Ministry of Higher Education & Scientific Research
University of Anbar
College of Science
Department of Applied Mathematics

lectures
Subject: <u>Vector analysis.</u>
2020-2021.
Stage: 2ˢᵗ.
The lecturer: Assist. Prof. Dr. Ali Rashid Ibrahim

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Example (22): Determine which of the following subsets of $\mathbb{R}^2$ are subspaces as shown in figure (2-a, 2-b, 2-c) below.
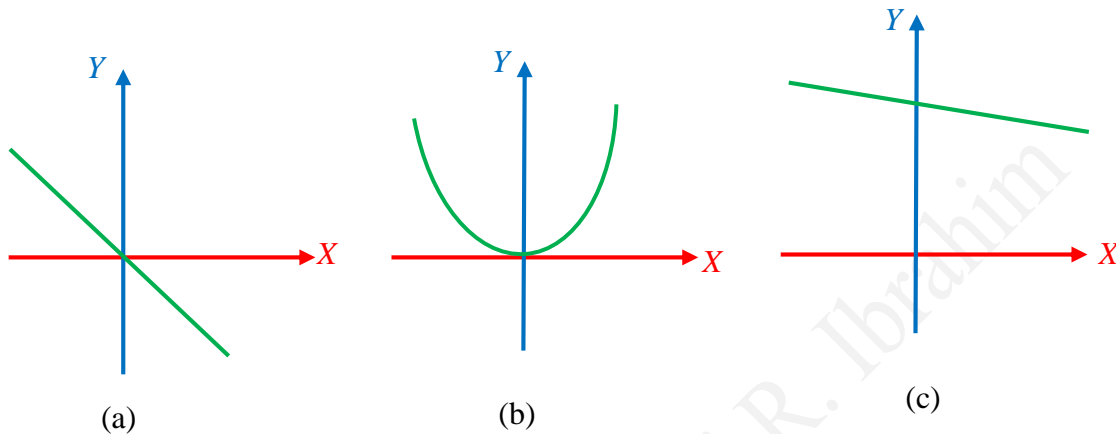

Figure (2)

Solution:

a) The line is a subspace of $\mathbb{R}^2$, because the line passes through origin and the addition of any two vectors on the line also lie on the line, and the scalar multiple of any vector on the line also lie on the line, thus the line represents a subset closed under addition and scalar multiplication, hence it is a subspace of $\mathbb{R}^2$.

b) Cannot be a subspace as it is not a line and if we take any two points on this parabola, their sum may not lie on the parabola, thus it cannot be a subspace of $\mathbb{R}^2$.

c) Is not a line through the origin and therefore, does not contain the zero vector and hence, by theorem (12) it cannot be a subspace of $\mathbb{R}^2$.

Example (23): Show that $W = \left\{ \begin{bmatrix} 1 \\ y \end{bmatrix} \in \mathbb{R}^2 : y \in \mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^2$.

Proof:

Let $\mathbf{u} = \begin{bmatrix} 1 \\ y_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ y_2 \end{bmatrix}$ are two vectors in W, such that $y_1, y_2 \in \mathbb{R}$, then

$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ y_1 + y_2 \end{bmatrix}$, since the vector form resulting from the sum of two vectors in W does not have the same form as the vectors in W, and therefore, $\mathbf{u} + \mathbf{v} \notin W$, and W is not closed under addition, hence W is not a subspace of $\mathbb{R}^2$.

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**Note:** _In this example as in examples (12), we could also have used theorem (12). If given subset does not contain the zero vector, it cannot be a subspace. We note the subset_ $W= \left\{ \begin{bmatrix} 1 \\ y \end{bmatrix} \in \mathbb{R}^2 : y \in \mathbb{R} \right\}$ _in this example cannot contain the zero vector as the first component is always 1._

Example (24): Let W be the set of vectors of the form $<a, a, a+2>$, $a \in \mathbb{R}$, show that W is not a subspace of $\mathbb{R}^3$.

Proof:

We can check whether the zero vector $<0, 0, 0>$ is in W.

The third component of the vector in W is $a+2$, and a represents the first and second component of the vector, and when $a= 0$, then the third component will not be zero, thus the zero vector is not in W. Hence W is not a subspace of $\mathbb{R}^3$.

Also we can prove as the following:

$a \in \mathbb{R}$, $<a, a, a+2> = <0, 0, 0>$, then

$a= 0$ and $a + 2= 0$, but this system of equations has no solution. Thus $<0, 0, 0> \notin W$. Hence, W is not a subspace of $\mathbb{R}^3$.

Example (25): (**Testing for a subspace**)

Prove whether the plane W passing through the origin point of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Proof:

Let **u** and **v** are two vectors in W, then

u + **v** $\in W$, because the sum of any two vectors in the plane represents the diagonal of the parallelogram determined by the two vectors **u** and **v**. Thus W is closed under addition.

Let k is any scalar $\in \mathbb{R}$ (real number), and **u** $\in W$, then k**u** must lie in W for any scalar k, because k**u** lies on a line through **u**. Thus W is closed under scalar multiplication, so it is a subspace of $\mathbb{R}^3$ as shown in figure (3).

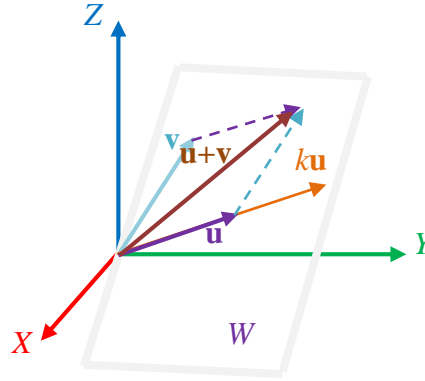**Ministry of Higher Education**
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Figure (3)

The vectors **u** + **v** and k**u** both lie in the plane as **u** and **v**

**Note:** The zero vector $\in W$, because W passes through the origin.

Subspaces of $M_{n \times n}$.

Theorem (13): If A and B are symmetric matrices with the same size, and if k is any scalar, then

1) A^T is symmetric matrix.
2) $A + B$ and $A - B$ are symmetric matrices.
3) kA is symmetric matrix.

We note the sum of two symmetric matrices is symmetric matrix, and a scalar multiple of a symmetric matrix is symmetric. Thus the set of $n \times n$ symmetric matrices is a subspace of the vector space $M_{n \times n}$ of all $n \times n$ matrices. Similarly, the set of $n \times n$ upper triangular matrices, the set of $n \times n$ lower triangular matrices, and the set of $n \times n$ diagonal matrices, all form subspaces of $M_{n \times n}$, since each of these sets is closed under addition and scalar multiplication.

Solution space of Homogeneous systems:

If A**x**= **b** a system of linear equations, then each vector **x** that satisfies these equations is called a **solution vector** of the system. The following theorem shows that the **solution vectors** of a Homogeneous linear system form a **vector space**, which we shall call the **solution space** of the system.

Theorem (14): If A**x**= **0** is a Homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$.

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Proof:

Let W be the set of solution vectors. There is at least one vector in W, namely **0** (zero vector).To show that W is closed under addition and scalar multiplication, we must show If $\mathbf{x}_1$ and $\mathbf{x}_2$ are any solution vectors and k is any scalar $\in \mathbb{R}$, then $\mathbf{x}_1 + \mathbf{x}_2$ and $k\mathbf{x}_1$ are also solution vectors ($\in W$). $\mathbf{x}_1$ and $\mathbf{x}_2$ are solution vectors, then

$A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$

$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$

$= \mathbf{0} + \mathbf{0}$

$= \mathbf{0};$

Thus, $\mathbf{x}_1 + \mathbf{x}_2$ is a solution vector, so $\mathbf{x}_1 + \mathbf{x}_2 \in W$ and W is closed under addition.

$A(k\mathbf{x}_1) = k(A\mathbf{x}_1) = k(\mathbf{0}) = \mathbf{0};$

Thus, $k\mathbf{x}_1$ is a solution vector, so $k\mathbf{x}_1 \in W$ and W is closed under scalar multiplication.

Hence, W is a subspace of $\mathbb{R}^n$.

Example (26): If we have the following linear systems (Homogeneous systems), find the solutions of these systems and show that each of these solutions represents a subspace of $\mathbb{R}^3$.

a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution (a):

$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix}$ -2R$_1$+R$_2 \rightarrow$ R$_2$, and -3R$_1$ + R$_3 \rightarrow$ R$_3$ $\sim$ $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x - 2y + 3z = 0 \rightarrow x = 2y - 3z$

Now, if $y = s$ and $z = t$, then the solutions are

$x = 2s - 3t$, $y = s$ and $z = t$

The equation $x - 2y + 3z = 0$ is the equation of the plane through the origin with $\mathbf{n} = <1, -2, 3>$ as normal vector. Thus the solution form a subspace of $\mathbb{R}^3$.

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Solution (b):

$x= -5y,\ y= -z$

Now, if $z= t$, then the solution are

$x= -5t,\ \ y= -t$ and $z= t$

The solutions represent parametric equations for the line through the origin parallel to the vector **v**= <−5, −1, 1>. Thus the solution form a subspace of $\mathbb{R}^3$.

Part (c) and (d) **homework**.

Example (27): Let P_3 the vector space, which two operations, addition and scalar multiplication are defined. If S the set of all polynomials of the form $p(x)= ax^3$ for $a \in \mathbb{R}$, show whether the set S is a subspace of P_3. (**Homework**)

Example (28): Determine which of the following sets is a subspace of $\mathbb{R}^3$. (**Homework**)

   a) $S= \{(x,y,z) \in \mathbb{R}^3 : x + 2y = 3z\}$.
   b) $S= \{(a,b,c) \in \mathbb{R}^3 : a,b,\ \text{and}\ c \geq 0\}$.
   c) $S= \{(0, \alpha, \alpha +1): \alpha \in \mathbb{R}\}$.
   d) $S= \{(k,m,n) \in \mathbb{R}^3 : k^2 = n^2\}$.

Example (29): Determine which of the following sets are subspace of $\mathbb{R}^3$. (**Homework**)

   a) The set of all vectors of the form <a, 0, 0>, ($a \in \mathbb{R}$).
   b) The set of all vectors of the form <a, 1, 1>, ($a \in \mathbb{R}$).
   c) The set of all vectors of the form <a, b, c>, where b= a + c, (a, b, and c $\in \mathbb{R}$).
   d) The set of all vectors of the form <>, where b= a + c + 1, (a, b, and c $\in \mathbb{R}$).
   e) The set of all vectors of the form <a, b, 0>, (a and b $\in \mathbb{R}$).

Example (30): Determine whether the following sets represent a subspace of $M_{n\times n}$. (**Homework**)

   a) The set of all $n\times n$ matrices A, such that A^T= -A.
   b) The set of all $n\times n$ matrices A, such that tr(A)= 0.
   c) The set of all $n\times n$ matrices A, such that the linear system A**x**= 0 has only the trivial solution.
   d) The set of all $n\times n$ matrices A, such that AB= BA for a fixed $n\times n$ matrix B.

Example (31): Determine which of the following sets are subspace of $M_{2\times 2}$. (**Homework**)

   a) The set of all 2×2 matrices with integer entries.
   b) The set of all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a + b + c + d= 0, (a, b, c, and c $\in \mathbb{R}$).

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c) The set of all 2×2 matrices A, such that $\det(A) = 0$.

d) The set of all matrices of the form $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$, $(a \in \mathbb{R})$.

Example (32): Let $S = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 0, x_1, x_2 \text{ and } x_3 \in \mathbb{R} \right\}$

a) Find three distinct element (vectors) of S.

b) Show that S is a subspace of $\mathbb{R}^3$.

c) Give a geometric interpretation of this results.

Solution (a):

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \in S.$

Solution (b):

i) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S.$

ii) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are two vectors $\in S$, then we must prove that $\mathbf{x} + \mathbf{y} \in S$.

$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix};$

Now, plugging the components of $\mathbf{x} + \mathbf{y}$ into the condition gives:

$2x_1 + 3x_2 - 4x_3 = 0 \rightarrow 2(x_1 + y_1) + 3(x_2 + y_2) - 4(x_3 + y_3)$

$\rightarrow 2x_1 + 2y_1 + 3x_2 + 3y_2 - 4x_3 - 4y_3$

$\rightarrow (2x_1 + 3x_2 - 4x_3) + (2y_1 + 3y_2 - 4y_3)$

$\rightarrow \quad 0 \quad + \quad 0 \quad = 0$

Therefore, $\mathbf{x} + \mathbf{y} \in S$, so S is closed under addition.

iii) Let $k \in \mathbb{R}$, and $\mathbf{x} \in S$, then we must prove $k\mathbf{x} \in S$.

$k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix};$

Now, plugging the components of $k\mathbf{x}$ into the condition gives:

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$$2x_1 + 3x_2 - 4x_3 = 0 \rightarrow 2(kx_1) + 3(kx_2) - 4(kx_3)$$

$$\rightarrow k(2x_1 + 3x_2 - 4x_3)$$

$$\rightarrow k(0) = 0$$

Therefore, $k\mathbf{x} \in S$, so S is closed under scalar multiplication.

Hence, S is a subspace of $\mathbb{R}^3$.

Solution (c):

Geometrically, the equation $2x_1 + 3x_2 - 4x_3 = 0$ represents equation of a plane through the origin, and planes passing through the origin give subspaces.

Example (33): If $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, \ x_1 \, and \, x_2 \in \mathbb{R} \right\}$, show whether S is a subspace of $\mathbb{R}^2$. (**Homework**)

Example (34): Determine Whether the set $W = \{ \langle a, \quad b, \quad c \rangle \in \mathbb{R}^3 : b \geq 0, \ a, b, \ and \ c \in \mathbb{R} \}$, is a subspace of $\mathbb{R}^3$. (**Homework**)

Example (35): Prove that $W = \{ \langle x_1, \ x_2, \dots, \ x_n \rangle \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0, \ x_i \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^n$. (**Homework**).

Theorem (15): If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, are vectors $\in V$ (vector space), such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.

Corollary (1): The zero vector and the additive inverse vector (for each vector) are unique.

References

1- Introductory linear algebra with applications, Bernard Kolman, first edition, 1976.

2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

3- Elementary Linear Algebra, Ninth Edition, Howard Anton, Chris Rorres, 2005.

4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.