**جامعة الانبار**

**كلية العلوم**

**قسم الرياضيات التطبيقيه**

**الماده : تفاضل وتكامل متقدم ٢**

**عنوان المحاضره**

# طريقه التعويض في التكامل الثنائي substitution in double integrals

**اسم التدريسي م . د. ناهض سليم السيف**

**المصدر : Calculus fourteenth edition**

**The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$**

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to $u$.
3. Replace $u$ by $g(x)$.

**EXAMPLE 3** Find $\int \sec^2(5x + 1) \cdot 5\,dx$

**Solution** We substitute $u = 5x + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5x + 1) \cdot 5\,dx = \int \sec^2 u\,du \qquad \text{Let } u = 5x + 1,\ du = 5\,dx.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5x + 1) + C. \qquad \text{Substitute } 5x + 1 \text{ for } u.$$

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u} \cdot \underbrace{2\,dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Let } u = 2x + 1,\ du = 2\,dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u.$$

# Substitutions in Double Integrals

**DEFINITION** The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{1}$$

**THEOREM 3—Substitution for Double Integrals**

Suppose that $f(x, y)$ is continuous over the region $R$. Let $G$ be the preimage of $R$ under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of $G$. If the functions $g$ and $h$ have continuous first partial derivatives within the interior of $G$, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \tag{2}$$

**EXAMPLE 1** Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and use Equation (2) to write the Cartesian integral $\iint_R f(x, y)\,dx\,dy$ as a polar integral.

**Solution** Figure 15.58 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle $G$: $0 \le r \le 1, 0 \le \theta \le \pi/2$, into the quarter circle $R$ bounded by $x^2 + y^2 = 1$ in the first quadrant of the $xy$-plane.

For polar coordinates, we have $r$ and $\theta$ in place of $u$ and $v$. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \ge 0$ when integrating in polar coordinates, $|J(r, \theta)| = |r| = r$, so that Equation (2) gives

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \tag{3}$$

This is the same formula we derived independently using a geometric argument for polar area in Section 15.4. ■

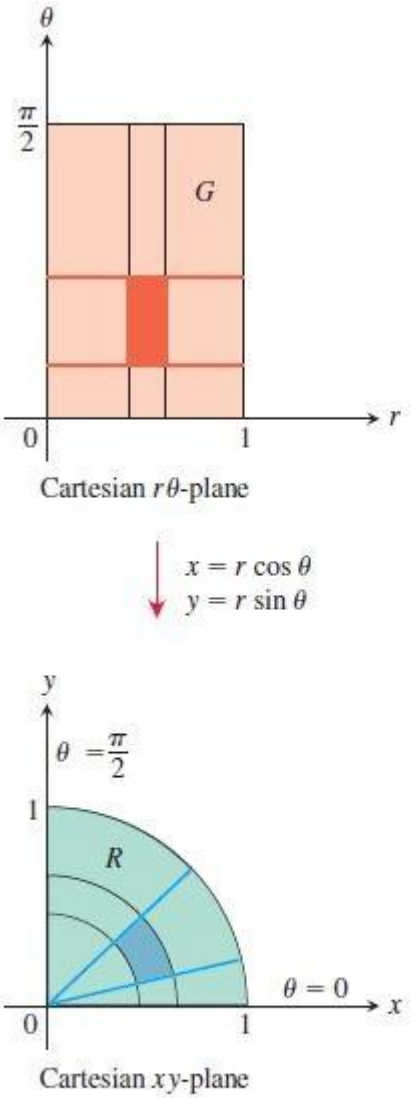


**FIGURE 15.58** The equations $x = r\cos\theta$, $y = r\sin\theta$ transform $G$ into $R$. The Jacobian factor $r$, calculated in Example 1,

**EXAMPLE 2** Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2} dx\, dy$$

by applying the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2} \tag{4}$$

and integrating over an appropriate region in the $uv$-plane.

**Solution** We sketch the region $R$ of integration in the $xy$-plane and identify its boundaries (Figure 15.59).

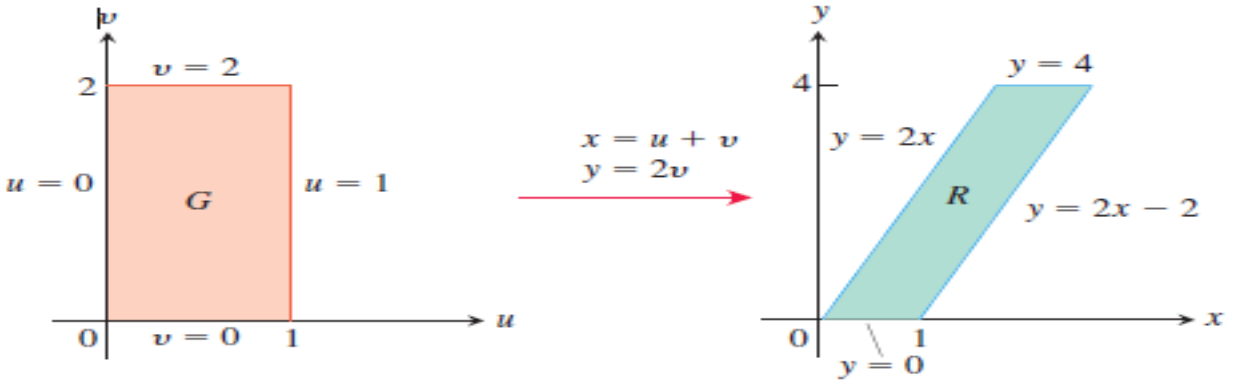


**FIGURE 15.59** The equations $x = u + v$ and $y = 2v$ transform $G$ into $R$. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms $R$ into $G$ (Example 2).

To apply Equation (2), we need to find the corresponding $uv$-region $G$ and the Jacobian of the transformation. To find them, we first solve Equations (4) for $x$ and $y$ in terms of $u$ and $v$. From those equations it is easy to find algebraically that

$$x = u + v, \qquad y = 2v. \tag{5}$$

$$x = u + v, \qquad y = 2v.$$

We then find the boundaries of $G$ by substituting these expressions into the equations for the boundaries of $R$ (Figure 15.59)

| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x = y/2$ | $u + v = 2v/2 = v$ | $u = 0$ |
| $x = (y/2) + 1$ | $u + v = (2v/2) + 1 = v + 1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |

From Equations (5) the Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (2):

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2} dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\, |J(u, v)|\, du\, dv$$

$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[u^2\Big]_0^1 dv = \int_0^2 dv = 2. \quad \blacksquare$$

**EXAMPLE 3** Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

**Solution** We sketch the region $R$ of integration in the $xy$-plane and identify its boundaries (Figure 15.60). The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Routine algebra produces $x$ and $y$ as functions of $u$ and $v$:

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}. \tag{6}$$

From Equations (6), we can find the boundaries of the $uv$-region $G$ (Figure 15.60).

| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x + y = 1$ | $\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$ | $u = 1$ |
| $x = 0$ | $\frac{u}{3} - \frac{v}{3} = 0$ | $v = u$ |
| $y = 0$ | $\frac{2u}{3} + \frac{v}{3} = 0$ | $v = -2u$ |

The Jacobian of the transformation in Equations (6) is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Equation (2), we evaluate the integral:

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2\,|J(u, v)|\,dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv\,du = \frac{1}{3}\int_0^1 u^{1/2}\left[\frac{1}{3}v^3\right]_{v=-2u}^{v=u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}(u^3 + 8u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9}u^{9/2}\Big]_0^1 = \frac{2}{9}.$$

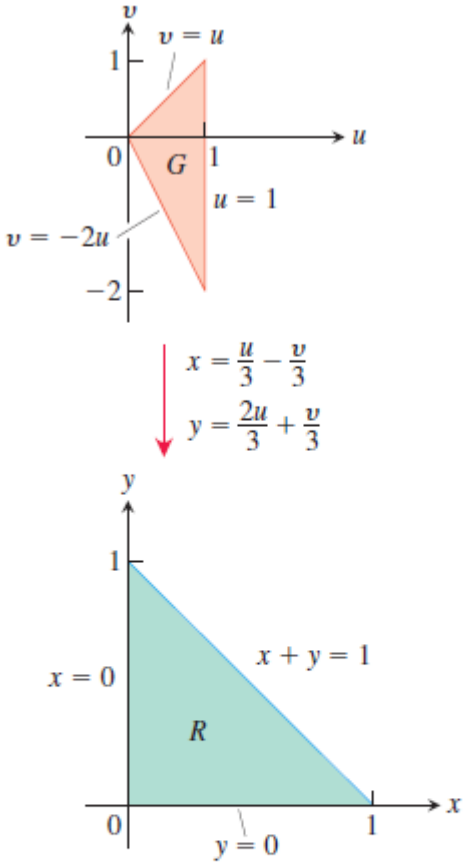


**FIGURE 15.60** The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform $G$ into $R$. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms $R$ into $G$ (Example 3).

**EXAMPLE 4** find translation to the integral and find the value of the jacobain

$$\int\int \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$$

**Solution** The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$. Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

If $G$ is the region of integration in the $uv$-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G ve^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$