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lectures
Subject: <u>Vector analysis.</u>
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An orthogonal projection (Principle of vector analysis):

In many applications it is of interest to **decompose** a vector (as $\mathbf{u}$) into a sum of two terms, one parallel to a specified nonzero vector (as $\mathbf{a}$) and the perpendicular to $\mathbf{a}$. if $\mathbf{u}$ and $\mathbf{a}$ are positioned so that their initial points coincide at a point Q, we can decompose the vector $\mathbf{u}$ as shown in **(figure 31)**. Drop a perpendicular from the tip (**terminal point**) of the vector $\mathbf{u}$ to the line through $\mathbf{a}$, and construct the vector $\mathbf{w}_1$ from Q to the tail of this perpendicular. We not that the deference between two vectors $\mathbf{u}$ and $\mathbf{w}_1$ forms a vector **perpendicular** (as $\mathbf{w}_2$) to the vector $\mathbf{a}$, which can be placed in the following formula:

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 \quad \ldots (1)$$

Thus, the vector $\mathbf{w}_1$ is **parallel** to the vector $\mathbf{a}$, and the vector $\mathbf{w}_2$ is **perpendicular** to the vector $\mathbf{a}$, and this can be formulated as follows:

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u} \quad \ldots (2)$$

The vector $\mathbf{w}_1$ is called the **orthogonal projection of u on a** or the **vector component of u along a** and it is denoted by:

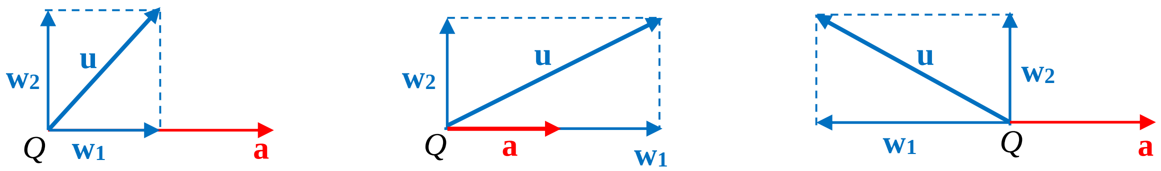
$$\boxed{\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u}} \quad \ldots (3)$$

Also The vector $\mathbf{w}_1$ is denoted by the symbol $\mathbf{u}\|$.

The vector $\mathbf{w}_2$ is called the **vector component of u orthogonal to a** and since $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$, then the vector $\mathbf{w}_2$ is denoted by:

$$\boxed{\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}} \quad \ldots (4)$$

Also The vector $\mathbf{w}_2$ is denoted by the symbol $\mathbf{u}\perp$.



(a) (b) (c)

Figure 31

(The vector $\mathbf{u}$ is the sum of $\mathbf{w}_1$ and $\mathbf{w}_2$, $\mathbf{w}_1$ is parallel to $\mathbf{a}$ and $\mathbf{w}_2$ is perpendicular to $\mathbf{a}$)

Now we take the theory through which we will get to know the formulas for calculating each of the two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$.

Theorem (3):

If $\mathbf{u}$ and $\mathbf{a}$ are two vectors in 2-or 3-space and $\mathbf{a} \neq 0$, then

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a})$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a}).$$

Proof:

Let $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$, since $\mathbf{w}_1$ is parallel to $\mathbf{a}$, it must be a scalar multiple of $\mathbf{a}$, so it can be written in the form $\mathbf{w}_1 = k\mathbf{a}$ (k any scalar). Thus

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = k\mathbf{a} + \mathbf{w}_2 \quad \dots (1)$$

Now we taking the dot product of both sides of (1) with $\mathbf{a}$, we obtain:

$$\mathbf{u} \cdot \mathbf{a} = (k\mathbf{a} + \mathbf{w}_2) \cdot \mathbf{a}$$

$$= k(\mathbf{a} \cdot \mathbf{a}) + \mathbf{w}_2 \cdot \mathbf{a}$$

$$= k\|\mathbf{a}\|^2 + \mathbf{w}_2 \cdot \mathbf{a} \quad \dots (2)$$

Since $\mathbf{w}_2$ is perpendicular to the vector $\mathbf{a}$, thus $\mathbf{w}_2 \cdot \mathbf{a} = 0$, and we rewrite (2) as follows:

$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}$$

Since $\text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{w}_1 = k\mathbf{a}$, we obtain

$$\boxed{\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}} \quad \dots (3)$$

(vector component of **u** *along* **a***)*

and,
$$\boxed{\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2}\,\mathbf{a}} \quad \dots (4)$$

(vector component of **u** *orthogonal to* **a***)*

$$\mathbf{u} = \mathbf{u}\| + \mathbf{u}\perp$$

$$\mathbf{u}\| = \text{proj}_{\mathbf{a}}\mathbf{u}$$

$$\mathbf{u}\perp = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$$

Example (40): Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a**, if **u**= <2, −1, 3> and **a**= <4, −1, 2>.

Solution:

The vector component of **u** along **a** (**u**‖) is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2}\,\mathbf{a}$, So in the beginning we should find both **u** . **a** and $\|\mathbf{a}\|^2$.

$$\mathbf{u}\,.\,\mathbf{a} = (2)(4) + (-1)(--1) + (3)(2) = 15;$$

$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21;$$

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{15}{21} <4, -1, 2>$$

$$= <\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}>.$$

The vector component of **u** orthogonal to **a** (**u**⊥) is $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2}\,\mathbf{a}$, thus

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = <2, -1, 3> - <\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}>$$

$$= <\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7}>.$$

We can check the solution as follow:

Using the vector addition principle:

$$\mathbf{u} = \mathbf{u}_{\|} + \mathbf{u}_{\perp}$$

$$= <\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}> + <\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7}>.$$

$$= <2, -1, 3> = \mathbf{u}.$$

Also, we can use the concepts of parallelism and orthogonality to validate the solution as shown below:

Using the principle of parallelism:

The vector component of **u** along **a**, it means that the two vectors $\text{proj}_{\mathbf{a}}\mathbf{u}$ and **a** are parallel and this is true if and only if $\text{proj}_{\mathbf{a}}\mathbf{u} \cdot \mathbf{a} = \|\text{proj}_{\mathbf{a}}\mathbf{u}\|\|\mathbf{a}\|$, when the two vectors are parallel and in the same direction, and if and only if $|\text{proj}_{\mathbf{a}}\mathbf{u} \cdot \mathbf{a}| = \|\text{proj}_{\mathbf{a}}\mathbf{u}\|\|\mathbf{a}\|$ when the two vectors are parallel and in the opposite direction.

Now, $\text{proj}_{\mathbf{a}}\mathbf{u} \cdot \mathbf{a} = <\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}> \cdot <4, -1, 2> = 15;$

$$\|\mathbf{a}\|^2 = 21 \rightarrow \|\mathbf{a}\| = \sqrt{21}, \quad \|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \sqrt{(\tfrac{20}{7})^2 + (\tfrac{-5}{7})^2 + (\tfrac{10}{7})^2} = \sqrt{10.714}$$

Since $\text{proj}_{\mathbf{a}}\mathbf{u} \cdot \mathbf{a} = \|\text{proj}_{\mathbf{a}}\mathbf{u}\|\|\mathbf{a}\|$, then the two vectors are **parallel and in the same direction**.

Also we can use this principle "two nonzero vectors are **parallel** if they are scalar multiples of one another" to check whether the two vectors $\text{proj}_{\mathbf{a}}\mathbf{u}$ and **a** are parallel.

Using the principle of orthogonality:

The vector component of **u** orthogonal to **a**, it means that the two vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and **a** are orthogonal, and this is true if and only if $(\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}) \cdot \mathbf{a} = 0$.

$$(\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}) \cdot \mathbf{a} = <\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7}> \cdot <4, -1, 2>$$

$= 0$, Thus the two vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and **a** are **orthogonal.**

Also, we can verify by the dot product of the vectors $\text{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$, because the product of their dot product must be zero if the solution is correct.

Example (41): find the vector component of the vector **v** along **u** and the vector component of **v** orthogonal to **u**, if **v**= <2, −1, 4> and **u**= <1, 2, 3>.

Solution:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\,.\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}, \quad \mathbf{v}\,.\,\mathbf{u} = 12, \quad \|\mathbf{u}\| = \sqrt{14} \rightarrow \|\mathbf{u}\|^2 = 14$$

$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{12}{14} <1, 2, 3> = <\frac{6}{7}, \frac{12}{7}, \frac{18}{7}>$ the vector component of **v** along **u** ($\mathbf{v}\|$).

$$\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v} = <2, -1, 4> - <\frac{6}{7}, \frac{12}{7}, \frac{18}{7}>$$

$= <\frac{8}{7}, \frac{-19}{7}, \frac{10}{7}>$ the vector component of **v** orthogonal to **u** ($\mathbf{v}\perp$).

It is possible to verify the validity of the solution by using one of the properties we mentioned in the previous example.

Example (42): If the vector **u**= 2**i** + 4**j** + 2**k**, is the sum of vectors parallel and perpendicular to **v**= **i** + 2**j** – **k**, find the vector component of **u** along **v** and the vector component of **u** orthogonal to **v** then check that the solution is correct . (***or find the parallel and perpendicular vectors***).

Solution:

We must find the parallel vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ and the perpendicular vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{u}\,.\,\mathbf{v} = (2)(1) + (4)(2) + (2)(-1) = 8, \quad \|\mathbf{v}\|^2 = 1^2 + 2^2 + (-1)^2 = 6;$$

$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\,.\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{4}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ or $\frac{4}{3} <1, 2, -1>$ the vector component of **u** along **v** ($\mathbf{u}\|$).

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = (2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) - \frac{4}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

or $\quad = <2, 4, 2> - \frac{4}{3} <1, 2, -1>$

$= \frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{10}{3}\mathbf{k} = \frac{2}{3}(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ or $= \frac{2}{3} <1, 2, 5>$. the vector component of **u** orthogonal to **v** ($\mathbf{u}\perp$).

Now to verify the correctness of the solution, we can perform the dot product of the two vectors $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

$\mathbf{u}_{\parallel} \,.\, \mathbf{u}_{\perp}=$

or $\quad \text{proj}_{\mathbf{v}}\mathbf{u} \,.\, (\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) = \frac{4}{3} <1, 2, -1> \,.\, \frac{2}{3} <1, 2, 5>$

$= 0$, So $\mathbf{u}_{\parallel}$ and $\mathbf{u}_{\perp}$ are orthogonal vectors.

Example (43): If **u**= <2, 4> and **v**= <−1, 2>, then:

1- find $\text{proj}_{\mathbf{v}}\mathbf{u}$.
2- analyze **u** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v** (perpendicular). (**Homework**).

A formula for the length of the vector component of any vector along another vector:

If **u** and **a** are two nonzero vector, then the formula for the length of the vector component of **u** along **a** can be obtained as following:

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \left\| \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2}\, \mathbf{a} \right\|$$

$$= \left| \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2} \right| \|\boldsymbol{a}\| \quad \{\text{ the length of the vector } \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2}\, \mathbf{a} \text{ is } \left| \frac{\mathbf{u}\,.\mathbf{a}}{\|\mathbf{a}\|^2} \right| \text{ times the length of } \mathbf{a}\}$$

$$= \frac{|\mathbf{u}\,.\mathbf{a}|}{\|\mathbf{a}\|^2} \|\boldsymbol{a}\| \quad \{\text{ Since } \|\mathbf{a}\|^2 \geq 0\}$$

Thus,

$$\boxed{\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\,.\mathbf{a}|}{\|\boldsymbol{a}\|}} \qquad \ldots (1)$$

We know if θ is the angle between the two vectors **u** and **a**, then the dot product **u** . **a**= $\|\mathbf{u}\|\|\boldsymbol{a}\| \cos\theta$, thus we can write (1) as:

$$\boxed{\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \|\mathbf{u}\||\cos\theta|} \qquad \ldots (2)$$

The figure (**32**) below shows the geometric explanation of this result.

$0 \le \theta < \pi/2$ $\qquad$ $\pi/2 < \theta \le \pi$

Figure 32

The geometric explanation for the length of the vector component of **u** along **a**, if θ is the angle between them

References

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4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.