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Direction of cross product:

If **u** and **v** are two nonzero vectors in 3-space, then to determine the direction of **u**×**v**, we can use the “right-hand rule” when **u**×**v** is orthogonal to both **u** and **v**. As shown in **(figure 36)** if θ is the angle between **u** and **v**, and if **u** is rotated through the angle θ until it coincides with **v** then the fingers of the right hand are cupped such that they point in the direction of rotation and the thumb indicate to the direction of **u**×**v**.

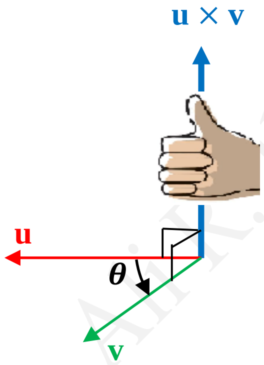

Figure 36

The direction of **u**×**v**

Example (49): (Relationships involving cross product and dot product)

Given the vectors **u**= **i** + **j** +**k** and **v**= 2**i** – **j** – **k**.

a) Show that **u** and **v** are orthogonal (perpendicular).
b) Find a unit vector perpendicular to both **u** and **v**.

Solution:

a) If **u** . **v**= 0, then **u** and **v** are perpendicular.
   u . **v**= <1, 1, 1> . <2, −1, −1>
   = 0 → **u** and **v** are perpendicular.
b) The cross product **u**×**v** is perpendicular to both **u** and **v**.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-1 + 1)\mathbf{i} - (-1 - 2)\mathbf{j} + (-1 - 2)\mathbf{k}$$

$$= 3\mathbf{j} - 3\mathbf{k}.$$

Now, the unit vector of **u** × **v** also perpendicular to the vectors **u** and **v**, thus

$$\mathbf{U}=\frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}=\frac{<3,-3>}{\sqrt{3^2+(-3)^2}}=\frac{<3,-3>}{3\sqrt{2}}=<\frac{3}{3\sqrt{2}},\frac{-3}{3\sqrt{2}}>=<\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}>=<\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}>.$$

Or $\mathbf{U}=\frac{\sqrt{2}}{2}\,\mathbf{j}-\frac{\sqrt{2}}{2}\,\mathbf{k}$ the unit vector that is perpendicular to both **u** and **v**.

Definition (13) (Scalar triple product u . (v×w)):

If **u**, **v** and **w** are three vectors in 3-space (3D), then **u** . (**v** × **w**) is called the **scalar triple product** of **u**, **v** and **w** and denoted by the following formula:

$$\boxed{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} \quad \dots (3)$$

Proof this formula:

We know that $\mathbf{v}\times\mathbf{w}=\begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}=\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k};$

Thus, $\quad \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k}\right)$

$$=\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}u_1-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}u_2+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}u_3$$

$$=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example (50): Calculate the scalar triple product **u** . (**v** × **w**), where **u**= 3**i** – 2**j** −5**k**, **v**= **i** + 4**j** −4**k**, and **w**= 3**j** + 2**k**.

Solution:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$=3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}+(-5)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$= 60 + 4 - 15$

$= 49$

Note: The symbol (**u** . **v**) × **w** makes no sense because we cannot form the cross product of a scalar and a vector. Thus no problem arises if we write **u** . **v** × **w** rather than **u** . (**v** . **w**).

Thus we can write the formula (3) in definition (13) (**Scalar triple product**) as follows:

$$\boxed{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})}$$

or $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (*Dot product is Commutative process*)

Example (51): If **u**= <1, 2, 3>, **v**= <2, 1, 1>, and **w**= <1, 1, 0>, find each of the following separately to show that they are equal in all cases.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}); \;\; \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}); \;\; \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$$

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 4$$

$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 4$ (Check)

$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = 4$ (Check)

Proof of Lagrange's identity:

Now we can proof **Lagrange's identity** (part c) in theorem (4) (**Relationships involving cross product and dot product**).

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof:

We know if **v** is any vector (in 2-or 3-space), then the dot product $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

Thus, $\|\mathbf{u} \times \mathbf{v}\|^2 = (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v})$.

Since, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$; (*scalar triple product*)

Thus, $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{u} \times \mathbf{v})]$;

Since, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (*relationships between cross and dot product*) (part 4 theorem 4).

Thus, $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{u} \times \mathbf{v})]$

$$= \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{v})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{v}]$$

$$= (\mathbf{v} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{u})$$

$$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{v} \cdot \mathbf{u})^2.$$

Geometric interpretation of cross product:

If **u** and **v** are two nonzero vectors in 3-space(3D), then the **norm** of $\mathbf{u} \times \mathbf{v}$ has a useful geometric explanation.

The magnitude (norm) or length of $\mathbf{u} \times \mathbf{v}$:

Using LaGrange's identity,

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \quad \ldots (1)$$

If θ denotes the angle between two vectors **u** and **v**, then

The dot product of the two vectors **u** and **v** is,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \quad \ldots (2)$$

(1) in (2) we obtain:

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2\theta$$

$$= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2\theta)$$

$$= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2\theta \quad \ldots (3)$$

Since $0 \le \theta \le \pi$, it is follows that $\sin\theta \ge 0$.

Taking the square root for two sides of (3), we obtain:

$$\boxed{\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta} \quad \ldots (4)$$

We note that the two vectors **u** and **v** are parallel if and only if $\mathbf{u} \times \mathbf{v} = 0$, because by the formula (4) we obtain:

$\sin\theta = \dfrac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}$, $\|\mathbf{u}\|\|\mathbf{v}\| > 0$ because **u** and **v** are nonzero vectors.

Thus $\sin\theta = 0 \rightarrow \theta = 0$ and the two vectors are parallel.

The norm of u × v and applications:

In the figure (37) below, we easily note that $\|\mathbf{v}\|\sin\theta$ is the altitude (height) of the parallelogram determined by **u** and **v**.

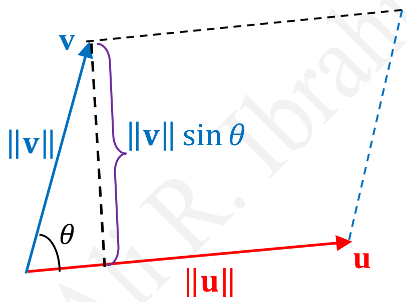

Figure 37

$\|\mathbf{v}\|\sin\theta$ is the altitude of the parallelogram.

Thus, the formula (4) above represent the area A of the parallelogram.

The area of the parallelogram (A_P) = (base)(altitude)= $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \|\mathbf{u}\times\mathbf{v}\|$.

Therefore we note that the area of the parallelogram determined by **u** and **v** is zero when these vectors are parallel (or collinear) because $\theta = 0$ and this leads to $\mathbf{u}\times\mathbf{v} = 0$.

Applications of cross product:

1) **To find the area of the parallelogram**.

Theorem (5): The area of parallelogram determined by two vectors in 3-space, is the magnitude (length) of the cross product of these vectors.

Or (if **u** and **v** are vectors in 3-space, then $\|\mathbf{u}\times\mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**.

If we have the parallelogram denoted by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ as shown in (figure 38):

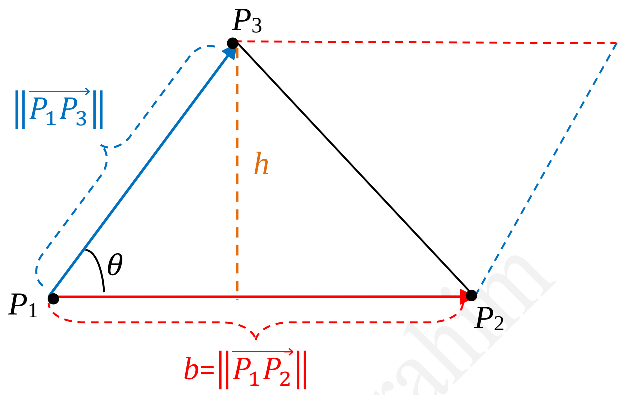

Figure 38

The parallelogram determined by two vectors

We know that the area of the parallelogram A_P= (base)(altitude)= bh.

The length of the vector $\overrightarrow{P_1P_2}$ represents the base of the parallelogram, thus

$$A_P = \|\overrightarrow{P_1P_2}\|h, \quad (\|\overrightarrow{P_1P_2}\| = \|P_2 - P_1\|)$$

Since, $\sin\theta = \dfrac{h}{\|\overrightarrow{P_1P_3}\|} \rightarrow h = \|\overrightarrow{P_1P_3}\|\sin\theta, \quad (\|\overrightarrow{P_1P_3}\| = \|P_3 - P_1\|)$

Therefore, $\qquad A_P = \|\overrightarrow{P_1P_2}\|\|\overrightarrow{P_1P_3}\|\sin\theta$

Since $\|\overrightarrow{P_1P_2}\|\|\overrightarrow{P_1P_3}\|\sin\theta$ represents the length (magnitude) of $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$, according to the formula (4), then **the area of the parallelogram is:**

$$\boxed{A_P = \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|}$$

2) To find the area of tr

If we have the triangle determined by the points P_1, P_2, and P_3 as shown in figure (39), then we can find the area of this triangle using the concepts of vectors.

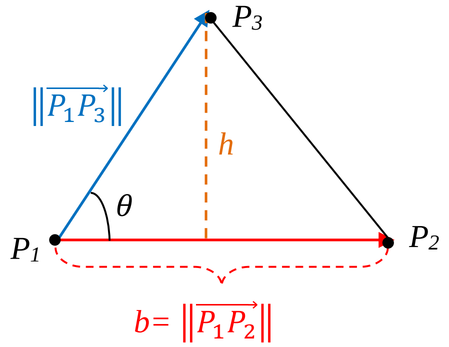

Figure 39

The area of the triangle $A_T = \frac{1}{2}$(base)(height)

$$= \frac{1}{2}bh$$

The length of the vector $\overrightarrow{P_1P_2}$ represents the base of the triangle, thus

$$A_T = \frac{1}{2}\|\overrightarrow{P_1P_2}\|h, \quad \|\overrightarrow{P_1P_2}\| = \|P_2 - P_1\|$$

Since, $\sin\theta = \dfrac{h}{\|\overrightarrow{P_1P_3}\|} \rightarrow h = \|\overrightarrow{P_1P_3}\|\sin\theta, \quad \|\overrightarrow{P_1P_3}\| = \|P_3 - P_1\|$

Thus, $$A_T = \frac{1}{2}\|\overrightarrow{P_1P_2}\|\|\overrightarrow{P_1P_3}\|\sin\theta,$$

Since $\|\overrightarrow{P_1P_2}\|\|\overrightarrow{P_1P_3}\|\sin\theta$ represents the length (magnitude) of $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$, according to the formula (4), then, **the area of the triangle is**:

$$\boxed{A_T = \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|}$$

We note that the area of the parallelogram A_P is equal multiple the area of the triangle and that means,

$$\boxed{A_P = 2A_T}$$

Example (52): Find the area of the parallelogram determined by $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, if $P_1 = (2, 2, 0)$, $P_2 = (-1, 0, 2)$ and $P_3 = (0, 4, 3)$.

Solution: we must find the area of the parallelogram in figure (40).

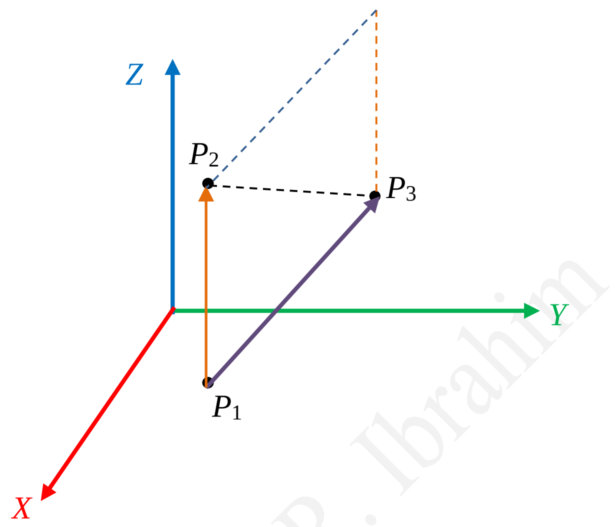

Figure 40

$A_P = \left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$

$\overrightarrow{P_1P_2} = P_2 - P_1 = (-1, 0, 2) - (2, 2, 0) = <-3, -2, 2>;$

$\overrightarrow{P_1P_3} = P_3 - P_1 = (0, 4, 3) - (2, 2, 0) = <-2, 2, 3>;$

Now we must find $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$.

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & -2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= (-6 - 4)\mathbf{i} - (-9 + 4)\mathbf{j} + (-6 - 4)\mathbf{k}$$

$$= -10\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$$

Or $= <-10, 5, -10>$

$A_P = \left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$

$= \sqrt{(-10)^2 + 5^2 + (-10)^2}$

$= \sqrt{225}$

$= 15$ the area of the parallelogram.

Example (53): Find the area of the triangle determined by the points, P_1= (2, 2, 0), P_2= (−1, 0, 2) and P_3= (0, 4, 3).

Solution:

$$A_T = \frac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$$

Since we use the same points in example (52), therefore $\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$= 15, and

$A_T = \frac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\| = \frac{1}{2}(15) = 7.5$ the area of the triangle.

Or directly using the formula, $A_P = 2A_T \rightarrow A_T = \frac{1}{2}A_P$, if it is required for the same points to find the area of the tringle and parallelogram.

Now we can formulate the following questions:

Question (1): Derive the formula that used to find the area of the parallelogram determined by two vectors.

Question (2): Derive the formula that used to find the area of the triangle determined by three points.

References

1- Introductory linear algebra with applications, Bernard Kolman, first edition, 1976.

2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

3- Elementary Linear Algebra, Ninth Edition, Howard Anton, Chris Rorres, 2005.

4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.