جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات / الفصل الاول

Graphs and Their Operations

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Lecture (4)

Graphs and Their Operations

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Outlines

- ✓ Union, Intersection and Ringsum of Graphs.
- ✓ The complement of a graph.
- ✓ The join of two graphs.
- ✓ Edge Deletion & Vertex Deletion in Graphs.
- ✓ Fusion of Vertices & Edge Contraction in Graphs.
- ✓ Subdivision of an Edge.
- ✓ Homeomorphic Graphs.
- ✓ Smoothing Vertices in Graphs.

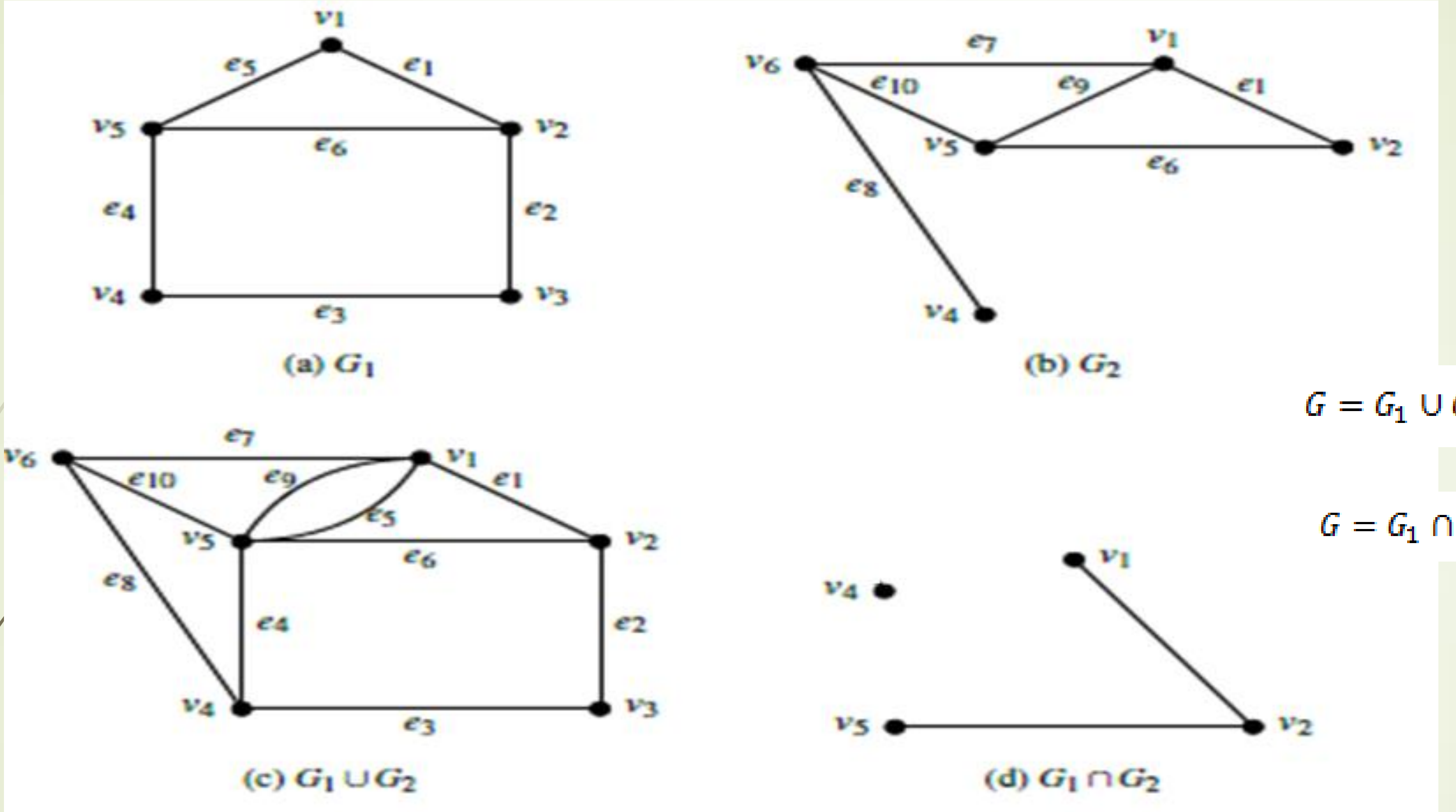
Union, Intersection and Ringsum of Graphs

Definition: The union of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \cup G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

Definition: The intersection of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \cap G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Definition: The ringsum of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \oplus G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \oplus E(G_2) = (E(G_1) \cup E(G_2)) - (E(G_1) \cap E(G_2))$.

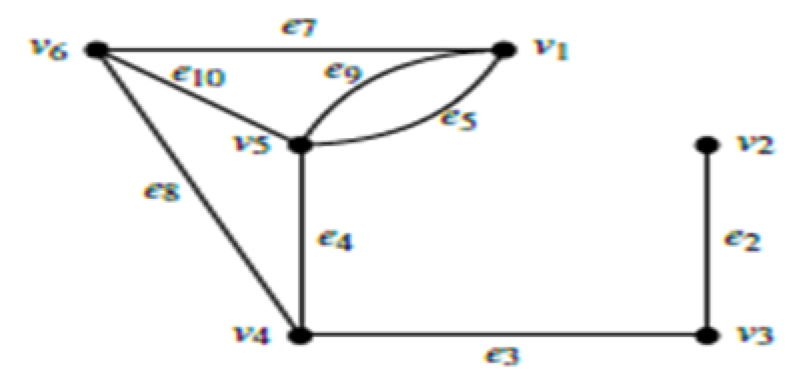
The following examples explain the union, intersection and ringsum of two given graphs.



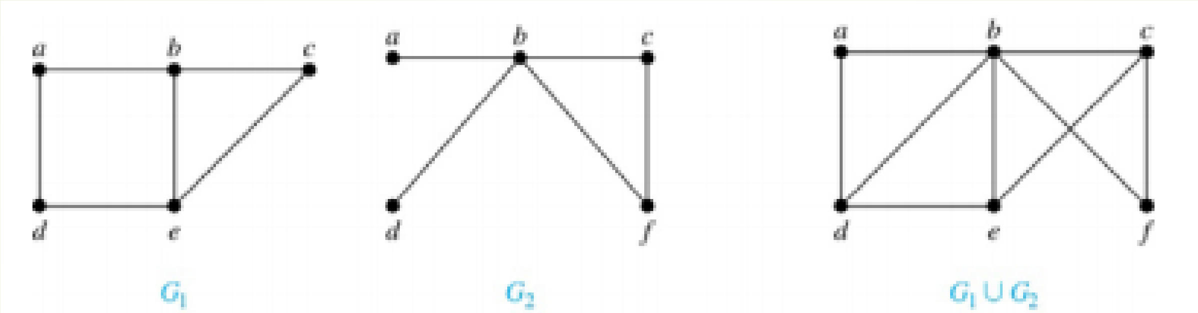
(a) G_1 (b) G_2 (c) $G_1 \cup G_2$ (d) $G_1 \cap G_2$

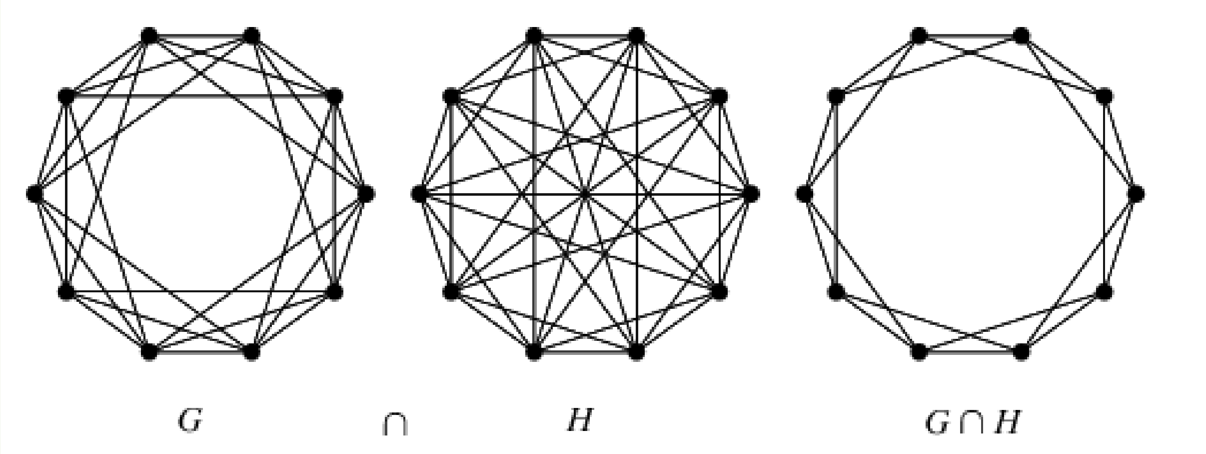
$$G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

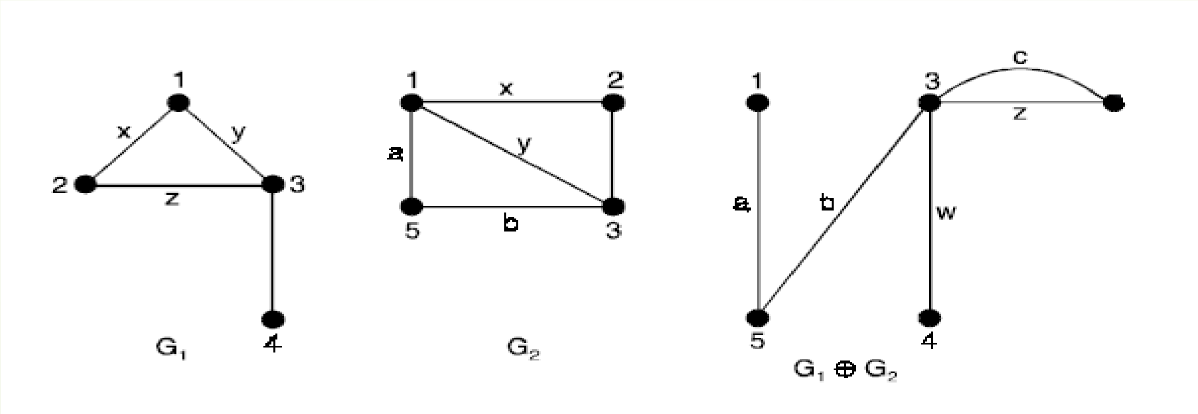
$$G = G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$



$$G = G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$$


a
b
c
d
e
G1
a
b
c
d
f
G2
a
b
c
d
e
f
G1 ∪ G2


G
∩
H
G ∩ H


1
x
y
2
z
3
4
G1
1
x
2
y
a
5
b
3
G2
1
3
c
z
a
b
w
5
4
G1 ⊕ G2

Note:

i) The union, intersection and ringsum operations of graphs are commutative.
That is, $G_1 \cup G_2 = G_2 \cup G_1$, $G_1 \cap G_2 = G_2 \cap G_1$ and $G_1 \oplus G_2 = G_2 \oplus G_1$.

ii) If G_1 and G_2 are edge-disjoint, then $G_1 \cap G_2$ is a null graph (the empty graph on 0 vertices), and $G_1 \oplus G_2 = G_1 \cup G_2$.

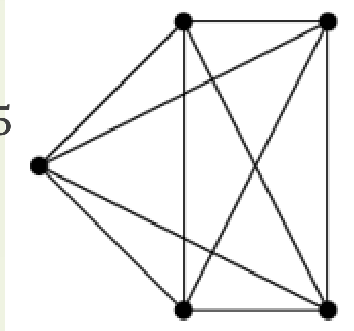
iii) If G_1 and G_2 are vertex-disjoint, then $G_1 \oplus G_2$ is empty.

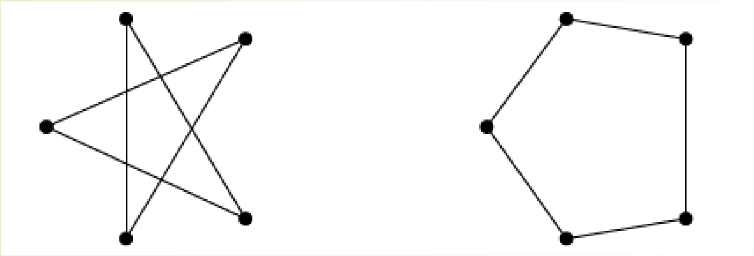
iv) For any graph G, $G \cap G = G \cup G$ and $G \oplus G$ is a null graph.

Definition: A graph G is said to be decomposed into two subgraphs G_1 and G_2, if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a empty graph.

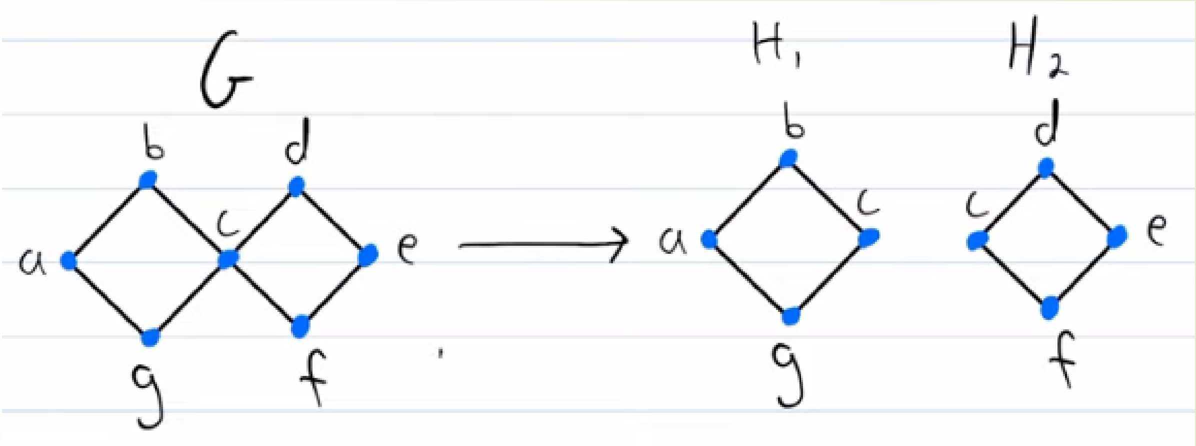
For example the graph K_5 decomposed into the following two subgraphs G_1 & G_2.

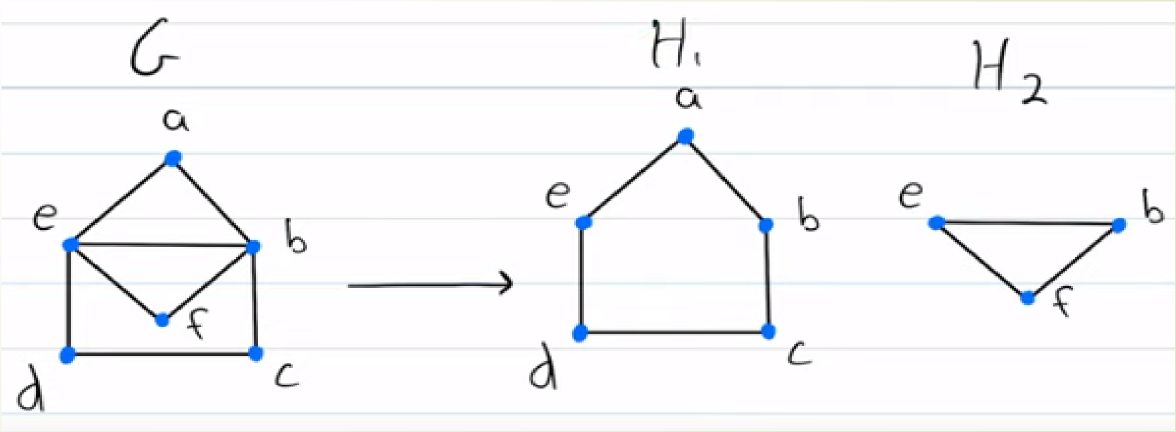
$G = K_5$





G_1 $\qquad$ G_2





Definition: The complement or inverse of a graph G, denoted by $\overline{G}$ is a graph with $V(G) = V(\overline{G})$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Note: that for a graph G and its complement $\overline{G}$, we have

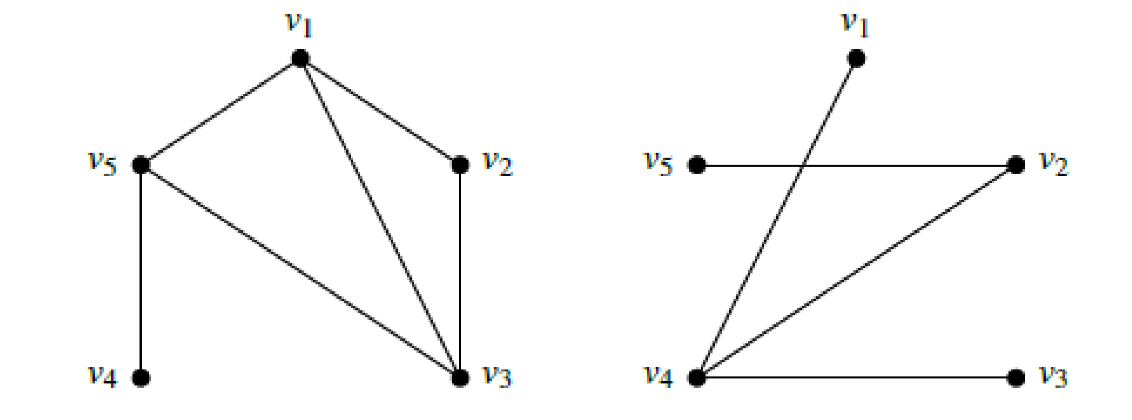
$i)\ G \cup \overline{G} = K_n;$

$ii)\ V(G) = V(\overline{G});$

$iii)\ E(G) \cup \mathrm{E}(\overline{G}) = E(K_n);$

$iv)\ |E(G)| + |\mathrm{E}(\overline{G})| = |E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}.$

A graph and its complement are illustrated below.

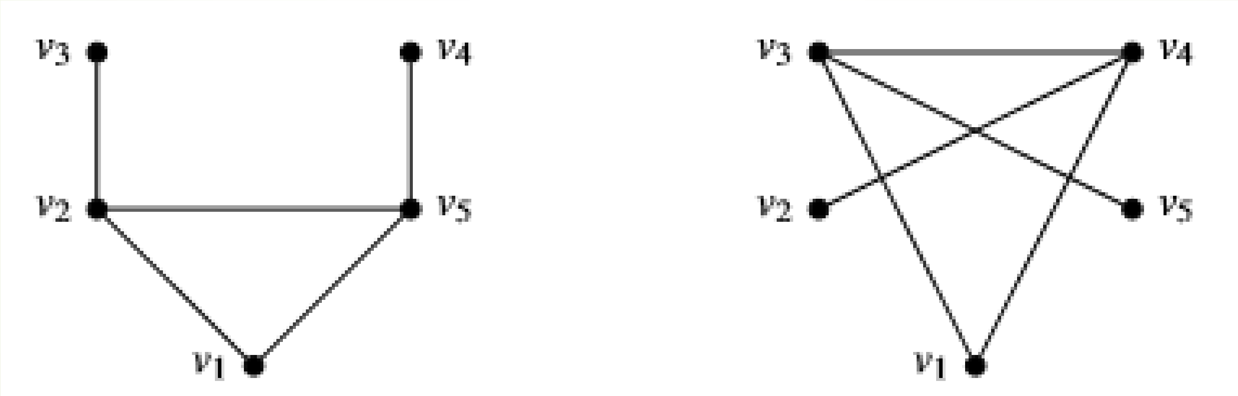


(a) G (b) $\bar{G}$

Definition: A graph G is said to be self-complementary if G is isomorphic to its complement $\overline{G}$. If G is self- complementary, then

$$|E(G)| = |E(\overline{G})| = \frac{1}{2}|E(K_n)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}.$$

The following is example of self-complementary graphs.

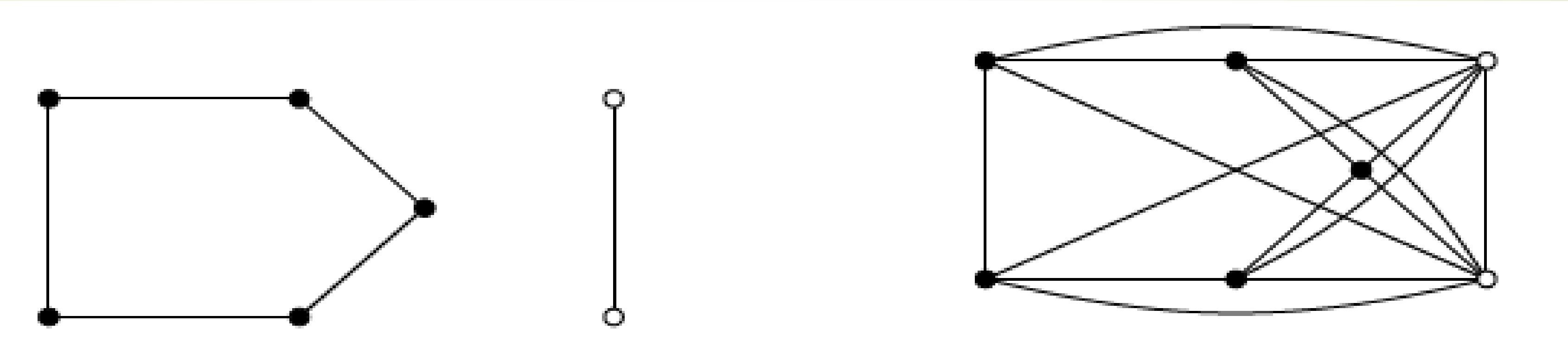


G $\qquad\qquad$ $\overline{G}$

$$f: \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & 3 & 5 & 2 & 4 \end{matrix}$$

Definition: The join of two graphs G and H, denoted by $G + H$ is the graph such that $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

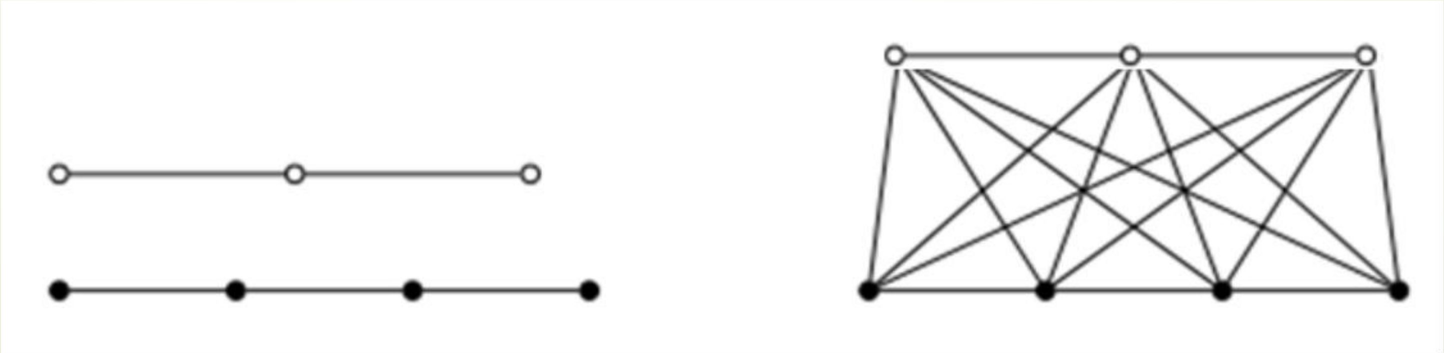
In other words, the join of two graphs G and H is defined as the graph in which every vertex of the first graph is adjacent to all vertices of the second graph.



G $\quad$ H $\quad$ $G + H$

P_3

P_4

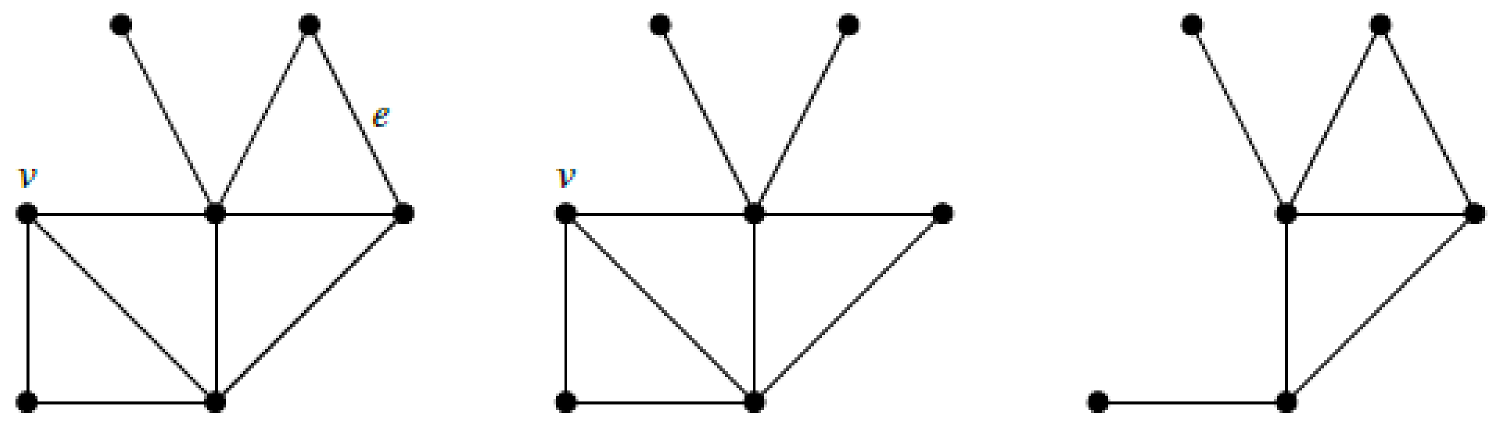


$P_3 + P_4$

Definition: (Edge Deletion in Graphs) If e is an edge of G, then $G - e$ is the graph obtained by removing the edge e of G. The subgraph of G thus obtained is called an edge-deleted subgraph of G. Clearly, $G - \text{e}$ is a spanning subgraph of G.

Definition: (Vertex Deletion in Graphs) If v is a vertex of G, then $G - v$ is the graph obtained by removing the vertex v and all edges G that are incident on v. The subgraph of G thus obtained is called a vertex-deleted subgraph of G. Clearly, $G - v$ will not be a spanning subgraph of G.

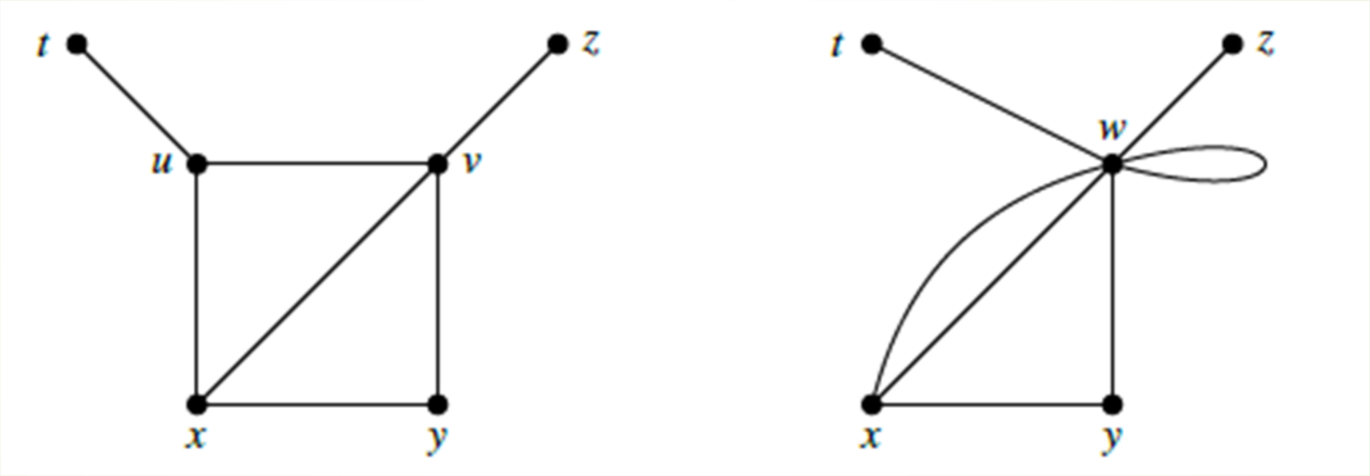
The following figure illustrates the edge deletion and the vertex deletion of a graph G.



(a) G (b) $G - e$ (c) $G - v$

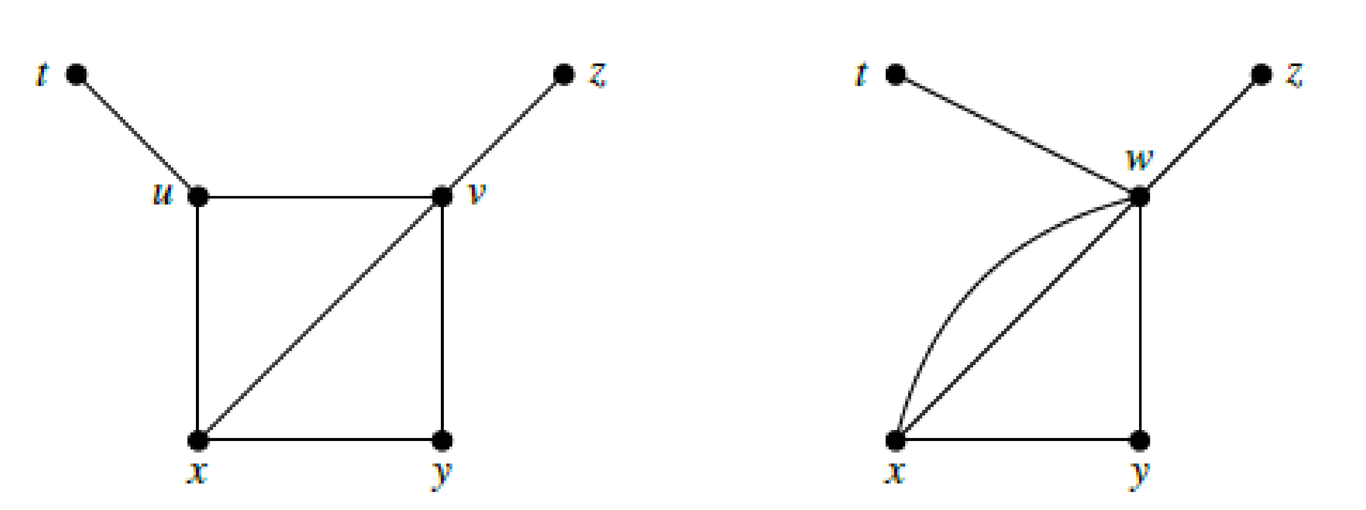
Definition: (Fusion of Vertices) A pair of vertices u and v are said to be fused (or merged) together if the two vertices are together replaced by a single vertex w such that every edge incident with either u or v is incident with the new vertex w.

Note that the fusion of two vertices does not change the number of edges, but reduces the number of vertices by 1.



Fusion of two vertices u, v.

Definition: (Edge Contraction in Graphs) An edge contraction is operation occurs relative to a particular edge e. The edge e is removed and its two incident vertices u and v are merged into a new vertex w, where the edges incident to w each correspond to an edge incident to either u or v. In other words The contraction of e results in a new graph $G' = (V', E')$, where $(V', E') = (V \setminus \{u, v\} \cup \{w\}, E \setminus \{e\})$. A graph obtained by contracting an edge e of a graph G is denoted by $G \circ e$. Vertex fusion is a less restrictive form of this operation.



Edge contraction of a graph $G \circ uv$.

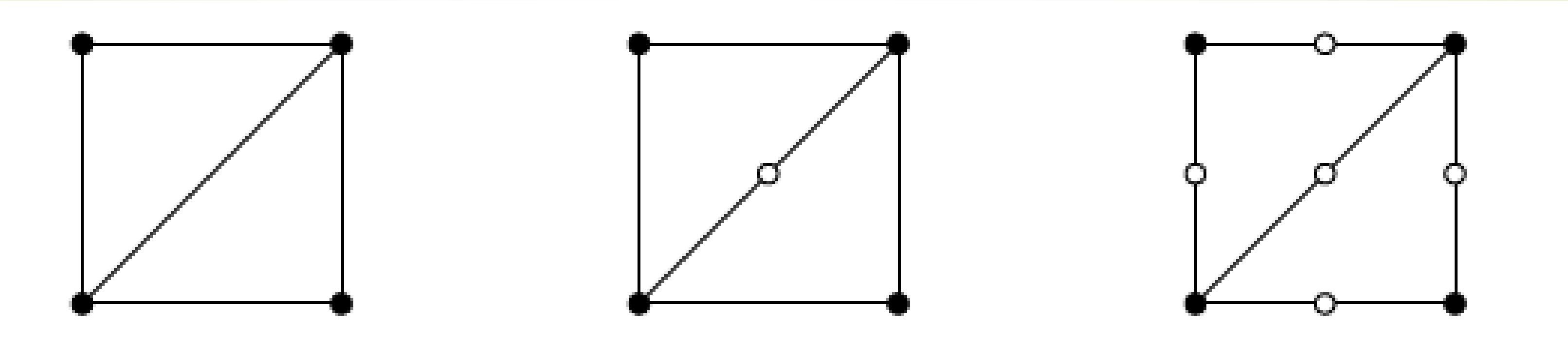
Definition: (Subdivision of an Edge) Let $e = uv$ be an arbitrary edge in G. The subdivision of the edge e yields a path of length 2 with end vertices u and v with a new internal vertex w (That is, the edge $e = uv$ is replaced by two new edges, uw and wv).

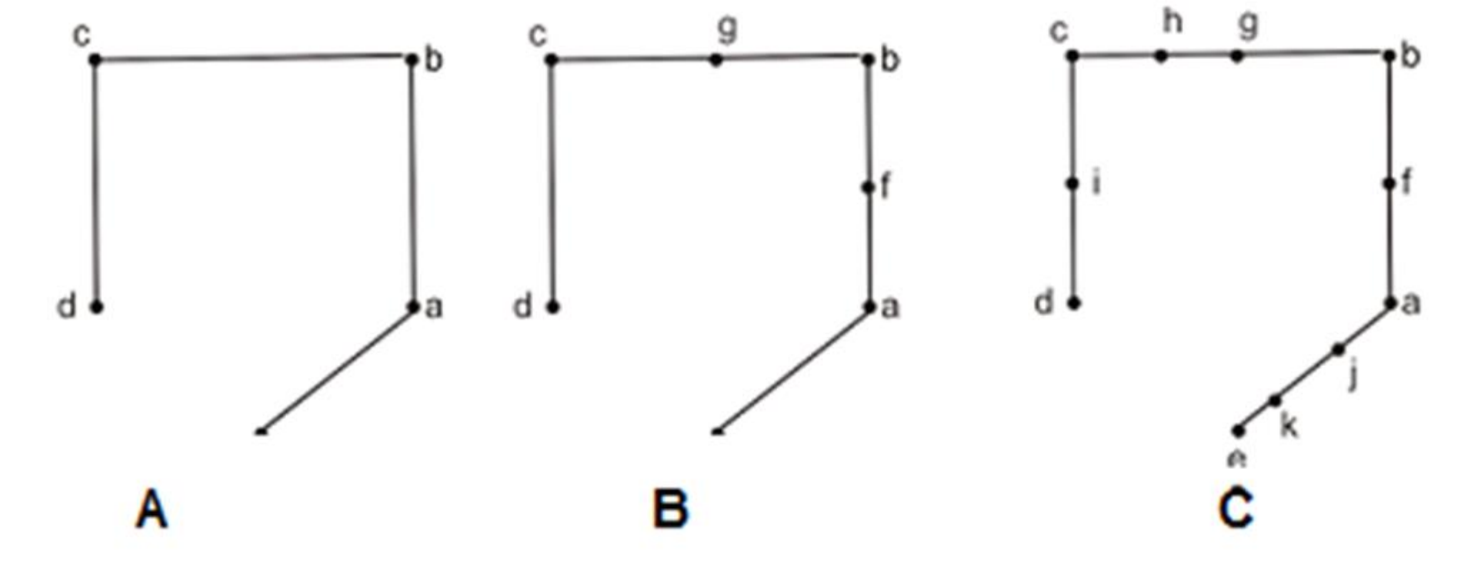
Subdivision of an edge

Definition: A subdivision of a graph G is a graph resulting from the subdivision of (some or all) edges in G. The newly introduced vertices in the subdivisions are represented by white vertices.

Definition: (Homeomorphic Graphs) Two graphs are said to be homeomorphic if both can be obtained by the same graph by subdivisions of edges.

In the following figure, the second and third graphs are homeomorphic, as they are obtained by subdividing the edges of the first graph in the figure.





The graph B and C are not the isomorphic graph, since they are obtained by the graph A, they are homeomorphic graphs.

Definition: (Smoothing Vertices in Graphs) The reverse operation, smoothing out or smoothing a vertex w of degree 2 with regards to the pair of edges (e_i, e_j) incident on w, removes w and replaces the pair of edges (e_i, e_j) containing w with a new edge e that connects the other endpoints of the pair (e_i, e_j) .

Smoothing of the vertex w

Thank You

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