Ministry of Higher Education
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lectures
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Distance between a point and a line:

The **distance between a point and a line,** is defined as the shortest distance between a fixed point and any point on the line. It is the length of the line segment that is perpendicular to the line and passes through the point.

We will use vector methods to derive a formula for the distance between a point in the plane to a line.

Now for example to find a formula for the distance between a point $P_0(x_0, y_0)$ and the line $ax + by + c=0$. Let $Q= (x_1, y_1)$ be any point on the line, and position the vector $\mathbf{u}= <a, b>$, so that its initial point is at Q as shown in (**figure 33**).

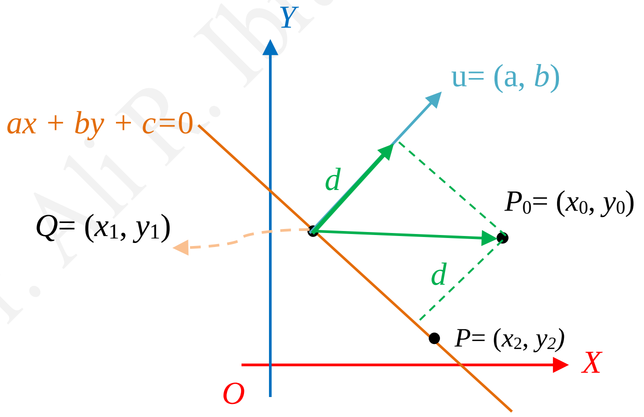

Figure 33

At the first we must show that the nonzero vectors **u** is perpendicular to the line $ax +by +c= 0$.

Let $P= (x_2, y_2)$ any point on the line and we have the point $Q= (x_1, y_1)$, also 0n the line, so that

$$ax_1 +by_1 +c= 0$$
$$ax_2 +by_2 +c= 0$$
$$\overline{}$$
$$a(x_2 - x_1) + b(y_2 - y_1)= 0$$

$$\text{so, } <a, b> . < x_2 - x_1, y_2 - y_1>= 0$$

$< x_2 - x_1, y_2 - y_1>$ is the vector $\overrightarrow{QP}$ along the line.

Thus $\mathbf{u} . \overrightarrow{QP}= 0$, and **u** is perpendicular to the line as shown in the figure (33).

In this figure we note that the distance d between the point P_0 and the line, is the **length of the orthogonal projection** of the vector $\overrightarrow{QP_0}$ on the vector **u**, thus

$$d = \|\text{proj}_{\mathbf{u}}\overrightarrow{QP_0}\| = \frac{|\overrightarrow{QP_0}.\mathbf{u}|}{\|\mathbf{u}\|};$$

$$\overrightarrow{QP_0} = \langle x_0 - x_1, y_0 - y_1 \rangle;$$

$$\overrightarrow{QP_0}\,.\,\mathbf{u} = a(x_0 - x_1) + b(y_0 - y_1);$$

$$\|\mathbf{u}\| = \sqrt{a^2 + b^2};$$

So,

$$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2+b^2}} \quad \dots (1)$$

$$d = \frac{|ax_0 - ax_1 + by_0 - by_1|}{\sqrt{a^2+b^2}} \quad \dots (2)$$

Since the point $Q = (x_1, y_1)$ lies on the line, then its coordinates satisfy the equation of the line, so

$$ax_1 + by_1 + c = 0 \rightarrow c = -ax_1 - by_1$$

substituting that in (2), we obtain the **formula of the distance from a point to a line**,

$$\boxed{d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}} \quad \dots (3)$$

Note: The distance formula can be reduced to a simpler form if the point is at the **origin** as:

$$d = \frac{|a(0) + b(0) + c)|}{\sqrt{a^2+b^2}} = \boxed{\frac{|c|}{\sqrt{a^2 + b^2}}} \quad \dots (4)$$

Example (44): Find the distance between the line $2x + 4y - 5 = 0$ and the point $P = (-3, 2)$.

Solution:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

$$= \frac{|2(-3)+4(2)-5|}{\sqrt{2^2+4^2}} = \frac{3}{2\sqrt{5}}.$$

Example (45): Find the distance *d* for the following cases.

a) From the point $P= (1, -2)$ to the line $3x + 4y - 6= 0$.
Solution:
$d= \frac{11}{5}$ **(check).**

b) Between the point $P= (-3, 7)$ and the line $y= \frac{6}{5}x + 2$. (**Homework**).

Example (46): Find the orthogonal projection of **u** on **a**, the vector component of **u** orthogonal to **a** and the length of the orthogonal projection $\text{proj}_{\mathbf{a}}\mathbf{u}$, for all the pair of the vectors in the following. (**Homework**).

a) **u**= <6, 2>, **a**= <3, −9>.
b) **u**= <−1, −2>, **a**= <−2, 3>.
c) **u**= <3, 1, −7>, **a**= <1, 0, 5>.
d) **u**= <1, 0, 0>, **a**= <4, 3, 8>.

Cross product:

We know that the dot product of two vectors in 2-or 3-space produces a scalar. Now we define a type of vector multiplication (only in 3-space) that produces a vector as the product and this is called the cross product.

Definition (12): If $\mathbf{u}= <u_1, u_2, u_3>$ and $\mathbf{v}= <v_1, v_2, v_3>$ are two vectors in 3-space, then the cross product $\mathbf{u} \times \mathbf{v}$ is the vector defined by:

$$\mathbf{u} \times \mathbf{v}= <u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1>$$

Or in determinant notation,

$$\mathbf{u} \times \mathbf{v}= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right) \quad \dots (1).$$

Also we can obtain the components of the vector $\mathbf{u} \times \mathbf{v}$ as follows:

Taking the 2 × 3 matrix $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$, first row contains the components of the vector **u** and the second row contain the components of the vector **v**.

Now to find the **first component** of the vector **u** × **v**, delete the first column and take the determinant; to find the **second component**, delete the second column and take **the negative** of the determinant; and to find the **third component**, delete the third column and take the determinant.

Example (47): find the cross product (**u** × **v**) of the tow vectors **u**= <1, -2, 2> and **v**= <3, 0, 1>.

Solution:

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} \right)$$

$$= <-2, 5, 6>.$$

Theorem (4): (Relationships involving cross product and dot product).

If **u**, **v** and **w** are vectors in 3-space, then:

a) **u** . (**u** × **v**)= 0 (**u** × **v** is **orthogonal** to **u**).
b) **v** . (**u** × **v**)= 0 (**u** × **v** is **orthogonal** to **v**).
c) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (**Lagrange's identity**).
d) **u** × (**v** × **w**)= (**u** . **w**)**v** – (**u** . **v**)**w** (**relationships between cross and dot product**).
e) (**u** × **v**) × **w**= (**u** . **w**)**v** – (**v** . **w**)**u** (**relationships between cross and dot product**).

Proof (a):

Let **u**= $<u_1, u_2, u_3>$ and **v**= $<v_1, v_2, v_3>$, then

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = <u_1, u_2, u_3> \cdot <u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)>$$

$$= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$

Proof (b): Similar to (a).

Proof (c): We will proof this part later, using the concept of scalar triple product that is

(**u** × **v**) . **w**= **u** . (**v** × **w**) or **u** . (**v** × **w**)= **w** . (**u** × **v**).

Example (47): <u>(**u** × **v** is perpendicular to **u** and **v**).</u>

If **u**= <1, 2, -2> and **v**= <3, 0, 1>, show that **u** × **v** is perpendicular to **u** and **v**.

Solution:

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u}\times\mathbf{v}=\left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\right)$$

$= <2, -7, -6>$

$\mathbf{u}\,.\,(\mathbf{u}\times\mathbf{v})= (1)(2) + (2)(-7) + (-2)(-6)$
$= 0$

$\mathbf{v}\,.\,(\mathbf{u}\times\mathbf{v})= (3)(2) + (0)(-7) + (1)(-6)$
$= 0$

$\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** an **v**.

Theorem (5) (properties of cross product):

If **u**, **v**, and **w** are three vectors in 3-space and let k is any scalar, then:

a) $\mathbf{u}\times\mathbf{v}= -(\mathbf{v}\times\mathbf{u})$
b) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})= (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$.
c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}= (\mathbf{u}\times\mathbf{w}) + (\mathbf{v}\times\mathbf{w})$.
d) $k(\mathbf{u}\times\mathbf{v})= (k\mathbf{u})\times\mathbf{v}= \mathbf{u}\times(k\mathbf{v})$.
e) $\mathbf{u}\times\mathbf{0}= \mathbf{0}\times\mathbf{u}= 0$.
f) $\mathbf{u}\times\mathbf{u}= 0$.

Standard unit vectors:

The vectors, $\mathbf{i}= <1, 0, 0>$, $\mathbf{j}= <0, 1, 0>$, $\mathbf{k}= <0, 0, 1>$ are called the standard unit vectors in 3-space and the length of each of these vectors is 1 and lie along coordinate axes as shown in (**figure 34**).

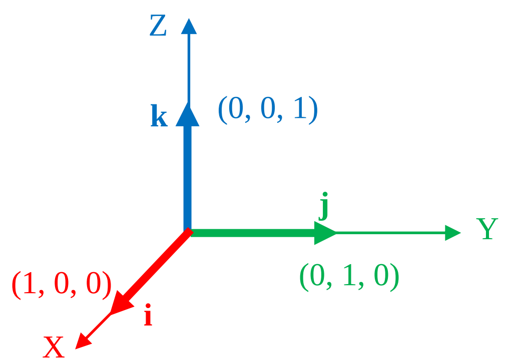

Figure 34

The standard unit vector

Every vector in 3-space is expressible in terms of **i**, **j**, and **k**, for example if **v**= <v_1, v_2, v_3>, then we can write this vector as follows:

$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = v_1\langle 1, 0, 0\rangle + v_2\langle 0, 1, 0\rangle + v_3\langle 0, 0, 1\rangle$$

$$= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \text{ (As linear combination).}$$

Thus if **v**= <4, 1, -7>, then we can write it in this form: **v**= 4**i** + **j** – 7**k**.

Now if we want to find the cross product of two unit vectors **i** and **j** using formula (1) in definition of cross product, we obtain:

$$\mathbf{i} \times \mathbf{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right)$$

$$= \langle 0, 0, 1\rangle$$

$$= \mathbf{k}.$$

Thus and using the same method we obtain the following results:

$$\mathbf{i} \times \mathbf{i} = 0 \qquad \mathbf{j} \times \mathbf{j} = 0 \qquad \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Figure 35 below, helps us to remember the above results, where the resulting vector is positive if the cross product of two consecutive goes clockwise, and is negative if the cross product goes counterclockwise.

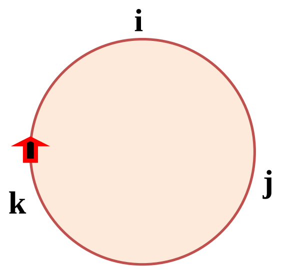

Figure 35

Direction of cross product

Determinant form of cross product:

The cross product can be represented symbolically in the form of a formal 3×3 determinant as follows:

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}=\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \quad \ldots (2)$$

Example (48): If **u**= <3, 1, 1> and **v**= <2, 1, 2>, find **u** × **v**.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{k}= \ = \mathbf{i}-4\mathbf{j}+\mathbf{k}.$$

Note: It is not true in general that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

For example, $\mathbf{i}\times(\mathbf{j}\times\mathbf{j})=\mathbf{i}\times 0=0$ and $(\mathbf{i}\times\mathbf{j})\times\mathbf{j}=\mathbf{k}\times\mathbf{j}=-\mathbf{i}$

Thus $\mathbf{i}\times(\mathbf{j}\times\mathbf{j})\neq(\mathbf{i}\times\mathbf{j})\times\mathbf{j}$.

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4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.