



Definition (9): (scalar multiplication):

If $\mathbf{u} = \langle a, b \rangle$ is a nonzero vector and k is a nonzero real number (scalar), then the product of k and \mathbf{u} is denoted as usual by:

$$k\mathbf{u} \text{ or } \mathbf{u}k \text{ and } k\mathbf{u} = k\langle a, b \rangle = \langle ka, kb \rangle$$

It is, however, more customary to place the scalar multiplier before the multiplicand \mathbf{u} .

This multiplication by a scalar is called **scalar multiplication**, and it follows the **associative law** $c(k\mathbf{u}) = (ck)\mathbf{u} = k(c\mathbf{u})$, (c and k any scalars) as in ordinary algebra and arithmetic.

A vector of the form $k\mathbf{v}$ is called a **scalar multiple** of \mathbf{v} .

Geometrically, if a vector \mathbf{v} is multiplied by a scalar k , the magnitude of the vector \mathbf{v} is multiplied by $|k|$.

If k is positive, then $k\mathbf{v}$ has the same direction as \mathbf{v} .

If k is negative, then $k\mathbf{v}$ has the opposite direction to that of \mathbf{v} , as show in (figure 10).

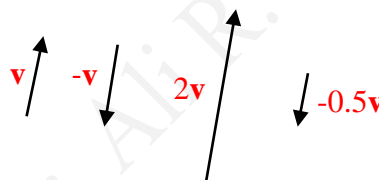


Figure 10

A vector of the form $k\mathbf{v}$ is called a scalar multiple of the vector \mathbf{v} .

Example (5): Let $\mathbf{u} = \langle 4, -3 \rangle$, $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle 0, 5 \rangle$, find: A) $\mathbf{u} + \mathbf{v}$, B) $-2\mathbf{u}$, C) $2\mathbf{u} - 3\mathbf{v}$, D) $3\mathbf{u} + 2\mathbf{v} - \mathbf{w}$.

Solution:

$$A) \mathbf{u} + \mathbf{v} = \langle 4, -3 \rangle + \langle 2, 3 \rangle = \langle 6, 0 \rangle$$

$$B) -2\mathbf{u} = -2 \langle 4, -3 \rangle = \langle -8, 6 \rangle$$

$$C) 2\mathbf{u} - 3\mathbf{v} = 2 \langle 4, -3 \rangle - 3 \langle 2, 3 \rangle = \langle 8, -6 \rangle + \langle -6, -9 \rangle = \langle 2, -15 \rangle$$

$$D) 3\mathbf{u} + 2\mathbf{v} - \mathbf{w} = 3 \langle 4, -3 \rangle + 2 \langle 2, 3 \rangle - \langle 0, 5 \rangle \\ = \langle 12, -9 \rangle + \langle 4, 6 \rangle - \langle 0, 5 \rangle \\ = \langle 16, 2 \rangle.$$

Definition (10): (Unit Vector) (in 2- space):

If the length of \mathbf{v} ($\|\mathbf{v}\| = 1$), then \mathbf{v} is called a **unit vector**. A unit vector can be formed from an arbitrary nonzero vector and a unit vector of \mathbf{v} with the same direction as \mathbf{v} .

If \mathbf{v} is a nonzero vector, then the unit vector of \mathbf{v} is: $\mathbf{U} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$ and it is with the same direction as \mathbf{v} .

Example (5): Given a vector $\mathbf{v} = \langle 1, -2 \rangle$, find a unit vector \mathbf{U} with the same direction as \mathbf{v} .

Solution:



$$\|\mathbf{v}\| = \sqrt{1^2 + (-2)^2}$$

$$= \sqrt{5}$$

$$\mathbf{U} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$

$$= \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$$

$$= \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle \text{ The unit vector of } \mathbf{v}.$$

$$\text{Checking: } |\mathbf{U}| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1 \text{ the length of unit vector.}$$

Example (6): Given a vector $\mathbf{v} = \langle 3, 1 \rangle$, find a unit vector \mathbf{U} with the same direction \mathbf{v} .
(homework).

The \mathbf{i} and \mathbf{j} unit vectors in 2-space:

We now define two very important unit vectors, the \mathbf{i} and \mathbf{j} unit vectors (in 2- space) as shown below in (figure 11).

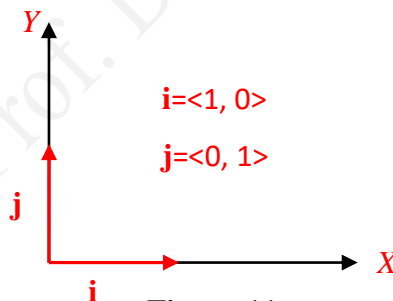


Figure 11

(The \mathbf{i} and \mathbf{j} unit vectors)

Why are the \mathbf{i} and \mathbf{j} unit vectors so important? One of the reasons is that any vector $\mathbf{v} = \langle a, b \rangle$ can be expressed as a linear combination (*we will detail it later*) of these vectors; that is as $a\mathbf{i} + b\mathbf{j}$.

$$\mathbf{v} = \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle$$

$$= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle$$

$$= a\mathbf{i} + b\mathbf{j} \text{ (Linear combination form of the vector).}$$

There is also the unit vector for vectors in 3-space and we will study the standard unit vectors in detail after studying vectors in 3-space.

Example (7): Express each vectors as linear combination of the \mathbf{i} and \mathbf{j} unit vectors.

a) $\langle -2, 4 \rangle$; b) $\langle 2, 0 \rangle$; c) $\langle 0, -7 \rangle$.

solution:



$$= -2\langle 1, 0 \rangle + 4\langle 0, 1 \rangle$$

$$= -2\mathbf{i} + 4\mathbf{j}$$

B and c (homework).

Definition (11): (subtracting a vector from another)

Let \mathbf{u} and \mathbf{v} are any two vectors, then the process of subtracting a vector \mathbf{v} from \mathbf{u} is defined by the following formula:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}), \text{ as showing in (figure 12).}$$

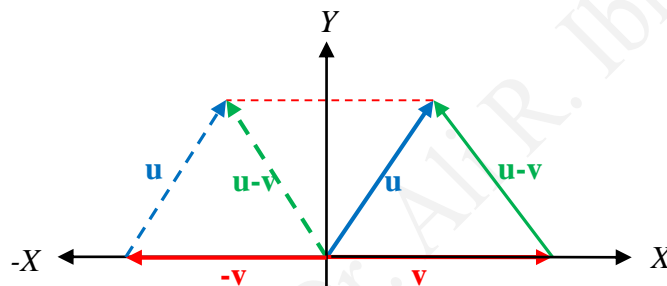


Figure 12

(Subtracting a vector \mathbf{v} from \mathbf{u})

We can obtain $\mathbf{u} - \mathbf{v}$ by the vector from the terminal point of \mathbf{v} to the terminal point of \mathbf{u} as shown in **figure (12)**.

Note: If $\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are two nonzero vectors (in 2-space), then the vectors are parallel if and only if $x_1 y_2 = x_2 y_1$.

Vectors in physics:

Force vectors:

When several forces influence an object, we can find one force, which is called the force yield or the resultant force, having the same effect, which can be represented using vectors as shown in **(figure 13)**.

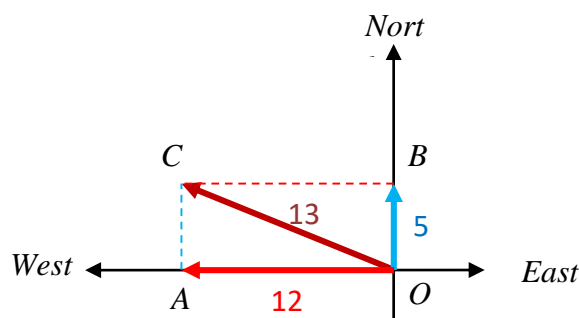


Figure 13



$$\vec{OC} = \vec{OA} + \vec{OB}$$

The magnitude of $\vec{OC} = \sqrt{(12)^2 + (5)^2} = 13$.

Velocity vectors:

Suppose that a boat goes east in a river at a rate of 4 miles per hour and the current of the river flows south at a rate of 3 miles per hour, the resultant of the velocity of the boat will be as shown in (figure 14).

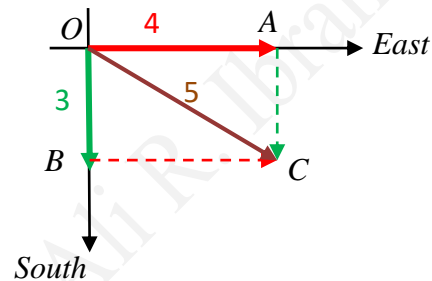


Figure 14

$$\vec{OC} = \vec{OA} + \vec{OB}$$

The magnitude of $\vec{OC} = \sqrt{(4)^2 + (3)^2} = 5$.

Vectors in 3- space:

Just as vector in the plane can be described by pairs of real numbers, vector in 3- space can be described by triples of real numbers introducing a **rectangular coordinate system**.

To construct such a coordinate system, select a point O , called the **origin**, and choose three mutually perpendicular lines, called **coordinate axes**, passing through the origin point. Label these axes X , Y and Z , and select a positive direction for each coordinate axes as well as a unit of length for measuring distance (figure 15-a).

Each pair of coordinate axes determines a plane called a **coordinate plane**. These are referred to as the **xy -plane**, the **xz -plane** and the **yz -plane**. To each point P in the **3-space** we assign a triple of numbers (x, y, z) , called **the coordinates of P** , as follows (figure 15-b), pass three planes through P parallel to the coordinate planes, and denote the points of intersection of these planes with the three coordinate axes by X , Y and Z .

The coordinates of P are defined to be the signed lengths $x=OX$, $y=OY$ and $z=OZ$.

In (figure 13-c) we have constructed the point whose coordinates are $(4, 5, 6)$ and in (figure 15-d), the point whose coordinates are $(-3, 2, -4)$.

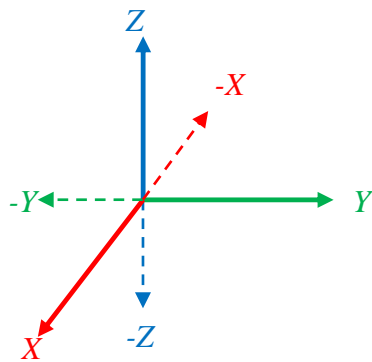


Figure 15-a

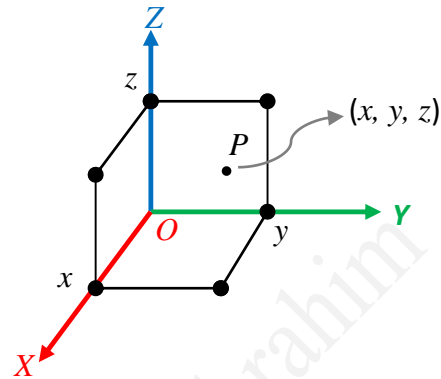


Figure 15-b

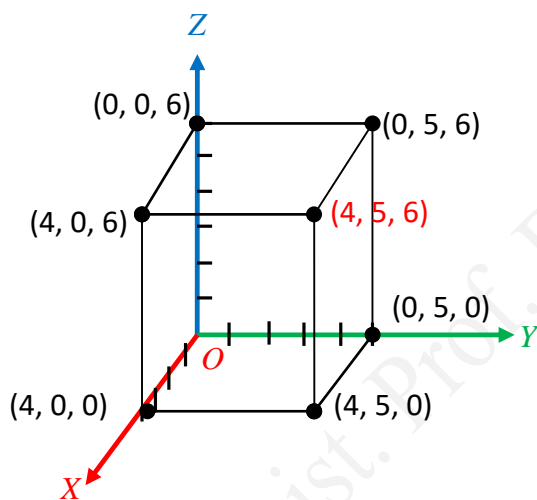


Figure 15-c

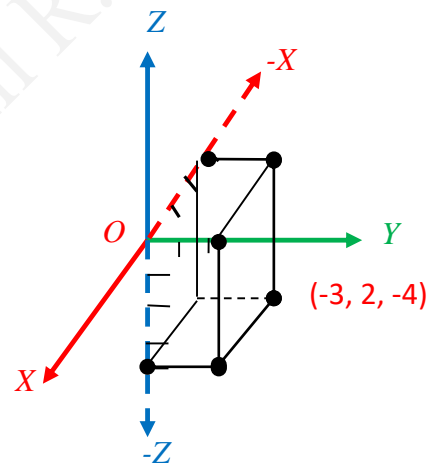


Figure 15-d

Components of the vectors in 3-space:

If a vector \mathbf{v} (in 3-space) is positioned so its initial point is at the origin of rectangular coordinate system, then the coordinates of the terminal point are called the **components** of \mathbf{v} , and we write $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ as shown in (figure 16).

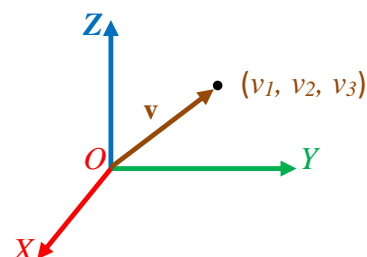


Figure 16



Vector addition and scalar multiplication (in 3-space):

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in 3-space and k is any scalar, then:

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$$

Example (8): If $\mathbf{v} = \langle 1, -3, 2 \rangle$ and $\mathbf{w} = \langle 4, 2, 1 \rangle$, then:

$$\mathbf{v} + \mathbf{w} = \langle 5, -1, 3 \rangle;$$

$$2\mathbf{v} = \langle 2, -6, 4 \rangle;$$

$$-\mathbf{w} = \langle -4, -2, -1 \rangle;$$

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = \langle -3, -5, 1 \rangle.$$

Note: Sometimes a vector is positioned so that its initial point is not at the origin point.

If the vector $\overrightarrow{P_1 P_2}$ has initial point $P_1 = (x_1, y_1, z_1)$ and terminal point $P_2 = (x_2, y_2, z_2)$, then we can find the scalar components of this vector by the following formula:

$$\overrightarrow{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The components of $\overrightarrow{P_1 P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point as shown in **figure 17**. (*We discussed this topic for vectors in 2-space, page 2, 3*)

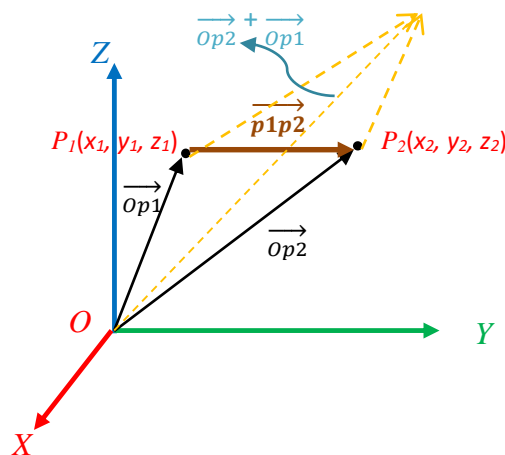


Figure 17

The vector $\overrightarrow{P_1 P_2}$ is the difference of vectors $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$, so

$$\begin{aligned} \overrightarrow{P_1 P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle \\ &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle. \end{aligned}$$



Example (9): Find the components of the vector $\mathbf{v} = \overrightarrow{P_1 P_2}$ with initial point $P_1 = (2, -1, 4)$ and terminal point $P_2 = (7, 5, -8)$.

Solution:

$$\begin{aligned}\mathbf{v} = \overrightarrow{P_1 P_2} &= \langle 7 - 2, 5 - (-1), (-8) - 4 \rangle \\ &= \langle 5, 6, -12 \rangle, \text{ as shown in (figure 18).}\end{aligned}$$

Note: The point $(5, 6, -12)$, represent the terminal point of the standard vector of $\overrightarrow{P_1 P_2}$ and if we denote this point by $P_3 = (5, 6, -12)$, then the standard vector is $\overrightarrow{O P_3}$, as shown in this figure.

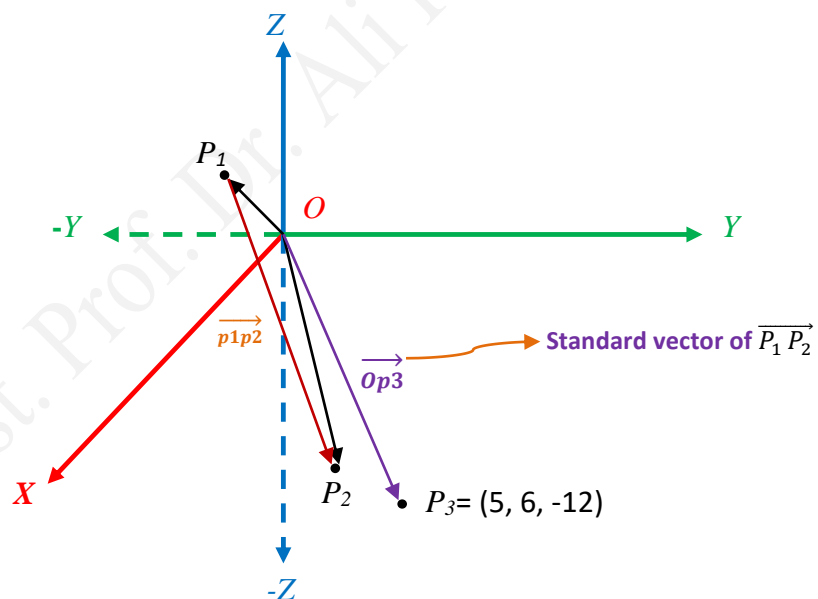


Figure 18

Note: In 2- space the vector with initial point $P_1 = (x_1, y_1)$ and terminal point $P_2 = (x_2, y_2)$ is:

$$\overrightarrow{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle. \text{ (We showed it, page 3 (first lecture)).}$$

Example (10): Find the components of the vector that has initial p_1 and terminal point p_2 for all of the following (with drawing).

- a) $P_1 = (4, 8)$, $P_2 = (3, 7)$; b) $P_1 = (3, -5)$, $P_2 = (-4, -7)$; c) $P_1 = (0, 0)$, $P_2 = (a, b)$; d) $P_1 = (3, -7, 2)$, $P_2 = (-2, 5, -4)$; e) $P_1 = (a, b, c)$, $P_2 = (0, 0, 0)$; f) $P_1 = (0, 0, 0)$, $P_2 = (a, b, c)$. **(Homework).**



Example (11): Find the standard vector and the magnitude of the vector \overrightarrow{AB} , where $A = (4, 3)$ and $B = (8, 6)$ and . (**Homework**).

References

- 1- Introductory linear algebra with applications, Bernard Kolman, first edition, 1976.
- 2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts, 1985.
- 3- Elementary Linear Algebra, Ninth Edition, Howard Anton, Chris Rorres, 2005.
- 4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.