**Example (13):** If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two vectors in the vector space $V$, and let $W$ the set of all vectors in $V$ of the form $a_1\mathbf{u}_1 + a_2\mathbf{u}_2$, where $a_1$ and $a_2$ are any scalars $\in \mathbb{R}$, show that the set $W$ is a subspace of $V$.

Proof:

1) Let $\mathbf{x}= a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ and $\mathbf{y}= b_1\mathbf{u}_1 + b_2\mathbf{u}_2$ are two vectors in $W$, then we must show that $\mathbf{x} + \mathbf{y} \in W$.
$\mathbf{x} + \mathbf{y}= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2) + (b_1\mathbf{u}_1 + b_2\mathbf{u}_2)$
$= (a_1 + b_1)\mathbf{u}_1 + (a_2 + b_2)\mathbf{u}_2;$

Since $a_1 + b_1$ and $a_2 + b_2$ are scalars $\in \mathbb{R}$ and the vector form resulting from the sum of the two vectors in $W$ has the same form as the vectors in $W$, then $\mathbf{x} + \mathbf{y} \in W$, and $W$ is closed under addition.

2) Let $k$ is any scalar $\in \mathbb{R}$ and $\mathbf{x} \in W$, then we must show that $k\mathbf{x} \in W$.
$k\mathbf{x}= k(a_1\mathbf{u}_1 + a_2\mathbf{u}_2)= (ka_1)\mathbf{u}_1 + (ka_2)\mathbf{u}_2;$

Since $ka_1$ and $ka_2$ are scalars $\in \mathbb{R}$ and the vector form resulting from the multiplication of the vector in $W$ by scalar has the same form as the vectors in $W$, then $k\mathbf{x} \in W$, and $W$ is closed under scalar multiplication.

Thus, $W$ is a subspace of $V$.

**Example (14):** Prove that $V=\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_3 = x_1 + x_2\right\}$ is a subspace of $\mathbb{R}^3$.

Proof:

1) Let $\mathbf{x}= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y}= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are two vectors $\in V$, such that $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, and $y_3 \in \mathbb{R}$ and $x_3= x_1 + x_2$ and $y_3= y_1 + y_2$, then
$\mathbf{x} + \mathbf{y}= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix};$

$x_3 + y_3= (x_1 + x_2) + (y_1 + y_2)= (x_1 + y_1) + (x_2 + y_2)$

Since, the vector form resulting from the sum of the two vectors in $V$ has the same form as the vectors in $V$, then $\mathbf{x} + \mathbf{y} \in V$, and $V$ is closed under addition.

2) Let $k$ is any scalar $\in \mathbb{R}$ and $\mathbf{x} \in V$, then

$$k\mathbf{x}= k\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix};$$

$kx_3= k(x_1 + x_2)= kx_1 + kx_2$

Since, the vector form resulting from the multiplication of the vector in $V$ by scalar has the same form as the vectors in $V$, then $k\mathbf{x} \in V$, and $V$ is closed under scalar multiplication.

Thus, $V$ is a subspace of $\mathbb{R}^3$.

**Example (15):** If $P_n$ is the set of all the polynomials of degree at most $n$ of the form $\boldsymbol{p}(x)= a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $a_0, a_1, a_2, \ldots$ and $a_n \in \mathbb{R}$, show whether $P_2$ (degree $\leq 2$)is a subspace of $P_n$ for $n \geq 3$.

Proof:

1) We must show that the set $P_2$ is closed under addition.
   Let $\boldsymbol{p}_1(x)= a_0 + a_1x + a_2x^2$ and $\boldsymbol{p}_2(x)= b_0 + b_1x + b_2x^2$ are two polynomials $\in P_2$, then
   $\boldsymbol{p}_1(x) + \boldsymbol{p}_2(x)= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$;

Since, $a_0 + b_0$, $a_1 + b_1$ and $a_2 + b_2$ are scalars $\in \mathbb{R}$ and $P_2$ is the set of all real valued polynomials of degree $\leq 2$, then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \in P_2$ for all these scalars, and $P_2$ is closed under addition.

2) We must show that the set $P_2$ is closed under scalar multiplication.
   Let $k$ is any scalar $\in \mathbb{R}$ and $\boldsymbol{p}_1(x) \in P_2$, then
   $k\boldsymbol{p}_1(x)= k(a_0 + a_1x + a_2x^2)= ka_0 + ka_1x + ka_2x^2$

Since, $ka_0$, $ka_1$, and $ka_2$ are scalars $\in \mathbb{R}$, then $k\boldsymbol{p}_1 \in P_2$ for all these scalars, and $P_2$ is closed under scalar multiplication.

Thus, the set $P_2$ is a subspace of $P_n$.

**Example (16):** Let $W$ be the set of all the vectors $<x, y> \in \mathbb{R}^2$, such that $x \geq 0$ and $y \geq 0$, show whether the set $W$ is a subspace of $\mathbb{R}^2$.

Or, $W= \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\right\}$.

Proof:

1) Let $\mathbf{u}= <x_1, y_1>$ and $\mathbf{v}= <x_2, y_2> \in W$ such that $x_1, y_1, x_2$, and $y_2 \geq 0$, then
   $\mathbf{u} + \mathbf{v}= < x_1 + x_2, y_1 + y_2>$;
   Since, $x_1, y_1, x_2$, and $y_2 \geq 0$, then $x_1 + x_2$ and $y_1 + y_2 \geq 0$.
   Thus, $\mathbf{u} + \mathbf{v} \in W$ and $W$ is closed under addition as shown in figure (1).

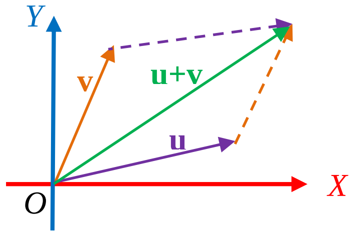


**Figure (1)**
The sum of two vectors

2) Let $k$ is any scalar $\in \mathbb{R}$ and $\mathbf{u} \in W$, then
$k\mathbf{u}= k<x_1, y_1>= <kx_1, ky_1>$;
Since, the components $kx_1$ and $ky_1$ of the vector resulting from the multiplication of the vector in $W$ by the scalar $k$ ($k\mathbf{u}$) are not $\geq 0$ for any scalar $k \in \mathbb{R}$, because if $k$ is a negative value then, these components will also be negative values.
So, $k\mathbf{u} \notin$ and $W$ is not closed under scalar multiplication.
Therefore, $W$ is not a subspace of $\mathbb{R}^2$.

***<u>Note:</u>*** Every nonzero vector space $V$ has at least two subspaces:

$V$ itself a subspace, and the set $\{\mathbf{0}\}$ consisting of just the zero vector in $V$ is a subspace, called the zero subspace.

**Theorem (12):** If $W$ is a subspace of a vector $V$, then $W$ contains the zero vector of $V$.

Proof:

Let $\mathbf{u}$ be arbitrary vector in $W$ and $\mathbf{0}$ be zero vector of $V$.

Let $0$ be the zero scalar, since $W$ is a subspace of $V$, and $W$ is itself a vector space and thus $W$ is closed under scalar multiplication, then

$0\mathbf{u} \in W$, but $0\mathbf{u}= \mathbf{0}$ the zero vector.

Thus, $\mathbf{0} \in W$.

***This theorem tells us that all subspaces of a vector space contain the zero vector.***

***Note:*** *Now if we go back to example (12), we notice that by using theory (12), we directly determine that the subset S in this example is not a subspace, because it does not contain the zero vector. The third component of the vector in S is always* 1**.**

**Example (17):** Let $W=\left\{\begin{bmatrix}x\\0\end{bmatrix}\in\mathbb{R}^2, x \text{ is a real number } (\in\ \mathbb{R})\right\}$, then $W$ is a subspace of $\mathbb{R}^2$.

Proof:

We note that the zero vector $\begin{bmatrix}0\\0\end{bmatrix}\in W$, since 0 is a real number. ($W$ contains zero vector).

Now to show $W$ is closed under addition and scalar multiplication:

1) Let $\mathbf{u}=\begin{bmatrix}x\\0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}y\\0\end{bmatrix}$ be vectors $\in W$, and

   $\mathbf{u}+\mathbf{v}=\begin{bmatrix}x\\0\end{bmatrix}+\begin{bmatrix}y\\0\end{bmatrix}=\begin{bmatrix}x+y\\0\end{bmatrix}$, since $x+y\in\ \mathbb{R}$, then $\mathbf{u}+\mathbf{v}\in W$.

Therefore, $W$ is closed under addition.

2) Let $k$ be any scalar $\in\mathbb{R}$ and $\mathbf{u}\in W$, then
   $k\mathbf{u}=k\begin{bmatrix}x\\0\end{bmatrix}=\begin{bmatrix}kx\\0\end{bmatrix}$, since $kx\in\ \mathbb{R}$,then $k\mathbf{u}\in W$.

Therefore, $W$ is closed under scalar multiplication. Hence, $W$ is a subspace of $\mathbb{R}^2$.

Geometrically, $W$ is the set of all vectors that lie on the $x$-axis and the sum of any two vectors is also lies on the x-axis, and the scalar multiple of any vector of $W$ also lies on $x$-axis. Thus x-axis is a subspace of the plane ($\mathbb{R}^2$).It is a <u>one-dimensional subspace</u> of $\mathbb{R}^2$.

Similarly If we have the subset $U=\left\{\begin{bmatrix}0\\y\end{bmatrix}\in\mathbb{R}^2, y \text{ is a real number } (\in\ \mathbb{R})\right\}$, is also a subspace of $\mathbb{R}^2$. Clearly $U$ represents $y$-axis on the plane.

***In general, it can be shown that any line passing through the origin point is a subspace of*** $\mathbb{R}^2$**.**

**All possible subspaces of $\mathbb{R}^2$ are listed below.**

1) Origin point is a subspace of $\mathbb{R}^2$. The dimensional of this subspace is zero.
2) The one-dimensional subspaces of $\mathbb{R}^2$ are lines through the origin.
3) The two-dimensional subspace of $\mathbb{R}^2$ is the whole space $\mathbb{R}^2$ itself.

The statements (1) and (3) are obvious and the proof of (2) <u>will be given later</u>.

***Note:*** To determine whether any subset of a vector space is a subspace of a vector space or not, it is enough merely to check that the subset of a vector space is closed under addition and scalar multiplication as shown in theory (11).

**Example (18):** Let $W=\left\{\begin{bmatrix}x\\0\\0\end{bmatrix} \in \mathbb{R}^3 : x \in \mathbb{R}\right\}$, show that $W$ is a subspace of $\mathbb{R}^3$.

Proof:

Let $\mathbf{u}=\begin{bmatrix}x_1\\0\\0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}x_2\\0\\0\end{bmatrix}$ are two vectors $\in W$, such that $x_1$ and $x_2 \in \mathbb{R}$, then

$\mathbf{u}+\mathbf{v}=\begin{bmatrix}x_1+x_2\\0\\0\end{bmatrix}$, since $x_1$ and $x_2 \in \mathbb{R}$, then $x_1+x_2$ and $\mathbf{u}+\mathbf{v} \in W$.

Thus, $W$ is closed under addition.

Let $k$ be any scalar $\in \mathbb{R}$, and $\mathbf{u} \in W$, then

$k\mathbf{u}=k\begin{bmatrix}x_1\\0\\0\end{bmatrix}=\begin{bmatrix}kx_1\\0\\0\end{bmatrix}$, since $kx_1 \in \mathbb{R}$, then $k\mathbf{u} \in W$.

Thus, $W$ is closed under scalar multiplication. Hence, $W$ is a subspace of $\mathbb{R}^3$.

Similarly, the subsets $\left\{\begin{bmatrix}0\\y\\0\end{bmatrix} \in \mathbb{R}^3 : y \in \mathbb{R}\right\}$ and $\left\{\begin{bmatrix}0\\0\\z\end{bmatrix} \in \mathbb{R}^3 : z \in \mathbb{R}\right\}$ of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

Geometrically, $W$ in this example represents $x$-axis and the subspaces $\left\{\begin{bmatrix}0\\y\\0\end{bmatrix} \in \mathbb{R}^3 : y \in \mathbb{R}\right\}$ and $\left\{\begin{bmatrix}0\\0\\z\end{bmatrix} \in \mathbb{R}^3 : z \in \mathbb{R}\right\}$ represent $y$-axis and $z$-axis respectively. These are <u>one-dimensional subspaces</u> of $\mathbb{R}^3$.

**Example (19):** Let $W=\left\{\begin{bmatrix}x\\y\\0\end{bmatrix} \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\}$, show that $W$ is a subspace of $\mathbb{R}^3$.

Proof:

Let $\mathbf{x}=\begin{bmatrix}x_1\\y_1\\0\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}x_2\\y_2\\0\end{bmatrix}$ be two vectors $\in W$, such that $x_1$, $y_1$, $x_2$, and $y_2 \in \mathbb{R}$ , then

$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} \in W$ and $W$ is closed under addition.

Let $c$ is any scalar $\in \mathbb{R}$ and $\mathbf{x} \in W$, then

$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cy_1 \\ 0 \end{bmatrix} \in W$ and $W$ is closed under scalar multiplication.

Thus , $W$ is a subspace of $\mathbb{R}^3$.

Geometrically, $W$ in this example represents $xy$-plane. This is a 2-dimensional subspace of $\mathbb{R}^3$.

Similarly, the $yz$-plane $\left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : y, z \in \mathbb{R} \right\}$ and $xz$-plane $\left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3 : x, z \in \mathbb{R} \right\}$ are also 2-dimensional subspaces of $\mathbb{R}^3$ (Verify). (**Homework**).

**All subspaces of $\mathbb{R}^3$ are listed below.**

1) The origin is a subspace of $\mathbb{R}^3$. The dimensional of this subspace is defined to be zero.
2) The one-dimensional subspaces of $\mathbb{R}^3$ are lines through the origin.
3) The two-dimensional subspaces of $\mathbb{R}^3$ are planes through the origin.
4) The whole spaces $\mathbb{R}^3$ is three-dimensional subspace of itself.

Statements (1) and (4) are obvious. We come to statement (2), we prove it using theorem (11) (subspace theorem), which is the line that passes through origin is a subspace of $\mathbb{R}^3$(and also of $\mathbb{R}^2$ as we have noted already in page 68).

Let ***l*** be a line passing through origin in $\mathbb{R}^3$, then the sum of any two vectors on this line also lies on the line ***l***, and that the scalar multiple of any vector on line also lies on the line ***l***.

Therefore, ***l*** is closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^3$. Thus the lines through origin are the only one-dimensional subspaces of $\mathbb{R}^3$.

**Example (20):** Let $V$ be a subset of $\mathbb{R}^3$ consisting of vectors of the form $<a, a, b>$, $a$, b $\in \mathbb{R}$. Show that $V$ is a subspace of $\mathbb{R}^3$. (*The first two components are the same*).

Or, $V = \{< a, a, b > \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$.

Proof: Let $\mathbf{u} = <a, a, b>$ and $\mathbf{v} = <c, c, d>$ be two vectors (elements) of $V$, such that $a, b, c, d \in \mathbb{R}$, then

$\mathbf{u} + \mathbf{v} = <a + c, a + c, b + d> \in V$, and $V$ is closed under addition.

Let $k$ be a scalar (real number) $\in \mathbb{R}$, and $\mathbf{u} \in V$, then

$k\mathbf{u} = \langle ka, ka, kb\rangle \in V$, and $V$ is closed under scalar multiplication.

Thus, $V$ is a subspace of $\mathbb{R}^3$.

Note that geometrically $V$ is a plane perpendicular to the $xy$-plane, through the line $\boldsymbol{y} = \boldsymbol{x}$, $\boldsymbol{z} = 0$.

**Example (21):** Let $S = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y = z,\ x, y, and\ z \in \mathbb{R}\right\}$, show whether, $S$ is a subspace of $\mathbb{R}^3$.

Proof:

Let $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be two vectors in $S$, such that $x_1, y_1, z_1, x_2, y_2$, and $z_2 \in \mathbb{R}$ and $x_1 + y_1 = z_1$, $x_2 + y_2 = z_2$ then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = z_1 + z_2.$$

Thus, $\mathbf{u} + \mathbf{v} \in S$, and $S$ is closed under addition.

Let $k$ be any scalar $\in \mathbb{R}$, and $\mathbf{u} \in S$, then

$$k\mathbf{u} = \begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix},\ kx_1 + kx_2 = k(x_1 + y_1) = kz_1.$$

Thus, $k\mathbf{u} \in S$, and $S$ is closed under scalar multiplication.

Hence, $S$ is a subspace of $\mathbb{R}^3$.

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