint pdf

$$f(x_1, x_2) = 2, \qquad 0 < x_1 < x_2 < 1.$$

Consider the transformation

$$Y_1 = \frac{X_1}{X_2}, \qquad Y_2 = X_2.$$

It is certainly easy enough to solve for x_1 and x_2, namely,

$$x_1 = y_1 y_2, \qquad x_2 = y_2,$$

and compute

$$J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

Let X_1 and X_2 be independent random variables, each with pdf

$$f(x) = e^{-x}, \qquad 0 < x < \infty.$$

Hence, their joint pdf is

$$f(x_1)f(x_2) = e^{-x_1-x_2}, \qquad 0 < x_1 < \infty, \ 0 < x_2 < \infty.$$

Let us consider

$$Y_1 = X_1 - X_2, \qquad Y_2 = X_1 + X_2.$$

Thus,

$$x_1 = \frac{y_1 + y_2}{2}, \qquad x_2 = \frac{y_2 - y_1}{2},$$

with

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

The region S_X is depicted . The line segments on the boundary, namely, $x_1 = 0$, $0 < x_2 < \infty$, and $x_2 = 0$, $0 < x_1 < \infty$, map into the line segments $y_1 + y_2 = 0$, $y_2 > y_1$ and $y_1 = y_2$, $y_2 > -y_1$, respectively. These are shown in Figure 5.2-2(b), and the support of S_Y is depicted there. Since the region S_Y is not bounded by horizontal and vertical line segments, Y_1 and Y_2 are dependent.

The joint pdf of Y_1 and Y_2 is

$$g(y_1, y_2) = \frac{1}{2}e^{-y_2}, \qquad -y_2 < y_1 < y_2, \ 0 < y_2 < \infty.$$

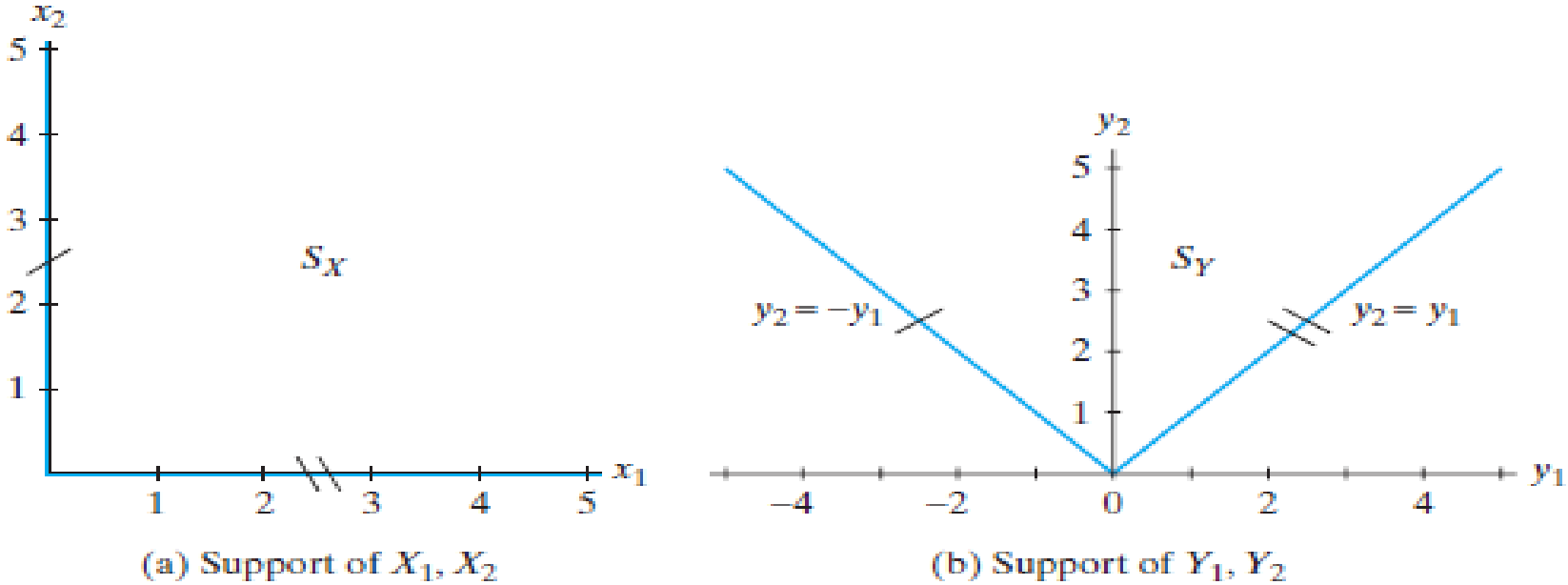


Figure Mapping from x_1, x_2 to y_1, y_2

The probability $P(Y_1 \geq 0, Y_2 \leq 4)$ is given by

$$\int_0^4 \int_{y_1}^4 \frac{1}{2} e^{-y_2}\, dy_2\, dy_1 \qquad \text{or} \qquad \int_0^4 \int_0^{y_2} \frac{1}{2} e^{-y_2}\, dy_1\, dy_2.$$

While neither of these integrals is difficult to evaluate, we choose the latter one to obtain

$$\begin{aligned}\int_0^4 \frac{1}{2} y_2 e^{-y_2}\, dy_2 &= \left[\frac{1}{2}(-y_2)e^{-y_2} - \frac{1}{2} e^{-y_2}\right]_0^4 \\ &= \frac{1}{2} - 2e^{-4} - \frac{1}{2} e^{-4} = \frac{1}{2}\left[1 - 5e^{-4}\right].\end{aligned}$$

The marginal pdf of Y_2 is

$$g_2(y_2) = \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2}\, dy_1 = y_2 e^{-y_2}, \qquad 0 < y_2 < \infty.$$

This is a gamma pdf with shape parameter 2 and scale parameter 1. The pdf of Y_1 is

$$g_1(y_1) = \begin{cases} \displaystyle\int_{-y_1}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = \frac{1}{2} e^{y_1}, & -\infty < y_1 \le 0, \\[2ex] \displaystyle\int_{y_1}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = \frac{1}{2} e^{-y_1}, & 0 < y_1 < \infty. \end{cases}$$

That is, the expression for $g_1(y_1)$ depends on the location of y_1, although this could be written as

$$g_1(y_1) = \frac{1}{2} e^{-|y_1|}, \qquad -\infty < y_1 < \infty,$$

which is called a **double exponential** pdf, or sometimes the **Laplace** pdf. ■

Example

Let X_1 and X_2 have independent gamma distributions with parameters α, θ and β, θ, respectively. That is, the joint pdf of X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} x_1^{\alpha-1} x_2^{\beta-1} \exp\left(-\frac{x_1 + x_2}{\theta}\right), \; 0 < x_1 < \infty, \; 0 < x_2 < \infty.$$

Consider

$$Y_1 = \frac{X_1}{X_1 + X_2}, \qquad Y_2 = X_1 + X_2,$$

or, equivalently,

$$X_1 = Y_1 Y_2, \qquad X_2 = Y_2 - Y_1 Y_2.$$

The Jacobian is

$$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2(1 - y_1) + y_1 y_2 = y_2.$$

Thus, the joint pdf $g(y_1, y_2)$ of Y_1 and Y_2 is

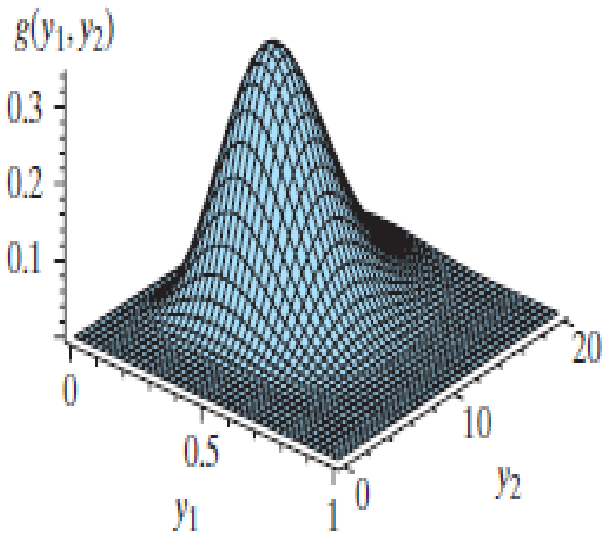
$$g(y_1, y_2) = |y_2| \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} (y_1 y_2)^{\alpha-1} (y_2 - y_1 y_2)^{\beta-1} e^{-y_2/\theta},$$

where the support is $0 < y_1 < 1$, $0 < y_2 < \infty$, which is the mapping of $0 < x_i < \infty$, $i = 1, 2$. To see the shape of this joint pdf, $z = g(y_1, y_2)$ is graphed in Figure 5.2-3(a) with $\alpha = 4$, $\beta = 7$, and $\theta = 1$ and in Figure (b) with $\alpha = 8$, $\beta = 3$, and $\theta = 1$. To find the marginal pdf of Y_1, we integrate this joint pdf on y_2. We see that the marginal pdf of Y_1 is

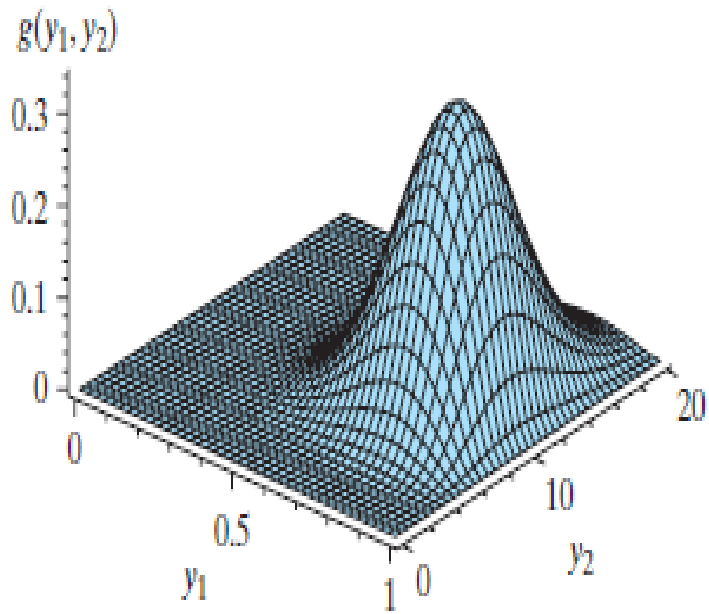
$$g_1(y_1) = \frac{y_1^{\alpha-1}(1 - y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{y_2^{\alpha+\beta-1}}{\theta^{\alpha+\beta}} e^{-y_2/\theta} \, dy_2.$$

But the integral in this expression is that of a gamma pdf with parameters $\alpha + \beta$ and θ, except for $\Gamma(\alpha + \beta)$ in the denominator; hence, the integral equals $\Gamma(\alpha + \beta)$, and we have

$$g_1(y_1) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1}(1 - y_1)^{\beta-1}, \qquad 0 < y_1 < 1.$$



(a) $\alpha = 4, \beta = 7, \theta = 1$



(b) $\alpha = 8, \beta = 3, \theta = 1$

Joint pdf of $z = g(y_1, y_2)$

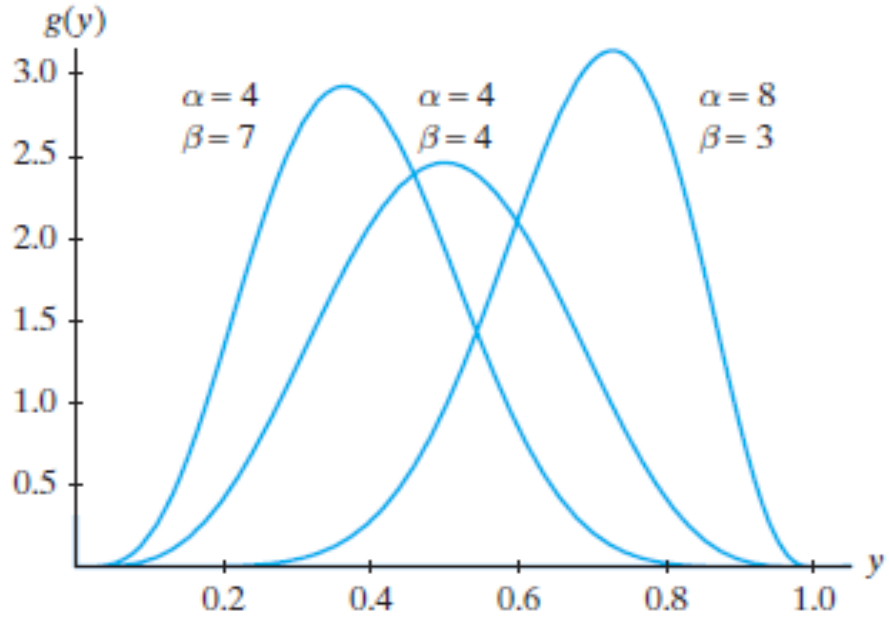


Figure Beta distribution pdfs

We say that Y_1 has a **beta pdf** with parameters α and β.

The next example illustrates the distribution function technique. You will calculate the same results in Exercise 5.2-2, but using the change-of-variable technique.

Example We let

$$F = \frac{U/r_1}{V/r_2},$$

where U and V are independent chi-square variables with r_1 and r_2 degrees of freedom, respectively. Thus, the joint pdf of U and V is

$$g(u,v) = \frac{u^{r_1/2-1}e^{-u/2}}{\Gamma(r_1/2)2^{r_1/2}}\frac{v^{r_2/2-1}e^{-v/2}}{\Gamma(r_2/2)2^{r_2/2}}, \qquad 0 < u < \infty,\ 0 < v < \infty.$$

In this derivation, we let $W = F$ to avoid using f as a symbol for a variable. The cdf $F(w) = P(W \le w)$ of W is

$$\begin{aligned} F(w) = P\left(\frac{U/r_1}{V/r_2} \le w\right) &= P\left(U \le \frac{r_1}{r_2} w V\right) \\ &= \int_0^\infty \int_0^{(r_1/r_2)wv} g(u,v)\,du\,dv. \end{aligned}$$

That is,

$$F(w) = \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)} \int_0^\infty \left[\int_0^{(r_1/r_2)wv} \frac{u^{r_1/2-1}e^{-u/2}}{2^{(r_1+r_2)/2}}\,du\right] v^{r_2/2-1}e^{-v/2}\,dv.$$

The pdf of W is the derivative of the cdf; so, applying the fundamental theorem of calculus to the inner integral, exchanging the operations of integration and differentiation (which is permissible in this case), we have

$$
\begin{aligned}
f(w) &= F'(w) \\
&= \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)} \int_0^\infty \frac{[(r_1/r_2)vw]^{r_1/2-1}}{2^{(r_1+r_2)/2}} e^{-(r_1/2r_2)(vw)} \left(\frac{r_1}{r_2} v\right) v^{r_2/2-1} e^{-v/2}\, dv \\
&= \frac{(r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)} \int_0^\infty \frac{v^{(r_1+r_2)/2-1}}{2^{(r_1+r_2)/2}} e^{-(v/2)[1+(r_1/r_2)w]}\, dv.
\end{aligned}
$$

In the integral, we make the change of variable

$$
y = \left(1 + \frac{r_1}{r_2} w\right) v, \qquad \text{so that} \qquad \frac{dv}{dy} = \frac{1}{1 + (r_1/r_2)w}.
$$

Thus, we have

$$
\begin{aligned}
f(w) &= \frac{(r_1/r_2)^{r_1/2}\Gamma[(r_1+r_2)/2] w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1 w/r_2)]^{(r_1+r_2)/2}} \int_0^\infty \frac{y^{(r_1+r_2)/2-1} e^{-y/2}}{\Gamma[(r_1+r_2)/2] 2^{(r_1+r_2)/2}}\, dy \\
&= \frac{(r_1/r_2)^{r_1/2}\Gamma[(r_1+r_2)/2] w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1 w/r_2)]^{(r_1+r_2)/2}},
\end{aligned}
$$

the pdf of the $W = F$ **distribution** with r_1 and r_2 degrees of freedom. Note that the integral in this last expression for $f(w)$ is equal to 1 because the integrand is like a pdf of a chi-square distribution with $r_1 + r_2$ degrees of freedom. Graphs of pdfs for the F distribution

If all n of the distributions are the same, then the collection of n independent and identically distributed random variables, $X_1, X_2, \ldots, X_n$, is said to be a **random sample of size n from that common distribution.** If $f(x)$ is the common pmf or pdf of these n random variables, then the joint pmf or pdf is $f(x_1)f(x_2)\cdots f(x_n)$.

Example Let X_1, X_2, X_3 be a random sample from a distribution with pdf

$$f(x) = e^{-x}, \qquad 0 < x < \infty.$$

The joint pdf of these three random variables is

$$f(x_1, x_2, x_3) = (e^{-x_1})(e^{-x_2})(e^{-x_3}) = e^{-x_1-x_2-x_3}, \qquad 0 < x_i < \infty, \ i = 1, 2, 3.$$

The probability

$$\begin{aligned} &P(0 < X_1 < 1, 2 < X_2 < 4, 3 < X_3 < 7) \\ &\qquad = \left(\int_0^1 e^{-x_1}\,dx_1\right)\left(\int_2^4 e^{-x_2}\,dx_2\right)\left(\int_3^7 e^{-x_3}\,dx_3\right) \\ &\qquad = (1 - e^{-1})(e^{-2} - e^{-4})(e^{-3} - e^{-7}), \end{aligned}$$

because of the independence of X_1, X_2, X_3. ■