BEAM-COLUMN BUCKLING

DIFFERENTIAL EQUATIONS OF BEAM-COLUMNS

Bifurcation-type buckling is essentially flexural behavior. Therefore, the free-body diagram must be based on the deformed configuration as the examination of equilibrium is made in the neighboring equilibrium position. Summing the forces in the horizontal direction in Fig. 1-4(a) gives:

$$\sum F_y = 0 = (V + dV) - V + q dx$$

$$\frac{dV}{dx} = V' = -q(x)$$

Summing the moment at the top of the free body gives

$$\sum M_{\text{top}} = 0 = (M + dM) - M + V dx + P dy - q(dx)\frac{dx}{2}$$

Taking derivatives on both sides of Eq.: $\frac{dM}{dx} + P\frac{dy}{dx} = -V$

Taking derivatives on both sides of Eq. above give:

$$M'' + (Py')' = -V'$$

Equation (3) is the fundamental beam-column governing differential equation. Consider the free-body diagram shown in Fig. 1-4(d). Summing forces in the y direction gives

$$\sum F_y = 0 = -(V + dV) + V + q dx \Rightarrow \frac{dV}{dx} = V' = q(x)$$

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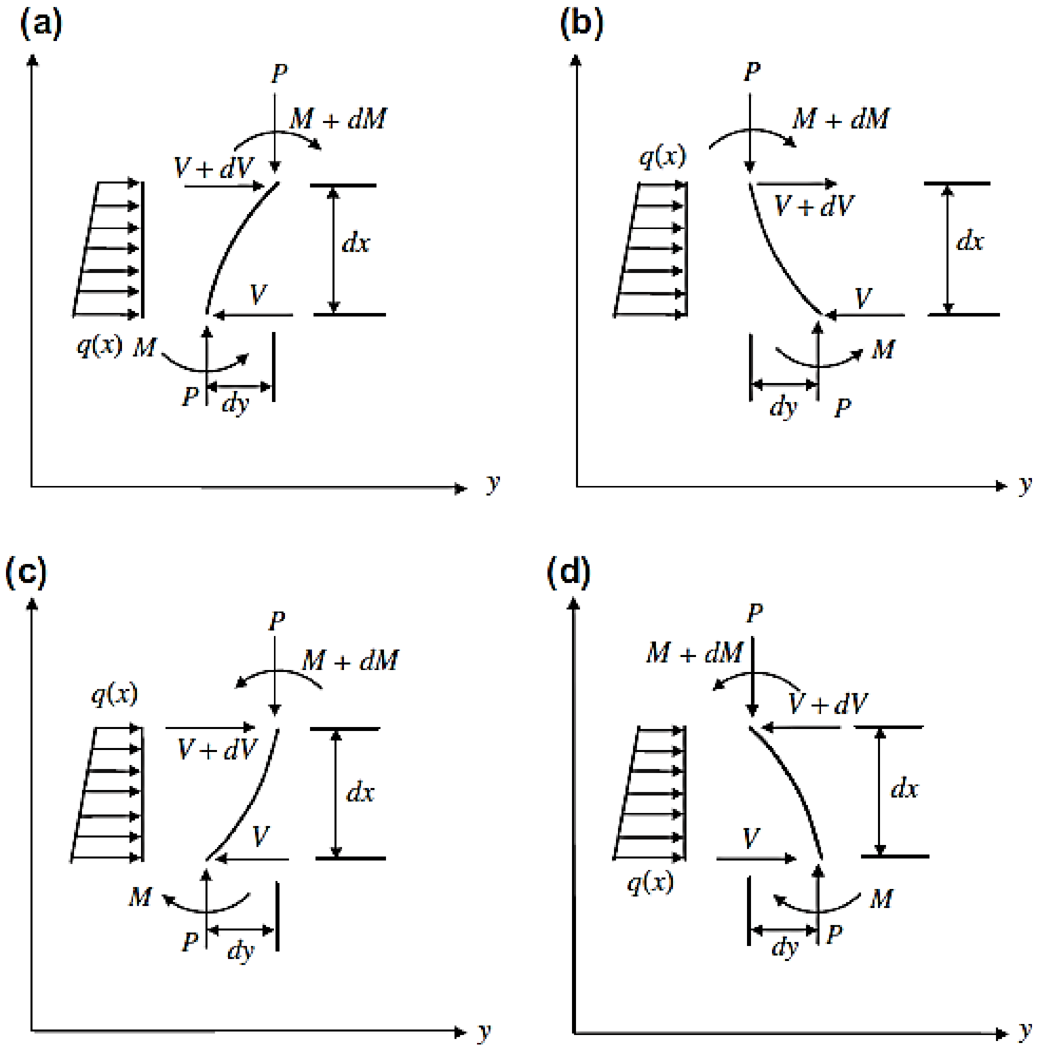


Figure 1-4 Free-body diagrams of a beam-column

Summing moments about the top of the free body yields

$$\sum M_{top} = 0$$

$$= -(M + dM) + M - Vdx - Pdy - \cancel{q\,dx\,dx/2} \Rightarrow$$

$$-\frac{dM}{dx} - P\frac{dy}{dx} = V$$

For the coordinate system shown in Fig. 1-4(d), the curve represents a decreasing function (negative slope) with the convex side to the positive y direction. Hence, *-EIy"=M(x)*. Thus,

$$-(-EIy'')' - (-Py') = V$$

which leads to

$$EIy''' + Py' = V \quad \text{or} \quad EIy^{iv} + Py'' = q(x)$$

It can be shown that the free-body diagrams shown in Figs. 1-4(b) and 1-4(c) will lead to Eq. (3). Hence, the governing differential equation is independent of the shape of the free-body diagram assumed. Rearranging Eq. (3) and if considered *q(x)*= 0 gives:

$$EIy^{iv} + Py'' = 0 \Rightarrow y^{iv} + k^2y'' = 0, \quad \text{where } k^2 = \frac{P}{EI}$$

Assuming the solution to be of a form $y = \alpha e^{mx}$, then $y' = \alpha m e^{mx}$, $y'' = \alpha m^2 e^{mx}$, $y''' = \alpha m^3 e^{mx}$, and $y^{iv} = \alpha m^4 e^{x}$. Substituting these derivatives back to the simplified homogeneous differential equation yields

$$\alpha m^4 e^{mx} + \alpha k^2 m^2 e^{mx} = 0 \Rightarrow \alpha e^{mx}(m^4 + k^2 m^2) = 0$$

Since $\alpha \neq 0$ *and* $e^{mx} \neq 0 \Rightarrow m^2(m^2 + k^2) = 0 \Rightarrow m = \pm 0, \ \pm ki$. Hence,

$$y_h = c_1 e^{kix} + c_2 e^{-kix} + c_3 x e^0 + c_4 e^0$$

Know the mathematical identities $\begin{cases} e^0 = 1 \\ e^{ikx} = \cos kx + i \sin kx \\ e^{-ikx} = \cos kx - i \sin kx \end{cases}$

Hence, $y_h = A \sin kx + B \cos kx + Cx + D$ where integral constants *A, B, C,* and *D* can be determined uniquely by applying proper boundary conditions of the structure.

TRANSVERSELY LOADED BEAM SUBJECTED TO AXIAL COMPRESSION

A slender member meeting the Euler-Bernoulli-Navier hypotheses under transverse loads and inplane compressive load (see Fig.1) is called a beam-column. An exact analysis of a beam-column can only be accomplished by solving the governing differential equation or its derivatives (for example, slope-deflection equations). Consider a very simple case of a beam-column shown in Fig. 1. The beam-column is subjected simultaneously to a transverse load Q at its mid-span and a concentric compressive force P. Since the response of a beam-column under these loads is no longer linear, the method of superposition does not apply even if the final results are within the elastic limit.

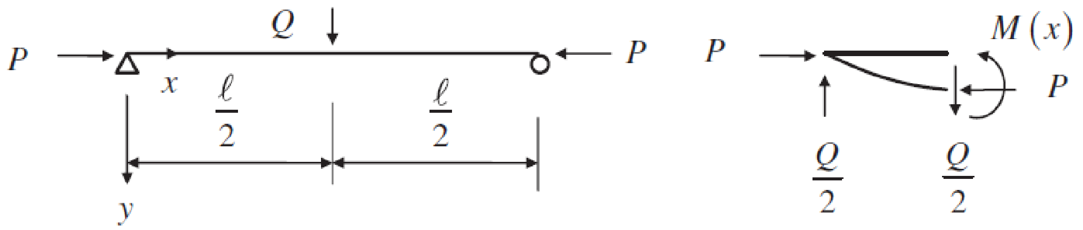


Figure 1: Simple beam-column

Summing moments at a point x from the origin gives

$$M(x) - Py - \frac{Q}{2}x = 0 \quad \text{for } 0 \leq x \leq \ell/2 \quad \text{with } M(x) = -EIy''$$

$$\text{or} \quad y'' + k^2 y = -\frac{Q}{2}\frac{x}{EI} = -\frac{Qx}{2P}k^2 \quad \text{with } k^2 = \frac{P}{EI}$$

The general solution to this differential equation is $y = y_h + y_P$. The homogeneous solution has been given earlier. The particular solution can be obtained by the method of undetermined coefficients. Assume the particular solution to be of the form

$$y_P = C + Dx \quad \text{with } y_P' = D, \; y_P'' = 0$$

Substituting these derivatives into the differential equation yields

$$0 + k^2(C + Dx) = -\frac{Qx}{2P}k^2$$

Hence,

$$C = 0 \quad \text{and} \quad D = -\frac{Q}{2P} \Rightarrow y_P = -\frac{Q}{2P}x$$

The total solution is

$$y = A\cos kx + B\sin kx - \frac{Qx}{2P}$$

The two constants of integration can be determined from the following boundary conditions:

$$y = 0 \quad \text{at } x = 0 \Rightarrow A = 0$$

$$y' = 0 \quad \text{at } x = \ell/2$$

$$(\text{Note : the boundary condition}, y = 0 \quad \text{at } x = \ell, \text{cannot be used}$$

$$\text{here as } 0 \leq x \leq \ell/2)$$

$$y' = Bk\cos kx - \frac{Q}{2P}, 0 = Bk\cos\frac{k\ell}{2} - \frac{Q}{2P} \Rightarrow B = \frac{Q}{2Pk\cos\dfrac{k\ell}{2}}$$

$$y = \frac{Q\sin kx}{2Pk\cos\dfrac{k\ell}{2}} - \frac{Qx}{2P} \quad \text{for } 0 \leq x \leq \frac{\ell}{2} \quad \text{with } P_{cr} = P_E = \frac{\pi^2 EI}{\ell^2}$$

By observation, the maximum lateral deflection occurs at the midspan.

$$y_{\max}\Big|_{x=\frac{\ell}{2}} = \frac{Q}{2Pk}\left(\tan\frac{k\ell}{2} - \frac{k\ell}{2}\right) \quad \text{with } u = \frac{k\ell}{2} = \frac{\ell}{2}\sqrt{\frac{P}{EI}}$$

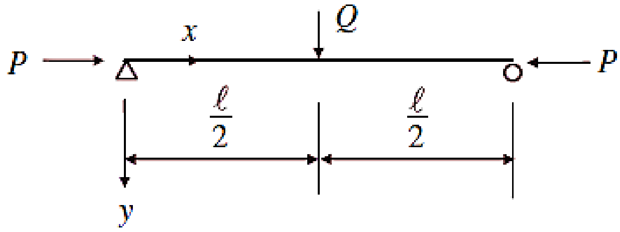
$$y_{\max}\Big|_{x=\frac{\ell}{2}} = \frac{Qk^3\ell^3}{16Pku^3}\left(\tan\frac{k\ell}{2} - \frac{k\ell}{2}\right) = \frac{Q\ell^3}{48EI}\left[\frac{3(\tan u - u)}{u^3}\right] = \frac{Q\ell^3}{48EI}X(u)$$

$$y_{max}\Big|_{x=\frac{\ell}{2}} = \delta_{max} = \frac{Q\ell^3}{48EI} \quad \text{when } P = 0$$

$$u^2 = \frac{\ell^2}{4}\frac{P}{EI} \Rightarrow P = \frac{4EIu^2}{\ell^2} \quad \text{and} \quad P_E = \frac{\pi^2 EI}{\ell^2}$$

$$\frac{P}{P_E} = \frac{4EIu^2}{\ell^2}\frac{\ell^2}{\pi^2 EI} = \frac{4u^2}{\pi^2}, \quad X(u) = \frac{3(\tan u - u)}{u^3}$$

The previous section showed that the deflection at the mid-span of a simple beam-column subjected to a lateral load shown in Fig. 3 is



$$\delta = y_{max} = \delta_0 \frac{3(\tan u - u)}{u^3}$$

Figure 3: simple beam-column subjected to a lateral load

where

$$\delta_0 = \frac{Q\ell^3}{48EI}, u = \frac{k\ell}{2}, \quad \text{and} \quad k = \sqrt{\frac{P}{EI}}$$

Recall the power series expansion of $\tan u$ given by

$$\tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \frac{17u^7}{315} + \dots$$

Hence,

$$\delta = \delta_0\left(1 + \frac{2u^2}{5} + \frac{17u^4}{105} + \dots\right)$$

Noting

$$u^2 = \frac{k^2\ell^2}{4} = \frac{P\ell^2}{4EI}\frac{\pi^2}{\pi^2} = 2.46\,\frac{P}{P_E}$$

$$\delta = \delta_0\left[1 + 0.984\,\frac{P}{P_e} + 0.998\left(\frac{P}{P_e}\right)^2 + \ldots\right]$$

$$\doteq \delta_0\left[1 + \frac{P}{P_E} + \left(\frac{P}{P_E}\right)^2 + \ldots\right]$$

$$= \delta_0\,\frac{1}{1 - \dfrac{P}{P_E}} \Leftarrow \text{from power series sum for } \frac{P}{P_E} < 1$$

where

$\dfrac{1}{1 - \dfrac{P}{P_E}}$ is called amplification factor or magnification factor.

The maximum bending moment is

$$M_{\max} = \frac{Q\ell}{4} + P\delta = \frac{Q\ell}{4} + \frac{PQ\ell^3}{48EI}\frac{1}{1 - \dfrac{P}{P_E}} = \frac{Q\ell}{4}\left(1 + \frac{P\ell^2}{12EI}\frac{1}{1 - \dfrac{P}{P_E}}\right)$$

$$= \frac{Q\ell}{4}\left(1 + 0.82\,\frac{P}{P_E}\,\frac{1}{1 - \dfrac{P}{P_E}}\right)$$

or

$$M_{\max} = \frac{Q\ell}{4}\left(\frac{1 - 0.18\,\dfrac{P}{P_E}}{1 - \dfrac{P}{P_E}}\right)$$

where

$$\left(\frac{1 - 0.18\dfrac{P}{P_E}}{1 - \dfrac{P}{P_E}}\right)$$

is amplification factor for bending moment due to a concentrated load.

The variation of δ with Q as given by the amplification factor is plotted on the left side of Fig. 4 for P = 0, P = 0.4P_{cr}, and P = 0.7Pcr. The curves show that the relation between Q and δ is linear even when P≠0, provided P is a constant. However, if P is allowed to vary, as is the case on the right side of Figure 4, the load-deflection relation is not linear. This is true regardless of whether Q remains constant (dashed curve) or increases

as P increases (solid curve). The deflection of a beam-column is thus a linear function of Q but a nonlinear function of P.

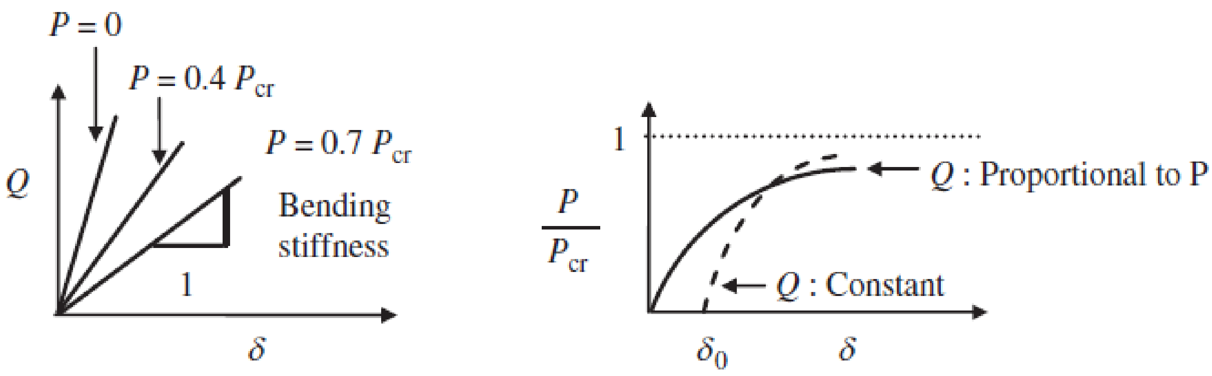
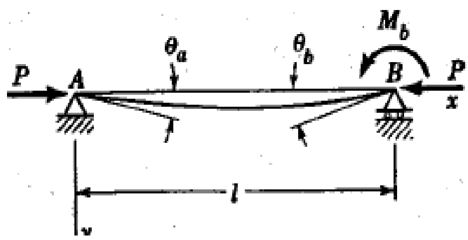


Figure 4: Lateral displacements of beam-column

BENDING OF BEAM-COLUMNS BY COUPLES

- Case 1: one end is subjected to moment



the deflection curve is obtained by:

$$y = \frac{M_b}{P}\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right)$$

$$\theta_a = \left(\frac{dy}{dx}\right)_{x=0} = \frac{M_b}{P}\left(\frac{k}{\sin kl} - \frac{1}{l}\right) = \frac{M_b l}{6EI}\frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$

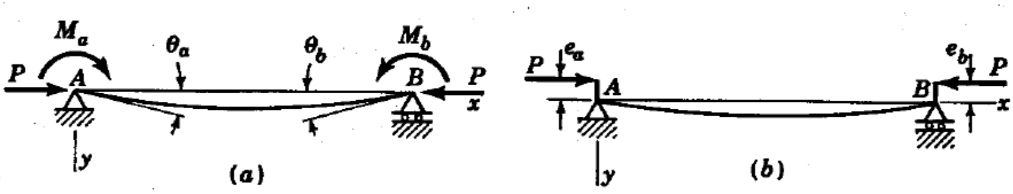
$$\theta_b = -\left(\frac{dy}{dx}\right)_{x=l} = -\frac{M_b}{P}\left(\frac{k\cos kl}{\sin kl} - \frac{1}{l}\right) = \frac{M_b l}{3EI}\frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

to simplify these expressions let:

$$\phi(u) = \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$

$$\psi(u) = \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

- Case 2: both ends are subjected to moments



By substituting M_a by M_b and x by $(l\text{-}x)$ in the same equation of case one. Adding the two results together, then the deflection curve for this case:

$$y = \frac{M_b}{P}\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right) + \frac{M_a}{P}\left[\frac{\sin k(l-x)}{\sin kl} - \frac{l-x}{l}\right]$$

Substituting $M_a = Pe_a$ & $M_b = Pe_b$ we obtain:

$$y = e_b\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right) + e_a\left[\frac{\sin k(l-x)}{\sin kl} - \frac{l-x}{l}\right]$$

$$\theta_a = \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u)$$

$$\theta_b = \frac{M_b l}{3EI}\psi(u) + \frac{M_a l}{6EI}\phi(u)$$

Case 3: both ends are subjected to equal moments ($M_a = M_b = M_o$)

$$y = \frac{M_0}{P\cos(kl/2)}\left[\cos\left(\frac{kl}{2} - kx\right) - \cos\frac{kl}{2}\right]$$

$$= \frac{M_0 l^2}{8EI}\,\frac{2}{u^2\cos u}\left[\cos\left(u - \frac{2ux}{l}\right) - \cos u\right]$$

The deflection at the center of the beam is obtained by substituting $x=l/2$

$$\delta = (y)_{x=l/2} = \frac{M_0 l^2}{8EI}\,\frac{2(1-\cos u)}{u^2\cos u} = \frac{M_0 l^2}{8EI}\lambda(u)$$

The slope at the ends are:

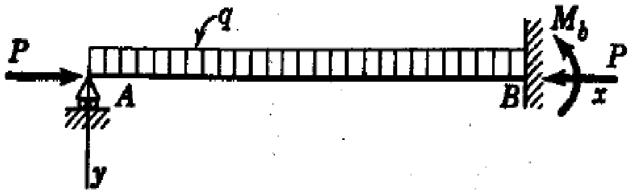
$$\theta_a = \theta_b = \left(\frac{dy}{dx}\right)_{x=0} = \frac{M_0 l}{2EI}\frac{\tan u}{u}$$

The max. bending moment which obtained at the middle of span:

$$M_{max} = -EI\left(\frac{d^2y}{dx^2}\right)_{x=l/2} = M_0 \sec u$$

BEAM-COLUMNS WITH BUILT UP ENDS

Case 1: one end is fixed

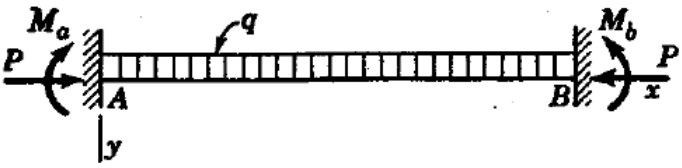


The rotation at the fixed due to the uniform load and the moment equal to zero

$$\frac{ql^3}{24EI}\chi(u) + \frac{M_0 l}{2EI}\frac{\tan u}{u} = 0 \qquad \chi(u) = \frac{3(\tan u - u)}{u^3}$$

$$M_0 = -\frac{ql^2}{12}\frac{\chi(u)}{(\tan u)/u}$$

Case 2: both ends are fixed



The deflection curve is symmetric and the moment at fixed ends are equals ($M_a = M_b = M_o$)

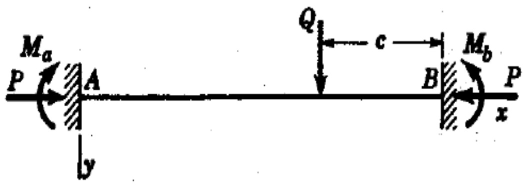
$$\frac{ql^3}{24EI}\chi(u) + \frac{M_0 l}{2EI}\frac{\tan u}{u} = 0$$

$$M_0 = -\frac{ql^2}{12}\frac{\chi(u)}{(\tan u)/u}$$

Case 3: unsymmetrical loaded beam

$$\theta_a = \theta_{0a} + \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u) = 0$$

$$\theta_b = \theta_{0b} + \frac{M_a l}{6EI}\phi(u) + \frac{M_b l}{3EI}\psi(u) = 0$$



SLOPE-DEFLECTION EQUATION WITHOUT AXIAL FORCE

A typical derivation process will be traced here as it will be used again in the development of the slope-deflection equations that include the effect of axial compression on the bending stiffness From the deformations of a beam shown in Fig.7, the moment at a distance x from the origin is expressed as:

$$M_x = M_{ab} - (M_{ab} + M_{ba}) \frac{x}{\ell}$$

Know $y'' = -\dfrac{M_x}{EI}$

Taking successive derivatives of the above equation gives

$$EIy^{iv} = 0$$

The general solution of the differential equation is

$$y = A + Bx + Cx^2 + Dx^3$$

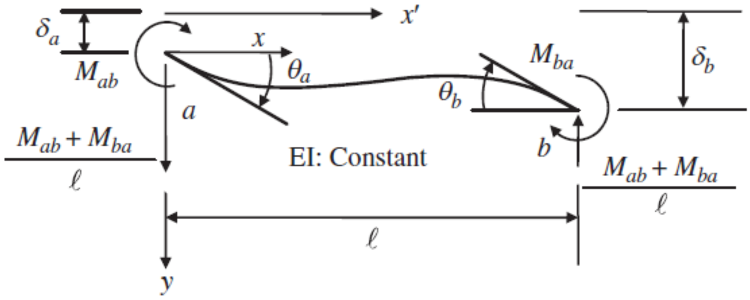


Figure 7: Deformations of beam

$$y' = B + 2Cx + 3Dx^2$$

$$y'' = 2C + 6Dx$$

The four kinematic boundary conditions available are

$$y = \delta_a \quad \text{at } x = 0 \quad \text{and} \quad y = \delta_b \quad \text{at } x = \ell$$

$$y' = \theta_a \quad \text{at } x = 0 \quad \text{and} \quad y' = \theta_b \quad \text{at } x = \ell$$

$$\delta_a = A, \quad \theta_a = B$$

$$\delta_b = \delta_a + \theta_a \ell + C\ell^2 + D\ell^3$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$2\delta_b = 2\theta_a \ell + 2C\ell^2 + 2D\ell^3 + 2\delta_a$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$2\delta_b - \theta_b \ell = 2\delta_a + \theta_a \ell - D\ell^3$$

from which

$$D = \frac{1}{\ell^3}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]$$

$$3\delta_b = 3\theta_a \ell + 3C\ell^2 + 3D\ell^3 + 3\delta_a$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$3\delta_b - \theta_b \ell = 3\delta_a + 2\theta_a \ell + C\ell^2$$

from which

$$C = \frac{1}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell]$$

$$y'' = \frac{2}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell] + \frac{6}{\ell^3}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]x$$

$$y''(0) = \frac{2}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell] = -\frac{M_{ab}}{EI}$$

$$y''(\ell) = \frac{2}{\ell^3}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell]\ell + \frac{6}{\ell^2}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]\ell$$

$$= -\frac{M_{ab}}{EI} + \frac{6}{\ell^2}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell] = \frac{M_{ba}}{EI}$$

$$M_{ab} = \frac{2EI}{\ell}\left[2\theta_a + \theta_b - \frac{3}{\ell}(\delta_b - \delta_a)\right]$$

$$M_{ba} = \frac{2EI}{\ell}\left[2\theta_b + \theta_a - \frac{3}{\ell}(\delta_b - \delta_a)\right]$$

If any fixed end moments exist prior to releasing the joint constraints such as M_{ab} fixed and M_{ba} fixed, then final member end moments become

$$M_{AB} = \frac{2EI}{\ell}\left[2\theta_a + \theta_b - \frac{3}{\ell}(\delta_b - \delta_a)\right] + M_{ab\ \ fixed}$$

$$M_{ba} = \frac{2EI}{\ell}\left[2\theta_b + \theta_a - \frac{3}{\ell}(\delta_b - \delta_a)\right] + M_{ba\ \ fixed}$$

EFFECTS OF AXIAL LOADS ON BENDING STIFFNESS

The classical slope-deflections equations that are introduced in any standard text on indeterminate structures give the moments, M_{ab} and M_{ba}, induced at the ends of member AB as a function of end rotations θ_a and θ_b and by a displacement Δ of one end to the other. In conventional linear structural analysis (first-order analysis), it is customary to ignore the effect of axial forces on the bending stiffness of flexural members. It can be shown that the effect of amplification is negligibly small as long as the axial load remains small in comparison with the critical load of the member. When the ratio of the axial load to the critical load becomes sizable, however, the bending stiffness is reduced markedly due to the axial compression, and it is no longer acceptable to neglect this reduction. As the first-order analysis results may become dangerously unconservative, modern design specifications call for a mandatory second-order analysis (AISC 2005).

It is expedient to introduce $\Delta=\delta_b$ - δ_a with δ_a= 0 to avoid the rigid body translation. The moment of the beam-column shown in Fig. 9 at a distance x from the origin is

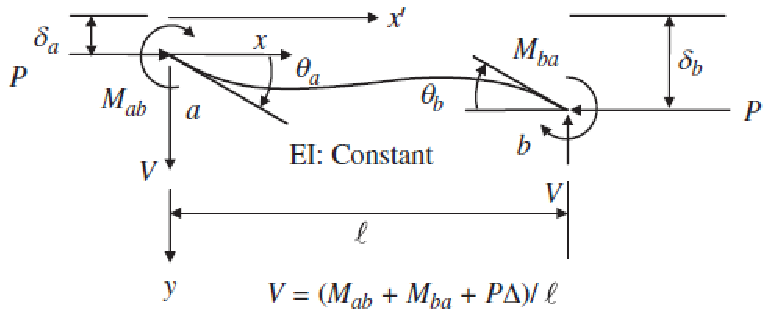


Figure 9: Deformations of beam-column

$$M_x = M_{ab} + Py - (M_b + M_{ba} + P\Delta)\frac{x}{\ell}$$

$$y'' = -\frac{M_x}{EI}$$

$$EIy'' + Py = -M_{ab} + (M_{ab} + M_{ba} + P\Delta)\frac{x}{\ell}$$

Taking successive derivatives on both sides yields

$$EIy^{iv} + Py'' = 0$$

Let $k^2 = \dfrac{P}{EI}$

The simplified differential equation is

$$y^{iv} + k^2 y'' = 0$$

for which the general solution is

$$y = A \sin kx + B \cos kx + Cx + D$$

The proper geometric boundary conditions are

$$y(0) = 0, \quad y(\ell) = \Delta, \quad y'(0) = \theta_a, \quad \text{and} \quad y'(\ell) = \theta_b$$

The proper natural boundary conditions are

$$y''(0) = -\frac{M_{ab}}{EI}, \quad \text{and} \quad y''(\ell) = \frac{M_{ba}}{EI}$$

Applying the geometric boundary conditions to eliminate the integral constants, A, B, C, D, and solving for M_{ab} and M_{ba} gives

$$0 = B + D$$

Let $\beta = k\ell$

$$\Delta = A \sin \beta + B \cos \beta + C\ell + D$$

$$\theta_a = Ak + C$$

The matrix equation for the integral constants becomes

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin \beta & \cos \beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k \cos \beta & -k \sin \beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ \theta_a \\ \theta_b \end{Bmatrix}$$

Applying Cramer's rule yields

$$A = \frac{\begin{vmatrix} 0 & 1 & 0 & 1 \\ \Delta & \cos \beta & \ell & 1 \\ \theta_a & 0 & 1 & 0 \\ \theta_b & -k \sin \beta & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 & 1 \\ \sin \beta & \cos \beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k \cos \beta & -k \sin \beta & 1 & 0 \end{vmatrix}} = \frac{D_a}{D_d}$$

$$D_a = \theta_a \begin{vmatrix} 1 & 0 & 1 \\ \cos\beta & \ell & 1 \\ -k\sin\beta & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ \Delta & \cos\beta & 1 \\ \theta_b & -k\sin\beta & 0 \end{vmatrix}$$

$$= \theta_a(\cos\beta + \beta\sin\beta - 1) + \theta_b - k\sin\beta\,\Delta - \theta_b\cos\beta$$

$$= \theta_a(\cos\beta + \beta\sin\beta - 1) + \theta_b(1 - \cos\beta) - k\sin\beta\ \Delta$$

$$D_d = -\begin{vmatrix} \sin\beta & \ell & 1 \\ k & 1 & 0 \\ k\cos\beta & 1 & 0 \end{vmatrix} - \begin{vmatrix} \sin\beta & \cos\beta & \ell \\ k & 0 & 1 \\ k\cos\beta & -k\sin\beta & 1 \end{vmatrix}$$

$$= -k + k\cos\beta - k(\cos^2\beta + \sin^2\beta) + k\,\beta\sin\beta + k\cos\beta$$

$$= -2k + 2k\cos\beta + k\ \beta\sin\beta = k(2\cos\beta + \beta\sin\beta - 2)$$

$$B = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ \sin\beta & \Delta & \ell & 1 \\ k & \theta_a & 1 & 0 \\ k\cos\beta & \theta_b & 1 & 0 \end{vmatrix}}{D_d} = \frac{D_b}{D_d}$$

$$D_b = -\begin{vmatrix} \sin\beta & \Delta & \ell \\ k & \theta_a & 1 \\ k\cos\beta & \theta_b & 1 \end{vmatrix}$$

$$= -\theta_a\sin\beta - \theta_b\beta - k\cos\beta\ \Delta + \theta_a\beta\cos\beta + k\ \Delta + \theta_b\sin\beta$$

$$= \theta_a(\beta\cos\beta - \sin\beta) + \theta_b(\sin\beta - \beta) + \Delta(k - k\cos\beta)$$

$$y' = Ak\cos kx - Bk\sin kx + C$$

$$y'' = -Ak^2\sin kx - Bk^2\cos kx$$

$$M_{ab} = -EIy''(0) = EIBk^2$$

$$= \left[\frac{EIk^2}{k(2\cos\beta + \beta\sin\beta - 2)}\right][(\beta\cos\beta - \sin\beta)\theta_a + (\sin\beta - \beta)\theta_b$$

$$+ (k - k\cos\beta)\Delta]$$

$$= \left[\frac{EI\beta}{\ell(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\theta_a + (\sin\beta - \beta)\theta_b\right.$$

$$\left. + (\beta - \beta\cos\beta)\frac{\Delta}{\ell}\right]$$

Let

$$S_1 = S = \frac{\beta(\beta \cos\beta - \sin\beta)}{2\cos\beta + \beta\sin\beta - 2}$$

$$S_2 = \frac{(\sin\beta - \beta)}{2\cos\beta + \beta\sin\beta - 2}$$

Recall identities

$$\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$$

$$\cos\beta = \cos^2(\beta/2) - \sin^2(\beta/2) = 1 - 2\sin^2(\beta/2)$$

Dividing the numerator and denominator of S_1 by $\sin\beta$ gives

$$S_1 = S = \frac{\beta(\beta\cot\beta - 1)}{2\cot\beta - \dfrac{2}{\sin\beta} + \beta} = \frac{\beta(\beta\cot\beta - 1)}{den1 + \beta}$$

where

$$den1 = 2\cot\beta - \frac{2}{\sin\beta} = \frac{2\cos\beta - 2}{\sin\beta} = \frac{2[1 - 2\sin^2(\beta/2) - 1]}{2\sin(\beta/2)\cos(\beta/2)}$$
$$= -2\tan(\beta/2)$$

$$S_1 = S = \frac{\beta(\beta\cot\beta - 1)}{-2\tan(\beta/2) + \beta}$$

$$S_1 = S = \frac{1 - \beta\cot\beta}{\dfrac{2\tan(\beta/2)}{\beta} - 1}$$

Let $S_2 = C = \dfrac{\beta(\sin\beta - \beta)}{2\cos\beta + \beta\sin\beta - 2}$

Taking the same procedure used above gives

$$S_2 = C = \frac{\beta(1 - \beta\operatorname{cosec}\beta)}{2\cot\beta - \dfrac{2}{\sin\beta} + \beta} = \frac{\beta(1 - \beta\operatorname{cosec}\beta)}{-2\tan(\beta/2) + \beta}$$

$$S_2 = C = \frac{\beta \operatorname{cosec} \beta - 1}{\dfrac{2 \tan(\beta/2)}{\beta} - 1}$$

$$\text{Let } S_3 = SC = \frac{\beta(\beta - \beta \cos \beta)/\ell}{2 \cos \beta + \beta \sin \beta - 2}$$

Again dividing the numerator and denominator of S_3 by $\sin \beta$ gives:

$$S_3 = SC = \frac{\beta(\beta \operatorname{cosec} \beta - \beta \cot \beta)/\ell}{2 \cot \beta - \dfrac{2}{\sin \beta} + \beta} = \frac{\beta(\beta \operatorname{cosec} \beta - \beta \cot \beta)/\ell}{-2 \tan(\beta/2) + \beta}$$

$$= \frac{(\beta \cot \beta - \beta \operatorname{cosec} \beta)/\ell}{\dfrac{2 \tan(\beta/2)}{\beta} - 1} = \frac{[-(1 - \beta \cot \beta) - (\beta \operatorname{cosec} \beta - 1)]/\ell}{\dfrac{2 \tan(\beta/2)}{\beta} - 1}$$

$$S_3 = SC = -\frac{S_1 + S_2}{\ell} = -\frac{S + C}{\ell}$$

Recall $M_{ab} = M(0) = -EIy''(0)$.

But $M_{ba} = -M(\ell) = EIy''(\ell)$ (note the negative sign!)

$$y'' = -Ak^2 \sin kx - Bk^2 \cos kx$$

$$M_{ba} = +EIy''(\ell)$$

$$= \left[\frac{-EIk^2}{k(2 \cos \beta + \beta \sin \beta - 2)}\right] \left\{ \begin{aligned} &\sin \beta[\theta_a(\cos \beta + \beta \sin \beta - 1) + \theta_b(1 - \cos \beta) - \Delta \ k \sin \beta] \\ &\quad + \cos \beta[\theta_a(\beta \cos \beta - \sin \beta) + \theta_b(\sin \beta - \beta) \\ &\quad + \Delta(k - k \cos \beta)] \end{aligned} \right\}$$

$$= \left(\frac{-EIk}{2 \cos \beta + \beta \sin \beta - 2}\right) \left[\begin{aligned} &\theta_a(\cos \beta \sin \beta + \beta \sin^2 \beta - \sin \beta + \beta \cos^2 - \cos \beta \sin \beta) \\ &\quad + \theta_b(\sin \beta - \cos \beta \sin \beta + \cos \beta \sin \beta - \beta \cos \beta) \\ &\quad + \Delta(k \cos \beta - k \cos^2 \beta - k \sin^2 \beta) \end{aligned} \right]$$

$$= \left(-\frac{EI\beta}{\ell}\right)\frac{[\theta_a(\beta - \sin\beta) + \theta_b(\sin\beta - \beta\cos\beta) + \Delta(k\cos\beta - k)]}{(2\cos\beta + \beta\sin\beta - 2)}$$

$$= \left(\frac{EI}{\ell}\right)\frac{[\theta_a\beta(\sin\beta - \beta) + \theta_b\beta(\beta\cos\beta - \sin\beta) + \Delta\beta(\beta - \beta\cos\beta)/\ell]}{(2\cos\beta + \beta\sin\beta - 2)}$$

$$M_{ab} = \frac{EI}{\ell}\left[S_1\theta_a + S_2\theta_b - (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

$$M_{ba} = \frac{EI}{\ell}\left[S_2\theta_a + S_1\theta_b - (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

If $M_{ab} = 0$ (when the support A is either pinned or roller), then

$$M_{ab} = \frac{EI}{\ell}\left[S_1\theta_a + S_2\theta_b - \left(S_1 + S_2\right)\frac{\Delta}{\ell}\right] = 0$$

$$\theta_a = \frac{1}{S_1}\left[-S_2\theta_b + (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

Substituting θ_a into M_{ba} yields

$$M_{ba} = \frac{EI}{\ell}\left[\left(S_1 - \frac{S_2^2}{S_1}\right)\theta_b - \left(S_1 + S_2\right)\left(1 - \frac{S_2}{S_1}\right)\frac{\Delta}{\ell}\right]$$

Let $\overline{S} = \dfrac{1}{S_1}(S_1^2 - S_2^2)$, then

$$\overline{M}_{ba} = \frac{EI}{\ell}\left[\overline{S}\theta_b - \overline{S}\frac{\Delta}{\ell}\right]$$

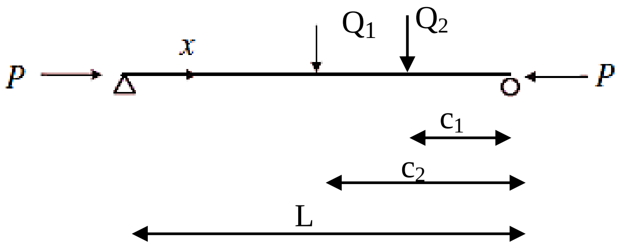
$$\overline{S} = \frac{1}{S_1}\left(S_1^2 - S_2^2\right)$$

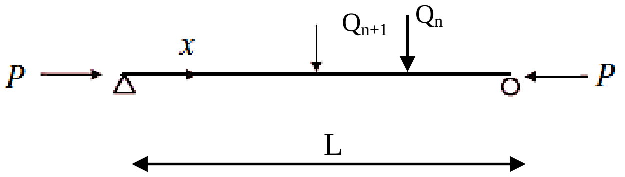
$$= \left[\frac{-2\tan(\beta/2) + \beta}{\beta(\beta\cot\beta - 1)}\right]\left[\frac{\beta^2(\beta\cot\beta - 1)^2}{(-2\tan(\beta/2) + \beta)^2} - \frac{\beta^2(1 - \operatorname{cosec}\beta)^2}{(-2\tan(\beta/2) + \beta)^2}\right]$$

$$= \frac{\beta}{(\beta\cot\beta - 1)[-2\tan(\beta/2) + \beta]}[(\beta\cot\beta - 1)^2 - (1 - \operatorname{cosec}\beta)^2]$$

$$= \frac{\beta^2}{(\beta\cot\beta - 1)[-2\tan(\beta/2) + \beta]}[-\beta + 2\tan(\beta/2)] = \frac{\beta^2}{1 - \beta\cot\beta}$$

H.W 1: Drive an expression for deflected curve for the following cases of compressed beam-column shown in Figures.





H.W 2: Drive an expression for deflected curve for the following cases of compressed beam-column shown in Figures by using high-order differential equation.