LECTURE (9)
CALCULUS 2

Integration of Rational Functions

المصادر: CALCULUS I
CALCULUS II

Integration of Rational Functions
تكاملات الدوال الكسرية النسبية(تجزئة كسور)

Definition: A rational function is a quotient of two polynomials, written as $R(X) = \frac{P_n(x)}{Q_m(x)}, Q_m(x) \neq 0$ where $P_n(x)$ and $Q_m(x)$ are polynomials Of degree *n* and *m* respectively.

(1) If $n \geq m$, we perform a long division until we obtain a rational function whose numerator degree less than to the denominator.

Example (1): Evaluate the integral $\int \frac{(x^2-3x+5)}{(x-2)}\, dx$

Solution: $\int \frac{(x^2-3x+5)}{(x-2)}\, dx = \int \left[(x-1) + \frac{3}{x-2}\right] dx$

$= \frac{1}{2}x^2 - x + 3Ln|x-2| + C$

$$
\begin{array}{r|l}
 & x-1 \\
\hline
x-2 & x^2-3x+5 \\
 & \underline{-x^2+2x} \\
 & x+5 \\
 & \underline{+x+2} \\
 & 3
\end{array}
$$

Example (2): Evaluate the integral $\int \frac{x^2+2}{x^2+1}\,dx$

Solution: $\int \frac{x^2+2}{x^2+1}\,dx = \int \left(1+\frac{1}{x^2+1}\right)dx$

$= x + tan^{-1}(x) + C$

$$\begin{array}{r|l} & 1 \\ \hline x^2+1 & x^2+2 \\ & \underline{-x^2-1} \\ & 1 \end{array}$$

(2) If $n < m$, we shall discuss the three cases of separating $\frac{P_n(x)}{Q_m(x)}$ into a sum of partial fractions.

Case (1): If the *m* factor of $Q_m(x)$ are all different and simple, that is , $Q_m(x) = (x-a_1)(x-a_2)\ldots(x-a_m)$. Then we assign the sum of *m* partial fractions to these factors as follows $\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_m}{(x-a_m)}$ where $A_1, A_2, \ldots, A_m$ are constants must be evaluated.

Example (3): Evaluate the integral $\int \frac{x^2+3x+3}{x^3-x}\,dx$

Solution: $\frac{x^2+3x+3}{x^3-x} = \frac{x^2+3x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$$= \frac{A(x-1)(x+1)+B(x)(x+1)+C(x)(x-1)}{x(x-1)(x+1)}$$

$$x^2 + 3x + 3 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$$

at $x = 0 \quad \rightarrow 3 = A(0-1)(0+1) + 0 + 0 \rightarrow A = -3$

at $x = 1 \quad \rightarrow 7 = 0 + B(1)(1+1) + 0 \rightarrow \quad B = \frac{7}{2}$

at $x = -1 \rightarrow 1 = 0 + 0 + C(-1)(-1-1) \rightarrow C = \frac{1}{2}$

او طريقة اخرئ

$$x^2 + 3x + 3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$$= (A + B + C)x^2 + (B - C)x - A$$

$$\left.\begin{array}{r} (A+B+C) = 1 \\ B - C = 3 \\ -A = 3 \end{array}\right\} \quad A = -3\,,\ B = \frac{7}{2},\quad C = \frac{1}{2}$$

$$\therefore \int \frac{x^2+3x+3}{x^3-x}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right)dx = \int \left(\frac{-3}{x} + \frac{\frac{7}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\right)dx = -3\ln x + \frac{7}{2}ln(x-1) + \frac{1}{2}ln(x+1) + C$$

Case (2): Repeated factors of $Q_m(x)$ suppose $(x-a)^r$ is the highest power of $(x-a)$which divides $Q_m(x)$.

Then we assign the sum of r partial fractions to these factors as follows:

$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_r}{(x-a)^r}$ where $A_1, A_2,$ … , A_r are constants must be evaluated.

For example $\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+2)^3}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}+\frac{D}{(x+1)}+\frac{E}{(x+2)}+\frac{F}{(x+2)^2}+\frac{G}{(x+2)^3}$

Example (4): Evaluate the integral $\int\frac{(x+5)}{(x+2)(x-1)^2}\,dx$

Solution: $\frac{x+5}{(x+2)(x-1)^2}=\frac{A}{(x+2)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$

$=\frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}$

$x+5=A(x-1)^2+B(x+2)(x-1)+C(x+2)$

at $x=1\ \rightarrow 6=0+0+3C\ \rightarrow C=2$

at $x=-2\ \rightarrow 3=9A+0+0\ \rightarrow A=\frac{1}{3}$

at $x=0\ \rightarrow 5=\frac{1}{3}-2B+4\ \rightarrow B=-\frac{1}{3}$

$\int\frac{(x+5)}{(x+2)(x-1)^2}\,dx=\int\left[\frac{\frac{1}{3}}{x+2}-\frac{\frac{1}{3}}{x-1}+\frac{2}{(x-1)^2}\right]dx=\frac{1}{3}\ln(x+2)-\frac{1}{3}\ln(x-1)-\frac{2}{(x-1)}+C$

Case (3): Quadratic factors of $Q_m(x)$ suppose $(x^2+ax+b)^r$ is the highest power of (x^2+ax+b) which divides $Q_m(x)$.

Then we assign the sum of r partial fractions to these factors as follows:

$\frac{A_1x+B_1}{(x^2+ax+b)}+\frac{A_2x+B_2}{(x^2+ax+b)^2}+\cdots+\frac{A_rx+B_r}{(x^2+ax+b)^r}$ where $A_1, A_2, \ldots, A_r,$ $B_1, B_2,$ … , are constants must be evaluated.

For example $\frac{x^2+2x-5}{x^2(x-1)\,(x^2+1)(x^2+2x+2)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}+\frac{Dx+E}{(x^2+1)}+\frac{Fx+G}{(x^2+2x+2)}+\frac{Hx+I}{(x^2+2x+2)^2}$

Example (5): Evaluate the integral $\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx$

Solution:

$$
\begin{array}{r|l}
 & x+1 \\
x^4-2x^3+2x^2-2x+1 & x^5-x^4-3x+5 \\
 & -(x^5-2x^4+2x^3-2x^2+x) \\
 & x^4-2x^3+2x^2-4x+5 \\
 & -(x^4-2x^3+2x^2-2x+1) \\
 & -2x+4
\end{array}
$$

$$\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \int \left(x+1+\frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\right)dx$$

$$= \frac{x^2}{2}+x+\int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,dx$$

$$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

at $x = 1 \rightarrow 2 = 0 + 2B + 0 \rightarrow B = 1$

at $x = 0 \rightarrow 4 = A(-1)(1) + B(1) + (D)(-1)^2$

$4 = -A + B + D \quad \rightarrow \quad 3 = -A + D \quad \ldots(1)$

at $x = -1 \rightarrow 6 = A(-2)(-2) + 2B + (-C+D)(4)$

$\rightarrow 1 = -A - C + D \quad \ldots(2)$

at $x = 2 \rightarrow 0 = A(1)(5) + B(5) + (2C+D)(1)$

$\rightarrow -5 = 5A + 2C + D \quad \cdots(3)$

From (2) and(3) we have

$1 = -A - C + D \quad \ldots(2) \quad \times 2$

$-5 = 5A + 2C + D \quad \cdots(3)$

$2 = -2A - 2C + 2D \quad \ldots(2)$

$-5 = 5A + 2C + D \quad \cdots(3)$

$-3 = 3A + 3D \qquad x\ \frac{1}{3} \qquad \rightarrow -1 = A + D \quad \ldots(4)$

From (1) and(4) we have

$3 = -A + D \quad \ldots(1)$

$-1 = A + D \quad \ldots(4)$

$2D = 2 \qquad \rightarrow D = 1$ and $A = -2$

From (2) we have $C = 2$

$\therefore \quad A = -2, \quad B = 1, \quad C = 2, \quad D = 1.$

Let $\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \frac{x^2}{2} + x + \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\right) dx$

$$= \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + \int \frac{2xdx}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + ln(x^2+1) + tan^{-1}x + C$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + ln(x^2+1) + tan^{-1}x + C$$

Exercises :

(1) $\int \frac{x^4+1}{x^3-x}\, dx$.

(2) $\int \frac{x^2+2x+3}{(x+1)(x-1)(x-2)}\, dx$.

(3) $\int \frac{x^2+3x+3}{(x+1)(x^2-1)}\, dx$.

(4) $\int \frac{x^3-2x-5}{x^2(x-1)^2(x+1)^2}\, dx$.

(5) $\int \frac{x^3+3x^2-2x+1}{x^4+5x^2+4}\, dx$.