Ministry of Higher Education
& Scientific Research
University of Anbar
College of Science
Department of Applied
Mathematics

lectures
Subject: Vector analysis.
2020-2021.
Stage: 2nd.
The lecturer: Assist. Prof. Dr.
Ali Rashid Ibrahim

Component form of the dot product:

For purposes of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors.

Let $\mathbf{u}= <u_1, u_2, u_3>$ and $\mathbf{v}= <v_1, v_2, v_3>$ are two nonzero vectors in 3-space, and θ is the angle between **u** and **v** as shown in **(figure23)**, then using the law of cosines yield, we can obtain the formula of dot product of two vectors as shown below:

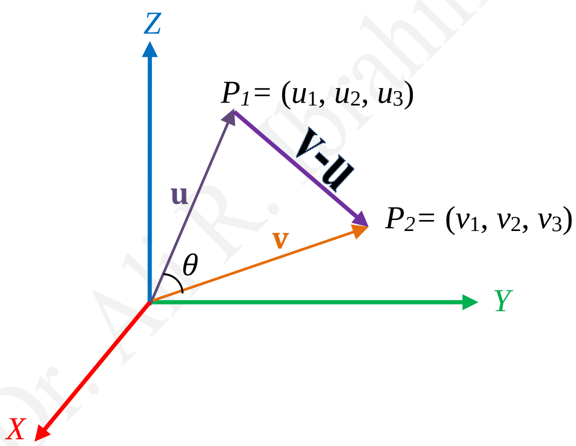

Figure 23

$$\|\overrightarrow{P_1 P_2}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \qquad \text{............ (2)}$$

Since $\overrightarrow{P_1 P_2} = \mathbf{v} - \mathbf{u}$, we will write (2) as:

$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$, and by (1) we obtain

$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$; and since,

$\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2$; $\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$; $\|\mathbf{v}-\mathbf{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$;

We obtain, $\boxed{\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3}$.

Similarly, for vectors in 2-space, if **u** and **v** are two nonzero vectors in 2-space then,

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2}\ .$$

The formula is also valid if $\mathbf{u} = 0$ *or* $\mathbf{v} = 0$.

Example (20): find the approximate angle between two vectors **x**= <2, 4> and **y**= <-1, 2>.

Solution:

$\mathbf{x}\,.\,\mathbf{y}= \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos\theta \;\rightarrow \cos\theta = \dfrac{\mathbf{x}\,.\,\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$,

$\mathbf{x}\,.\,\mathbf{y}= (2)(-1) + (4)(2)$

$= 6,$

$\|\mathbf{x}\|= \sqrt{(2)^2+(4)^2}$

$= \sqrt{20}$;

$\|\mathbf{y}\|= \sqrt{(-1)^2+(2)^2}$

$= \sqrt{5}$;

$\cos\theta = \dfrac{6}{\sqrt{20}\,\sqrt{5}} = 0.6 \;\rightarrow \theta = \cos^{-1}(0.6) \;\rightarrow\; \theta \approx 53°.$

Example (21): find the approximate angle between the two vectors **u**= <2, 3, 5> and **v**= <1, 6, -4>.

Solution:

$\mathbf{u}\,.\,\mathbf{v}= (2)(1) + (3)(6) + (5)(-4)$

$= 2 + 18 - 20$

$= 0$

$\|\mathbf{u}\|= \sqrt{(2)^2+(3)^2+(5)^2}$; $\|\mathbf{v}\|= \sqrt{(1)^2+(6)^2+(-4)^2}$;

$= \sqrt{4+9+25}$ $\quad = \sqrt{1+36+16}$

$= \sqrt{38}$ $\quad = \sqrt{53}$

$\cos\theta = \dfrac{0}{\sqrt{38}\,\sqrt{53}} = 0 \rightarrow \theta = \cos^{-1}(0) \;\rightarrow\; \theta = 90.$

Note: If the angle between any two nonzero vectors **u** and **v** is equal 90° then by formula (1), $\cos\theta = 0$ and **u . v**=0. The opposite if **u . v**=0, then $\cos\theta = 0$ and the two vectors are perpendicular (orthogonal) since θ= 90°.

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Example (22): If **u**= <2, −4> and **v**= <4, 2> are two nonzero vectors in 2-space, find the approximate angle between the two vectors.

Solution:
Using formula (1),

$$\cos\theta = \frac{\mathbf{u}\,.\,\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\;;$$

$$\mathbf{u}\,.\mathbf{v} = (2)(4) + (-4)(2)$$
$$= 0$$

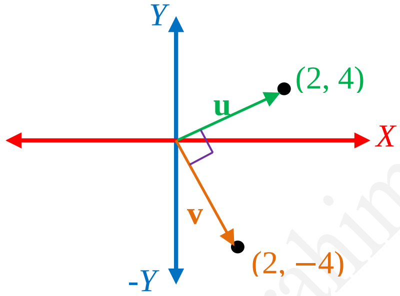

Figure 24

Now according to the note, $\cos\theta = 0$ and we don't need to find the magnitude of the two vectors, because the dot product of these vectors is equal to zero, so the angle between them is 90°and they are orthogonal as shown in **(figure 24)**.

Example (23): Find the approximate angle between the given vectors. **(Homework).**

a) **u**= <1, 2, 3>, **v**= <−3, −1, 4>.
b) **u**= <2, −1, 1>, **v**= <1, 1, 2>.
c) **u**= <0, 1, 0, 1>, **v**= <1, 0, 0, 1>.

Example (24): If **u**= <3, 4>, **v**= <5, −1> and **w**= <7, 1> are three vectors in 2-space, evaluate the expressions: **(Homework).**

a) $\mathbf{u}\,.\,(7\mathbf{v} + \mathbf{w})$.
b) $\|(\mathbf{u}\,.\,\mathbf{w})\,\mathbf{w}\|$.
c) $\|\mathbf{u}\|\,(\mathbf{v}\,.\,\mathbf{w})$.
d) $(\|\mathbf{u}\|\mathbf{v}).\mathbf{w}$.

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Example (25): a) Show that the components of the vector **v**= <v_1, v_2> in **(figure 25)** are v_1= ||**v**|| $\cos\theta$ and v_2= ||**v**|| $\sin\theta$.

b) Let **u** and **v** be the vectors in **(figure 26)**. Use the result in part (a) to find the components of 4**u** – 5**v**. **(Homework).**

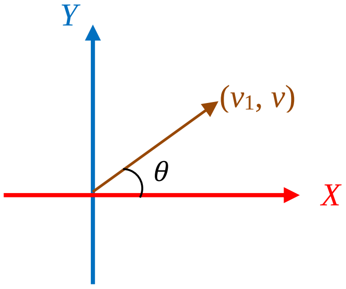

Figure 25

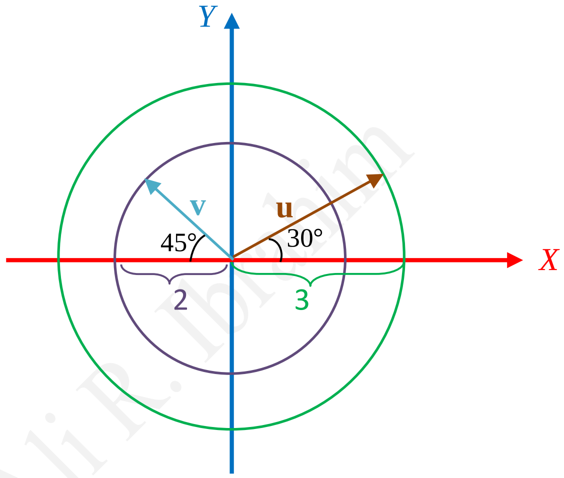

Figure 26

Example (26): Let **u**= <2, −1, 1> and **v**= <1, 1, 2> be two vectors in 3-space, find **u .v** and determine the angle θ between **u** and **v**. **(Homework).**

Example (26): Find the angle between a diagonal of a cube and one of its edges.

Solution:

Let k be the length of any edge of a cube as shown in **(figure 27)**.

Let $\mathbf{u}_1$= <k, 0, 0>, $\mathbf{u}_2$= <0, k, 0> and $\mathbf{u}_3$= <0, 0, k>, then the sum of these vectors represents a vector, **d** such that,

$$\mathbf{d} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$
$$= \langle k, 0, 0\rangle + \langle 0, k, 0\rangle + \langle 0, 0, k\rangle$$
$$= \langle k, k, k\rangle \text{ is the diagonal of a cube.}$$

The angle between the vector **d** and any edge (let the edge $\mathbf{u}_1$) satisfies the following formula:

$$\cos\theta = \frac{\mathbf{u}_1 \,.\, \mathbf{d}}{\|\mathbf{u}_1\|\;\|\mathbf{d}\|}, \text{ and } \mathbf{u}_1 \,.\, \mathbf{d} = \langle k, 0, 0\rangle \,.\, \langle k, k, k\rangle \rightarrow \mathbf{u}_1 \,.\, \mathbf{d} = k^2;$$

$$\|\mathbf{u}_1\| = \sqrt{k^2+0+0} \rightarrow \|\mathbf{u}_1\| = k$$

$$\|\mathbf{d}\| = \sqrt{k^2+k^2+k^2} \rightarrow \|\mathbf{d}\| = \sqrt{3k^2} = k\sqrt{3}$$

Thus $$\cos\theta = \frac{k^2}{k^2\sqrt{3}} \rightarrow \cos\theta = \frac{1}{\sqrt{3}} \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \rightarrow \theta \approx 54.74°.$$

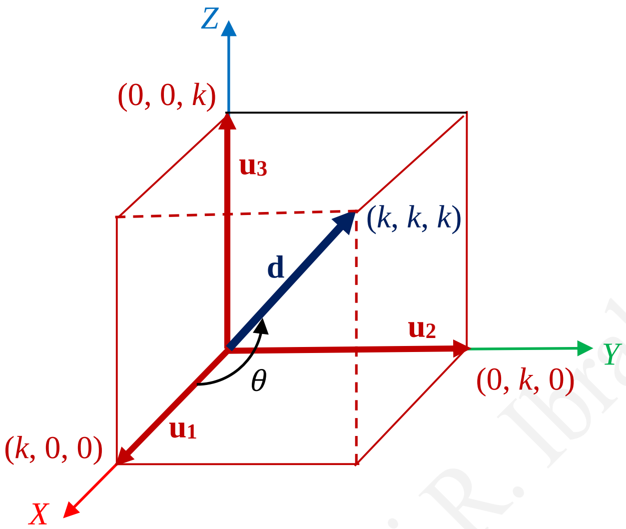

Figure 27

Theorem (2): Let **u** and **v** be vectors in 2- or 3-space.

a) $\mathbf{v}\,.\,\mathbf{v} = \|\mathbf{v}\|^2$; that is, $\|\mathbf{v}\| = (\mathbf{v}\,.\,\mathbf{v})^{1/2}$.

b) If the vectors **u** and **v** are two nonzero vectors and θ the angle between them, then

θ is acute if and only if $\mathbf{u}\,.\,\mathbf{v} > 0$.

θ is obtuse if and only if $\mathbf{u}\,.\,\mathbf{v} < 0$.

$\theta = \pi/2$ if and only if $\mathbf{u}\,.\,\mathbf{v} = 0$.

Proof (a):

The angle between **v** and **v** is 0, and by formula (1), we have

$$\mathbf{v}\,.\,\mathbf{v} = \|\mathbf{v}\|\|\mathbf{v}\| \cos\theta$$

$$= \|\mathbf{v}\|^2 \cos 0$$

$$= \|\mathbf{v}\|^2.$$

Proof (b):

Using formula (1):

$$\mathbf{u}\,.\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \rightarrow \cos\theta = \frac{\mathbf{u}\,.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|},\ \|\mathbf{u}\| > 0 \text{ and } \ \|\mathbf{v}\| > 0.$$

Since θ satisfies $0 \le \theta \le \pi$, it becomes clear the following:

1- θ is **acute** if and only if $\cos\theta > 0$ and this true if and only if **u . v** > 0.
2- θ is **obtuse** if and only if $\cos\theta < 0$ and this true if and only if **u . v** < 0.
3- $\theta = \pi/2$ (right angle) if and only if $\cos\theta = 0$ and this true if and only if **u . v** = 0.

*We can see that in **(figure 22)** page 17.*

Orthogonal vectors:

By theorem 2 (b) we denote that the two nonzero vectors **u** **a**nd **v** are **orthogonal (perpendicular)** if and only if their dot product is zero (**u . v**= 0) and that is true also if either (or both) of these vectors is zero, therefore we can state without exception that two vectors **u** and **v** are orthogonal if and only if **u . v**= 0 and denoted by (**u** ⊥ **v**).

Example (27): If **u**= <1, −2, 3>, **v**= <−3, 4, 2> and **w**= <3, 6, 3> are three vectors in 3-space, determine the type of the angle between each pair of these vectors.

Solution:

u . v= $(1)(-3) + (-2)(4) + (3)(2) = -5 \rightarrow$ the angle between **u** and **v** is obtuse.

u . w= $(1)(3) + (-2)(6) + (3)(3) = 0 \rightarrow$ the angle between **u** and **w** is a right angle (90°), **u** and **w** are perpendicular vectors (**u** ⊥ **w**).

v . **w**= $(-3)(3) + (4)(6) + (2)(3) = 21 \rightarrow$ the angle between **v** and **w** is acute.

Example (28): (A vector perpendicular to a line):

Show that the vector **u**= <a, b> in 2-space is perpendicular to the line $ax + by + c = 0$.

Solution:

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be any two points on the line such that.

$$ax_1 + by_1 + c = 0 \qquad \ldots\ldots (1)$$

$$ax_2 + by_2 + c = 0 \qquad \ldots\ldots (2)$$

We can obtain the vector $\overrightarrow{P_1 P_2}$ by subtracting the coordinates of P_1 from the coordinates of P_2, such that $\overrightarrow{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$ and this vector passes along the line.

Now to show that the vector **u** is perpendicular to a line, we must only proof that **u** . $\overrightarrow{P_1 P_2} = 0$

Subtracting the equation 1 from 2 we obtain:

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

and this represents the dot product of the two vectors **u** and $\overrightarrow{P_1 P_2}$.

Thus, **u .** $\overrightarrow{P_1 P_2} = 0$ and the vector is perpendicular to the line as shown in **(figure 28).**

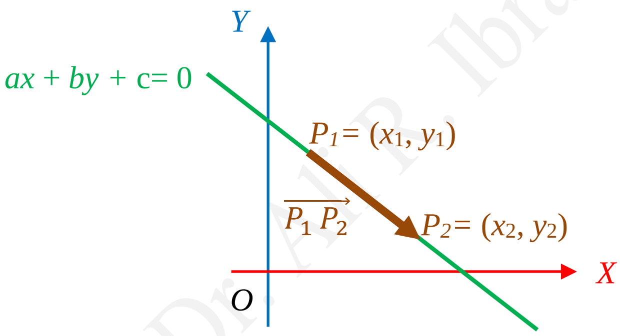

Figure 28

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4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.