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vector spaces:

Definition (1): Let V be an arbitrary nonempty set of objects on which two operations are defined, addition and multiplication by scalars (numbers), that is If **u**, **v**, and **w** are vectors in V and k, c are any scalars in $\mathbb{R}$, then if the addition of two vectors **u** + **v** (is called the **sum** of **u** and **v**) and the scalar multiplication k**u** (is called the **scalar multiple** of **u** by k) are defined then we call V (the set of vectors) or (V, ⊕, ⊙) is a **vector space** if the following ten vector space axioms are satisfied.

The notations ⊕ and ⊙ for vector addition and scalar multiplication to distinguish these operations from addition and multiplication of real numbers.

Five addition axioms:

1- If **u** and **v** are two vectors in V, then **u** + **v** ∈ V (**Closed under addition**).
2- **u** + **v**= **v** + **u** (**Commutative**).
3- **u** + (**v** + **w**)= (**u** + **v**) + **w** (**Associative**).
4- ∃ zero vector **0** (**an addition identity**) in V such that, for all **u** ∈ V, **0** + **u**= **u** + **0**= **u**.
5- ∀ **u** ∈ V, ∃ −**u** ∈ V (the negative (or an additive inverse) of **u**) such that, **u** + (-**u**)= (-**u**) + **u**= **0**.

Five scalar multiplication axioms:

6- If k is any scalar ($k \in \mathbb{R}$) and **u** ∈ V, then k**u** ∈ V (**Closed under scalar multiplication**).
7- k(**u** + **v**)= k**u** + k**v** (**Distributive**).
8- (k + c)**u**= k**u** + c**u**.
9- (kc)**u**= k(c**u**)= c(k**u**).
10- 1**u**= **u**.

Note: Depending on the application, scalars may be real numbers or complex numbers. Vector spaces in which the scalars are complex numbers are called **complex vector spaces**, and those in which the scalars are real numbers are called **real vector spaces**, which are the subject of our study at the present time.

Examples of vector spaces:

- $\mathbb{R}$ The set of real numbers.
- $\mathbb{R}^2$ The set of all ordered pairs (or ordered 2-tuples) of real numbers (the vectors in the plane (2-space)).
- $\mathbb{R}^3$ The set of all ordered triple (ordered 3-tuples) of real numbers (the vectors in 3-space).
- $\mathbb{R}^n$ The set of all ordered n-tuples of real numbers.
- $\mathbb{C}^n$ The set of all n-tuples of complex numbers.
- P_n The set of all polynomials of degree ≤ n.
- $M_{m\times n}(\mathbb{R})$ The set of all m×n matrices.

- $M_n(\mathbb{R})$ The set of all n×n square matrices.
- $C^k[a, b]$ the set of all continuous functions defined on $[a, b]$ that have at least k continuous derivatives.

Example (1): Show that $V= P_2$ (the set of all real valued polynomials of degree ≤ 2) and $F= \mathbb{R}$ (real numbers) with standard definition and scalar multiplication, forms a vector space.

Solution (proof):

The vectors can be written in the form of polynomials with degree at most 2, as $a_0 + a_1x + a_2x^2$, where a_2, a_1, and $a_0 \in \mathbb{R}$. Let $\boldsymbol{p}(x)$ and $\boldsymbol{q}(x)$ are two polynomials $\in P_2$. At first we must show that the vector addition $\boldsymbol{p} + \boldsymbol{q}$ (polynomial addition) and scalar multiplication $k\boldsymbol{p}$ (multiplying a polynomial by a scalar) are defined, where k is any scalar in $\mathbb{R}$, and after that we show whether P_2 is a vector space if and only if the five addition axioms and the five scalar multiplication are satisfied.

Let $\boldsymbol{p}(x)= a_0 + a_1x + a_2x^2$ and $\boldsymbol{q}(x)= b_0 + b_1x + b_2x^2$ are two polynomials in P_2, where a_0, a_1, a_2, b_0, b_1, and $b_2 \in \mathbb{R}$, and let k is any scalar $\in \mathbb{R}$, then

$\boldsymbol{p}(x) + \boldsymbol{q}(x)= \boldsymbol{p} + \boldsymbol{q}(x)= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$.

Since, $a_0 + b_0$, $a_1 + b_1$ and $a_2 + b_2$ are scalars $\in \mathbb{R}$ and the set of all real valued polynomials of degree ≤ 2, then $\boldsymbol{p} + \boldsymbol{q} \in P_2$ for all these scalars. *(Closed under addition)*

$k\boldsymbol{p}(x)= k(a_0 + a_1x + a_2x^2)= ka_0 + ka_1x + ka_2x^2$.

Since, ka_0, ka_1, and ka_2 are scalars $\in \mathbb{R}$, then $k\boldsymbol{p} \in P_2$. *(Closed under scalar multiplication)*

Now we must show whether the 10 axioms are satisfied.

The five addition axioms:

1) $\boldsymbol{p} + \boldsymbol{q} \in P_2$? *We showed it above*.
2) $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{q} + \boldsymbol{p}$?
   $\boldsymbol{p} + \boldsymbol{q} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$
   $= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2$ (The addition operation is commutative).
   $= \boldsymbol{q} + \boldsymbol{p}$.
3) If $\boldsymbol{r}(x)= r_0 + r_1x + r_2x^2$ is any polynomial $\in P_2$, then
   $(\boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r} = \boldsymbol{p} + (\boldsymbol{q} + \boldsymbol{r})$?
   $(\boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r} = [(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2] + r_0 + r_1x + r_2x^2$
   $= ((a_0 + b_0) + r_0) + ((a_1 + b_1) + r_1)\, x + ((a_2 + b_2) + r_2)\, x^2$
   $= (a_0 + (b_0 + r_0)) + (a_1 + (b_1 + r_1))\, x + (a_2 + (b_2 + r_2))\, x^2$ (Associative).
   $= a_0 + a_1x + a_2x^2 + [\,(b_0 + r_0) + (b_1 + r_1)\, x + (b_2 + r_2)\, x^2]$
   $= \boldsymbol{p} + (\boldsymbol{q} + \boldsymbol{r})$.
4) If $\boldsymbol{0}= d_0 + d_1x + d_2x^2$ is a zero polynomial $\in P_2$, such that $d_0= d_1= d_2= 0$, , then

$\mathbf{0} + \boldsymbol{p}(x) = \boldsymbol{p}$?

$\mathbf{0} + \boldsymbol{p} = (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2$

$= a_0 + a_1x + a_2\,x^2$

$= \boldsymbol{p}$.

5) If $-\boldsymbol{p}(x) = -a_0 - a_1x - a_2x^2$ is any polynomial $\in P_2$, then

$\boldsymbol{p}(x) + (-\boldsymbol{p}(x)) = \mathbf{0}$?

$\boldsymbol{p}(x) + (-\boldsymbol{p}(x)) = (a_0 - a_0) + (a_1 - a_1)x + (a_2 - a_2)x^2$

$= \mathbf{0}$ zero polynomial.

Five scalar multiplication axioms:

6) For all $k \in \mathbb{R}$ and $\boldsymbol{p}(x) \in P_2$, $k\boldsymbol{p}(x) \in P_2$? <u>*We showed it above*</u>.

7) $k(\boldsymbol{p} + \boldsymbol{q}) = k\boldsymbol{p} + k\boldsymbol{q}$? ($k$ is any scalar$\in \mathbb{R}$)

$k(\boldsymbol{p} + \boldsymbol{q}) = k[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2]$

$= k(a_0 + b_0) + k(a_1 + b_1)x + k(a_2 + b_2)x^2$

$= (ka_0 + kb_0) + (ka_1 + kb_1)x + (ka_2 + kb_2)x^2$

$= (ka_0 + ka_1x + ka_2x^2) + (kb_0 + kb_1x + kb_2x^2)$

$= k\boldsymbol{p} + k\boldsymbol{q}$.

8) If k and c are any scalars $\in \mathbb{R}$, then $(k + c)\boldsymbol{p}(x) = k\boldsymbol{p}(x) + c\boldsymbol{p}(x)$?

$(k + c)\boldsymbol{p}(x) = (k + c)(\,a_0 + a_1x + a_2x^2)$

$= (k + c)a_0 + (k + c)a_1x + (k + c)a_2x^2$

$= ka_0 + ca_0 + ka_1x + ca_1x + ka_2x^2 + ca_2x^2$

$= (ka_0 + ka_1x + ka_2x^2) + (ca_0 + ca_1x + ca_2x^2)$

$= k\boldsymbol{p}(x) + c\boldsymbol{p}(x)$.

9) $(kc)\boldsymbol{p}(x) = k(c\boldsymbol{p}(x))$?

$(kc)\boldsymbol{p}(x) = kc(a_0 + a_1x + a_2x^2)$

$= k(ca_0 + ca_1x + ca_2x^2)$

$= k(c\boldsymbol{p}(x))$.

10) $1\boldsymbol{p}(x) = \boldsymbol{p}(x)$?

$1\boldsymbol{p}(x) = 1(a_0 + a_1x + a_2x^2)$

$= a_0 + a_1x + a_2x^2$

$= \boldsymbol{p}(x)$.

The 10 axioms are satisfied, therefore P_2 is a vector space.

Example (2): Determine whether the set V of all vectors in $\mathbb{R}^2$ (2-space) of the form $\begin{bmatrix} x \\ x \end{bmatrix}$ with the usual definition of vector addition and scalar multiplication is a vector space.

Proof:

Let $\mathbf{u} = \begin{bmatrix} x \\ x \end{bmatrix}$=, $\mathbf{v} = \begin{bmatrix} y \\ y \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} z \\ z \end{bmatrix}$ are vectors $\in V$, k and c are any scalars $\in \mathbb{R}$.

u + **v**= $\begin{bmatrix} x+y \\ x+y \end{bmatrix}$ ∈ V. *(Closed under addition)*

k**u**= $k\begin{bmatrix} x \\ x \end{bmatrix}$= $\begin{bmatrix} kx \\ kx \end{bmatrix}$ ∈ V. *(Closed under scalar multiplication)*

1) **u** + **v** ∈ V *as shown above*.
2) **u** + **v**= $\begin{bmatrix} x+y \\ x+y \end{bmatrix}$= $\begin{bmatrix} y+x \\ y+x \end{bmatrix}$= **v** + **u**.
3) (**u** + **v**) + **w**= $\begin{bmatrix} x+y \\ x+y \end{bmatrix}$ + $\begin{bmatrix} z \\ z \end{bmatrix}$= $\begin{bmatrix} (x+y)+z \\ (x+y)+z \end{bmatrix}$= $\begin{bmatrix} x+(y+z) \\ x+(y+z) \end{bmatrix}$= $\begin{bmatrix} x \\ x \end{bmatrix}$ + $\begin{bmatrix} y+z \\ y+z \end{bmatrix}$= **u** + (**v** + **w**).
4) If **0**= $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then **u** + **0**= $\begin{bmatrix} x+0 \\ x+0 \end{bmatrix}$= $\begin{bmatrix} x \\ x \end{bmatrix}$= **u**.
5) For every **u** ∈ V ∃ −**u**, such that
   u + (−**u**)= $\begin{bmatrix} x \\ x \end{bmatrix}$ + $\begin{bmatrix} -x \\ -x \end{bmatrix}$= $\begin{bmatrix} x-x \\ x-x \end{bmatrix}$= $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$= **0** zero vector.
6) k**u** ∈ V *as shown above.*
7) k(**u** + **v**)= $k\begin{bmatrix} x+y \\ x+y \end{bmatrix}$= $\begin{bmatrix} k(x+y) \\ k(x+y) \end{bmatrix}$= $\begin{bmatrix} kx+ky \\ kx+ky \end{bmatrix}$= $\begin{bmatrix} kx \\ kx \end{bmatrix}$ + $\begin{bmatrix} ky \\ ky \end{bmatrix}$= k**u** + k**v**.
8) (k + c)**u**= (k + c)$\begin{bmatrix} x \\ x \end{bmatrix}$= $\begin{bmatrix} (k+c)x \\ (k+c)x \end{bmatrix}$= $\begin{bmatrix} kx+cx \\ kx+cx \end{bmatrix}$= $\begin{bmatrix} kx \\ kx \end{bmatrix}$ + $\begin{bmatrix} cx \\ cx \end{bmatrix}$= k**u** + c**u**.
9) k(c**u**)= $k\begin{bmatrix} cx \\ cx \end{bmatrix}$= $\begin{bmatrix} k(cx) \\ k(cx) \end{bmatrix}$= $\begin{bmatrix} (kc)x \\ (kc)x \end{bmatrix}$= $kc\begin{bmatrix} x \\ x \end{bmatrix}$= ($kc$)**u**.
10) 1**u**= $1\begin{bmatrix} x \\ x \end{bmatrix}$= $\begin{bmatrix} x \\ x \end{bmatrix}$= **u**.

The 10 axioms are satisfied; therefore, V is a vector space.

Example (3): Show whether V= $\left\{\begin{bmatrix} x \\ x \end{bmatrix} : x \geq 0 \text{ and } y \geq 0\right\}$ is a vector space.

Proof:

Let **u**= $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, **v**= $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ ∈ V, such that x_1, y_1, x_2 and $y_2 \geq 0$, then

u + **v**= $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ + $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$= $\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}$ ∈ V, because $x_1 + x_2$ and $y_1 + y_2 \geq 0$.

Therefore, V is closed under addition.

Let k any scalar ∈ ℝ, then

k**u**= $k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$= $\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix}$ ∉ V, because $kx_1, ky_1 < 0$ when k is negative.

Therefore, V is not closed under multiplication by scalar, hence V is not a vector space.

Example (4): Determine, whether the set S of all 2-nd degree polynomials (degree only 2), is a vector space.

Proof:

Let $\boldsymbol{p}(x)= a_0 + a_1x + a_2x^2$ and $\boldsymbol{q}(x)= b_0 + b_1x + b_2x^2$ are two polynomials in P_2, where a_0, a_1, a_2, b_0, b_1, and $b_2 \in \mathbb{R}$, then

$\boldsymbol{p}(x) + \boldsymbol{q}(x)= \boldsymbol{p} + \boldsymbol{q}(x)= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$.

Since, $a_0 + b_0$, $a_1 + b_1$ and $a_2 + b_2$ are scalars $\in \mathbb{R}$ and S is the set of all real valued polynomials of second degree, then $\boldsymbol{p} + \boldsymbol{q} \notin P_2$ for all these scalars because, if $a_2= -b_2$, then we obtain a polynomial that is not from the second degree but, from the first degree, therefore, the set S is not Closed under addition, hence S *is* not a vector space.

For example: If $\boldsymbol{p}(x)= 2x^2 + x + 5$ and $\boldsymbol{q}(x)= -2x^2 + 3x - 7$ are two polynomials in S, then

$\boldsymbol{p}(x) + \boldsymbol{q}(x)= 2x^2 + x + 5 + (-2x^2 + 3x - 7)$

$= 4x - 2 \notin S.$

Example (5): Let $M=\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : (a-b)c = 0\right\}$, find two nonzero elements (vectors) of M and show M is not closed under vector addition.

Solution:

Let $\mathbf{u}= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are two vectors in M, such that $(1-1)1= 0$ and $(1-2)0= 0$.

$\mathbf{u} + \mathbf{v}= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \notin M$, since $(2\text{-}3)1= -1 \neq 0$

Therefore, M is not closed under vector addition.

Example (6): Let $S=\left\{\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 2x_1 - 3x_2^3 + 4x_3^2 = 0\right\}$, find nonzero vectors of S to show whether S is not a vector space.

Solution:

Let $\mathbf{u}= \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \in S$, such that

$2(4) – 3(2)^3 + 4(2)^2= 0$ and $2(-2) – 3(0)^3 + 4(1)^2= 0$

$$\mathbf{u} + \mathbf{v}= \begin{bmatrix}4\\2\\2\end{bmatrix} + \begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\3\end{bmatrix} \notin M, \text{ since } 2(2) – 3(2)^3 + 4(3)^2= 16\neq 0.$$

Therefore, S is not closed under vector addition, hence S is not a vector space.

Example (7): Let $S=\left\{\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3: 2x_1 - 3x_2^3 + 4x_3^2 = 0\right\}$, find nonzero vector of S to show S is not closed under multiplication by scalar. (**Homework**).

Example (8): If V the set of all vectors of the form $\begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$ (2-space or xy-plane), such that $xy\geq 0$, find nonzero vector of V to show whether V is a vector space. (**Homework**).

Example (9): Show that the set V of all 2×2 matrices with real entries is a vector space if addition is defined to be matrix addition and scalar multiplication is defined to be matrix scalar multiplication. (**Homework**).

Example (10): Show that $\mathbb{R}^n$ is a vector space. (**Homework**).

Subspaces:

Definition (2): A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem (11): If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions holds.

a) If **u** and **v** are vectors in W, then $\mathbf{u} + \mathbf{v} \in W$.
b) If k is any scalar $\in \mathbb{R}$ and **u** is any vector $\in W$, then $k\mathbf{u} \in W$.

That means the subset W is closed under addition and closed under scalar multiplication.

Example (11): Let W be the subset of a vector space V, which is consist of all 2×3 matrices of the form $\begin{bmatrix}a & b & 0\\0 & c & d\end{bmatrix}$, where a, b, c, and $d \in \mathbb{R}$, prove that W is a subspace of V.

Proof:

1) Let $\mathbf{u}= \begin{bmatrix}a_1 & b_1 & 0\\0 & c_1 & d_1\end{bmatrix}$ and $\mathbf{v}= \begin{bmatrix}a_2 & b_2 & 0\\0 & c_2 & d_2\end{bmatrix}$ are any two vectors in W, then we must show that $\mathbf{u} + \mathbf{v} \in W$.

$$\mathbf{u} + \mathbf{v}= \begin{bmatrix}a_1+a_2 & b_1+b_2 & 0+0\\0+0 & c_1+c_2 & d_1+d_2\end{bmatrix}= \begin{bmatrix}a_1+a_2 & b_1+b_2 & 0\\0 & c_1+c_2 & d_1+d_2\end{bmatrix};$$

Since, $a_1 + a_2$, $b_1 + b_2$, $c_1 + c_2$, and $d_1 + d_2$ are scalars $\in \mathbb{R}$ and the vector form resulting from the sum of the two vectors in *W* has the same form as the vectors in *W*. Therefore, **u** + **v** ∈ *W*, hence, *W* is closed under addition.

2) Let k is any scalar $\in \mathbb{R}$ and $\mathbf{u} \in W$, then we must show that $k\mathbf{u} \in W$.

$$k\mathbf{u} = k\begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & 0 \\ 0 & kc_1 & kd_1 \end{bmatrix};$$

Since, ka_1, kb_1, kc_1, and kd_1 are scalars $\in \mathbb{R}$ and the vector form resulting from the multiplication of the vector in *W* by scalar has the same form as the vectors in *W*, then $k\mathbf{u} \in W$. Thus, *W* is closed under scalar multiplication. Hence, *W* is a subspace of *V*.

Example (12): Let *S* be the subset of $\mathbb{R}^3$, which is consisting from all the vectors of the form $<a, b, 1>$, where $a, b \in \mathbb{R}$, show whether the subset *S* is a subspace of $\mathbb{R}^3$.

Or: $S = \{< a,\ b,\ 1 > \in\ \mathbb{R}^3 : a, b, \in\ \mathbb{R}\ \}$

Proof:

1) Let $\mathbf{u} = <a_1, b_1, 1>$ and $\mathbf{v} = <a_2, b_2, 1> \in S$, then we must show that $\mathbf{u} + \mathbf{v} \in S$.
$\mathbf{u} + \mathbf{v} = <a_1, b_1, 1> + <a_2, b_2, 1> = <a_1 + a_2, b_1 + b_2, 2>$
$a_1 + a_2$ and $b_1 + b_2 \in \mathbb{R}$, but the vector form resulting from the sum of the two vectors in *S* does not have the same form as the vectors in *S*, because the third component is not equal to 1, thus $\mathbf{u} + \mathbf{v} \notin S$, therefore, *S* is not closed under addition.
Hence, *S* is not a subspace of $\mathbb{R}^3$.

<u>References</u>

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