UNIVERSITY OF ANBAR

جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات / الفصل الاول

Connectedness of Graphs

م. د. امين شامان امين

Lecture (5)

Connectedness of Graphs

Dr. Ameen Sh. Ameen
Dept. of Applied Mathematics.
College of Science \ University of Anbar.

Outlines

- ✓ Walk , Path and Distance in Graphs.
- ✓ Connected Graphs.
- ✓ Edge Deleted and Vertex Deleted Subgraphs.
- ✓ Cut-Edges and Cut-Vertices.

Walk , Path and Distance in Graphs

Definition: A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges ($W = v_0e_1v_1e_2 \ldots e_kv_k$) . If the starting and end vertices of a walk are the same (i.e. $v_0 = v_k$) , then such a walk is called a closed walk.

A walk can end on the same vertex on which it began or on a different vertex. A walk can travel over any edge and any vertex any number of times.

Definition: A trail is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time.

Definition: A circuit is a trail that begins and ends on the same vertex. (Closed Trail)

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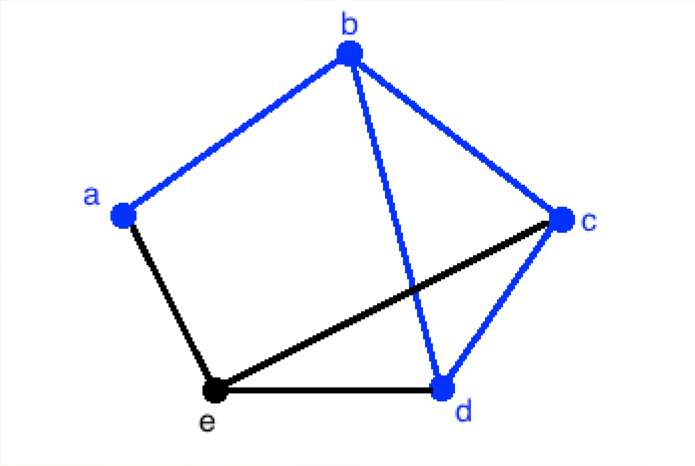
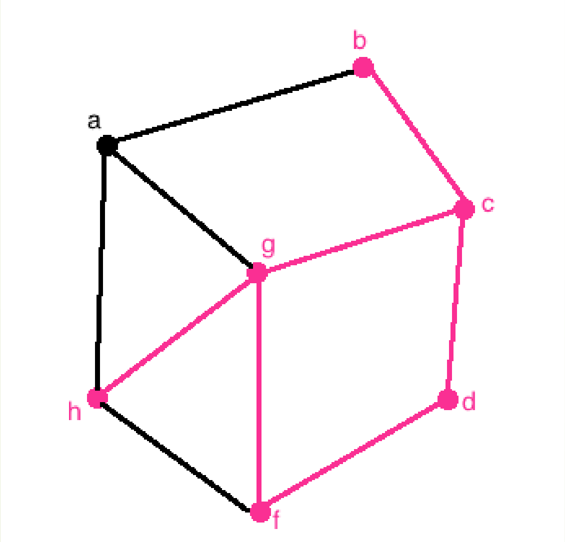
A walk in a graph G is an alternating sequence of vertices and connecting edges in G.

A trail is a walk that does not pass over the same edge twice.

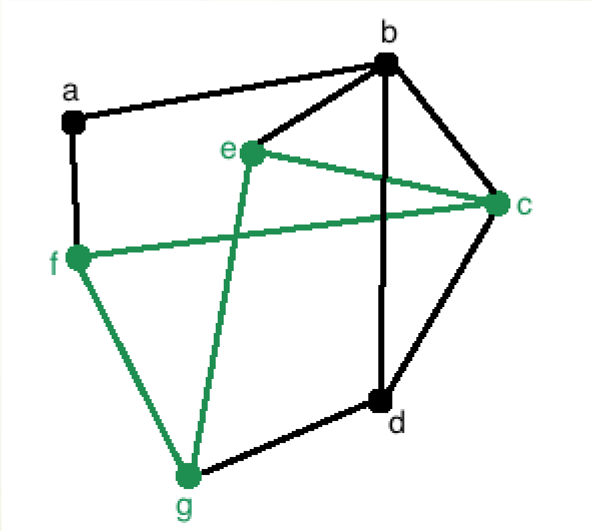
A circuit is a trail that begins and ends on the same vertex. (Closed Trail)



Open walk & Trail not circuit



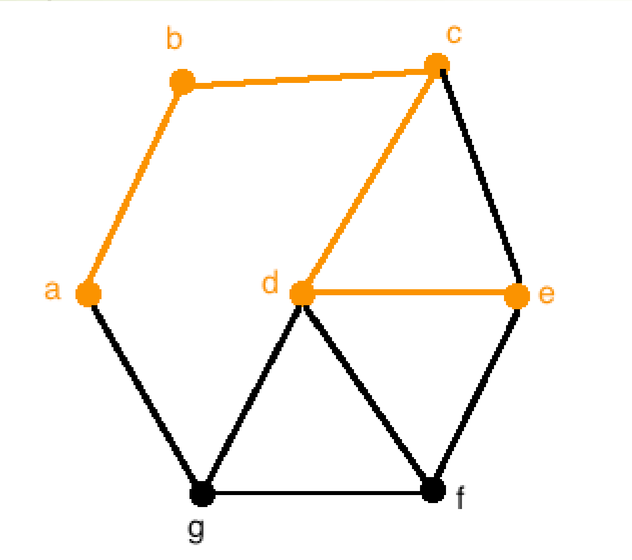
Open walk & Not trail (Repeat one edge)



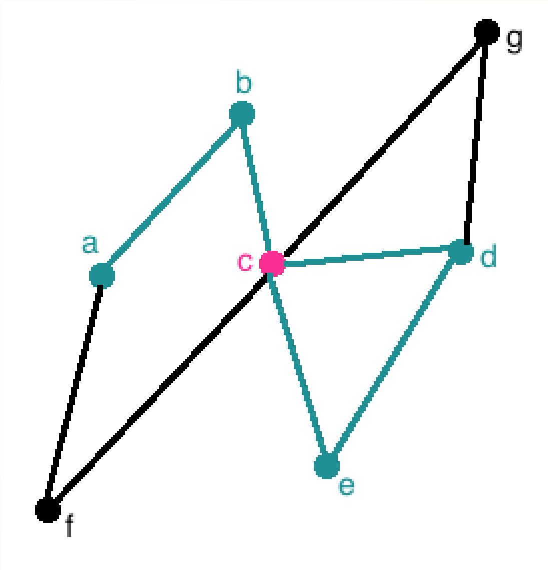
Closed walk , closed Trail , Circuit

Definition: A path is a trail that does not include any vertex twice, except that its first vertex might be the same as its last.

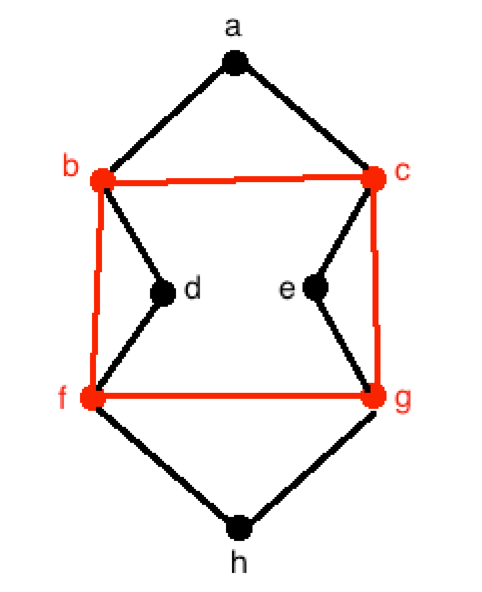
Definition: A cycle is a path that begins and ends on the same vertex. (Closed path)



path



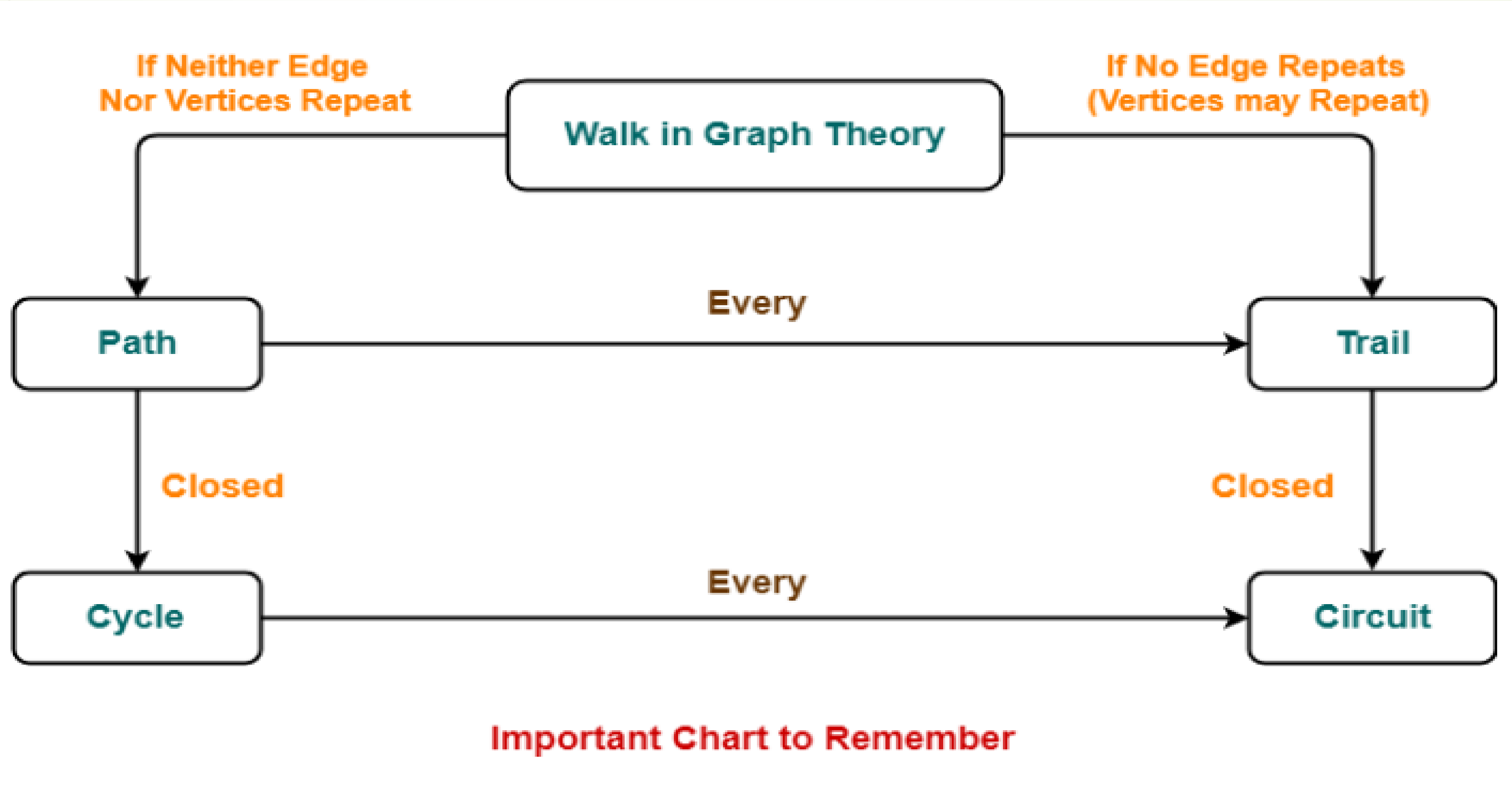
Not path
Vertex *c* repeat twice



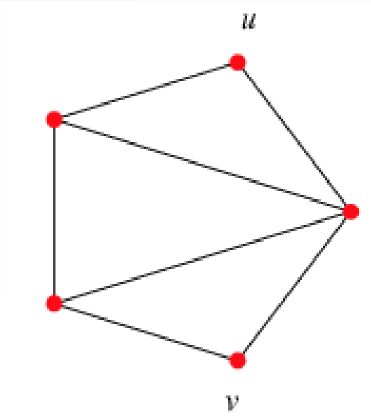
cycle

Note: i) The length of (walk , circuit , path and cycle) is the number of edges in it.

ii) A path of order n is denoted by P_n and a cycle of order n is denoted by C_n. Every edge of G can be considered as a path of length 1. Note that the length of a path on n vertices is $n-1$.

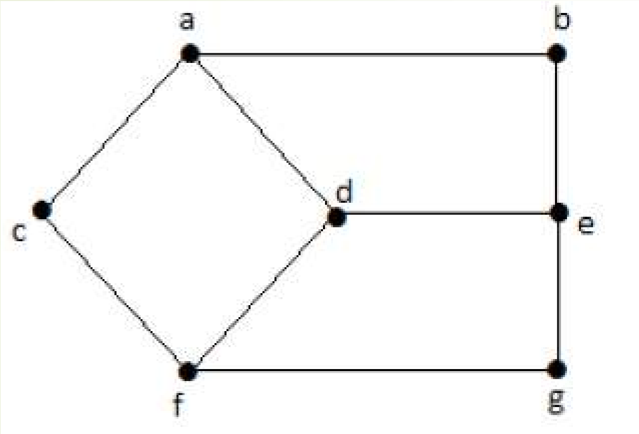


Definition: The distance between two vertices u and v in a graph G, denoted by $d_G(u,v)$ or simply $d(u,v)$, is the length (number of edges) of a shortest path connecting them.



$d(u, v) = 2$

Definition: The eccentricity of a vertex v, denoted by $\varepsilon(v)$, is the maximum distance between v and any other vertex.



The distance from vertex a to b is 1

The distance from vertex a to c is 1

The distance from vertex a to f is 2

The distance from vertex a to d is 1

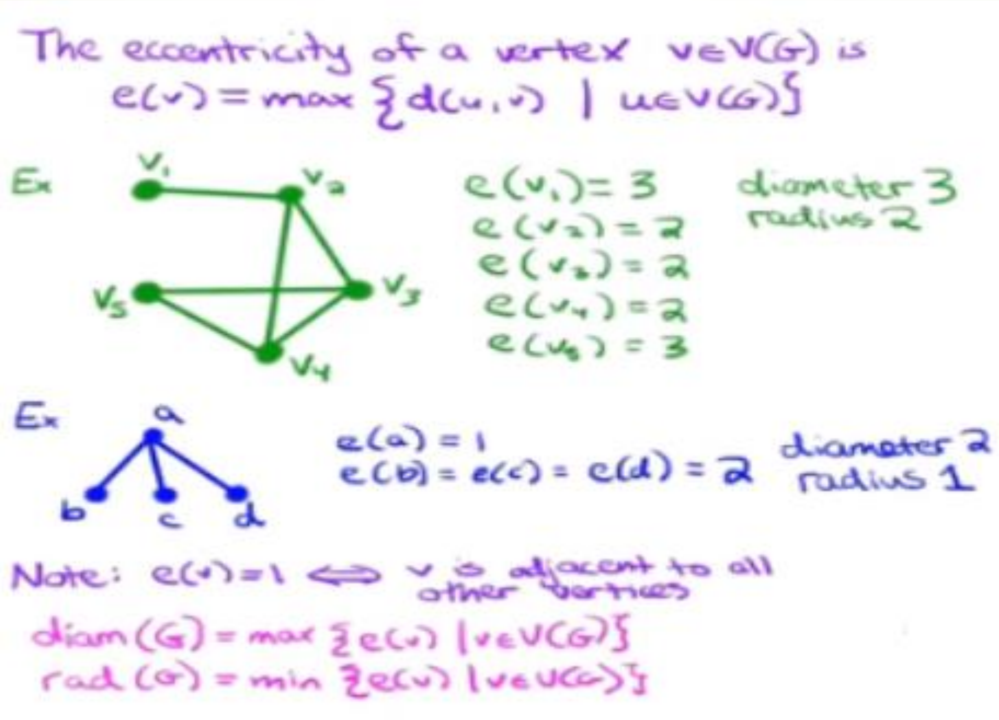
The distance from vertex a to e is 2

The distance from vertex a to g is 3

Hence, the maximum eccentricity of vertex 'a' is 3,

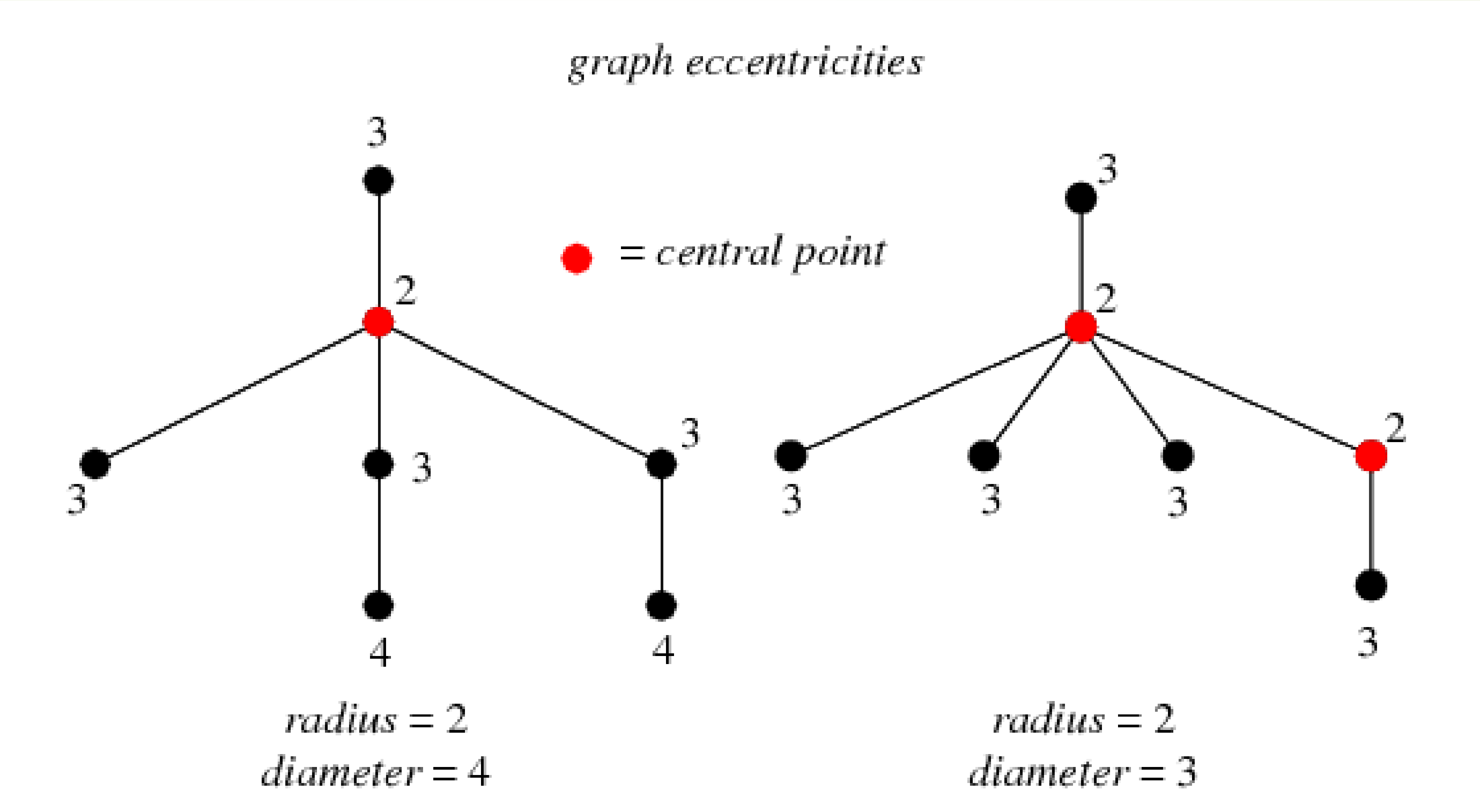
Definition: The radius r of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex in the graph. That is, $rad(G) = min_{v \in V(G)}\, \varepsilon(v)$.

Definition: The diameter of a graph G, denoted by $diam(G)$ is the maximum eccentricity of any vertex in the graph. That is, $diam(G) = max_{v \in V(G)}\, \varepsilon(v)$.



Definition: A center of a graph G is a vertex in G whose eccentricity equal to the radius of G.

Definition: A vertex v of G is called peripheral vertex of a graph G, if $\varepsilon(v) = diam(G)$.



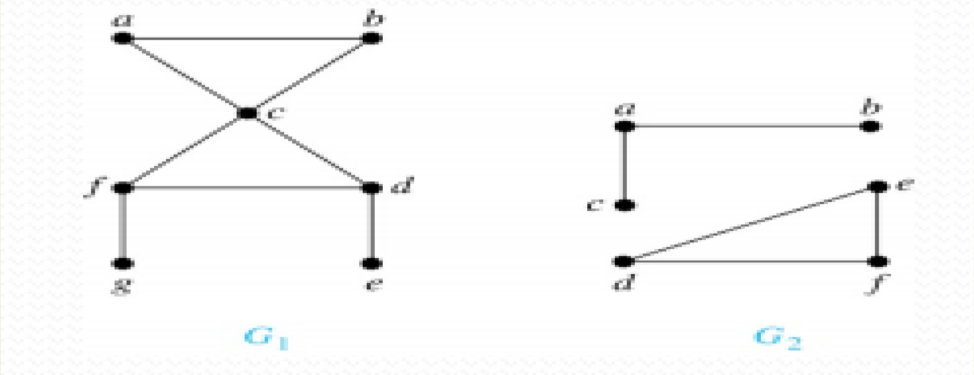
Connected Graphs

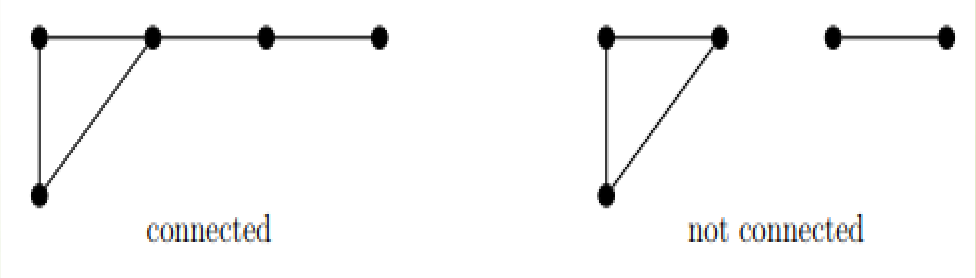
Definitions:

1) Two vertices u and v are said to be connected if there exists a path between them.

2) A graph G is said to be connected if any two vertices of G are connected.

Note that Complete graphs, complete bipartite graphs, cyclic graphs and path graphs are a few examples of connected graphs. A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no vertex in common. These disjoint connected subgraphs are called the connected components of the graph.





A direct application of the definition of a connected or disconnected graph gives the following results and hence the proof is omitted.

Theorem 1: A graph is disconnected if and only if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and other is in V_2.

Theorem 2: If a graph (connected or disconnected) has exactly two vertices of odd degree then there must be a path joining these vertices.

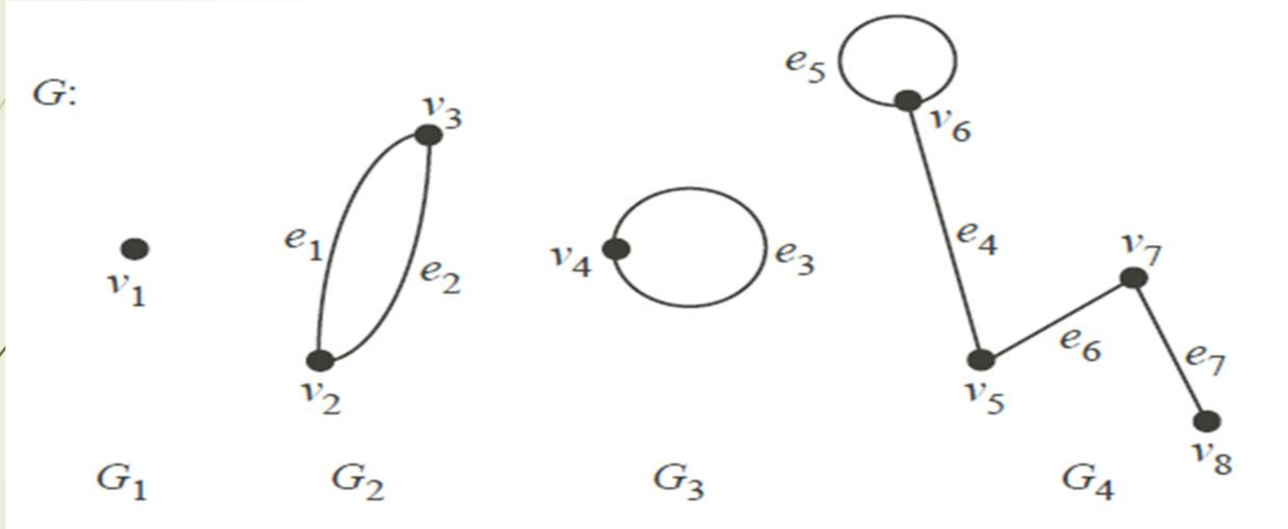
Theorem 3: A simple graph with n vertices and k components can have at most have $(n-k)(n- k +1)/2$ edges.

As a direct application of Theorem 3, we have the following result.

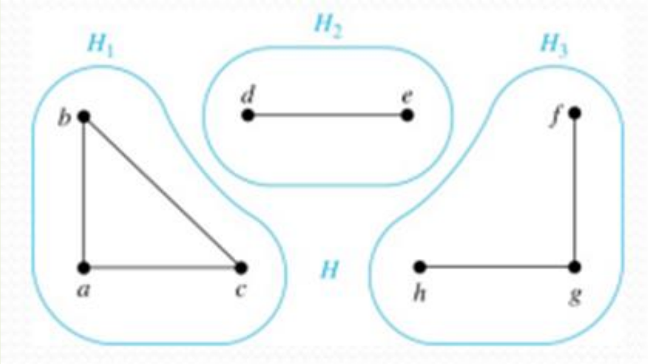
Corollary : Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.

Theorem 4: A connected graph G is bipartite if and only if G has no odd cycles.

Definition: A connected component of a graph is a maximal subgraph in which the vertices are all connected, and there are no connections between the subgraph and the rest of the graph. The number of components of a graph G is denoted by $\omega(G)$.

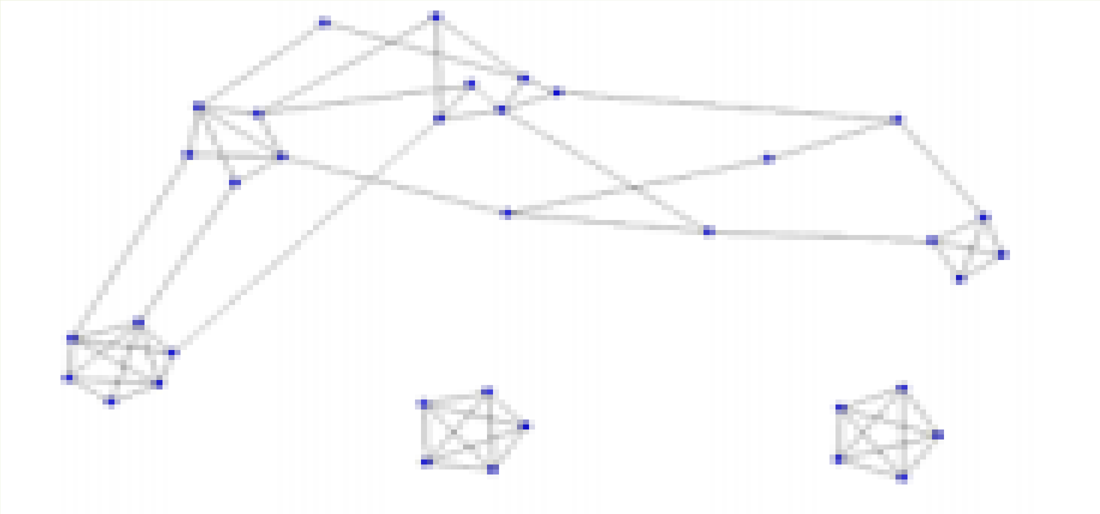


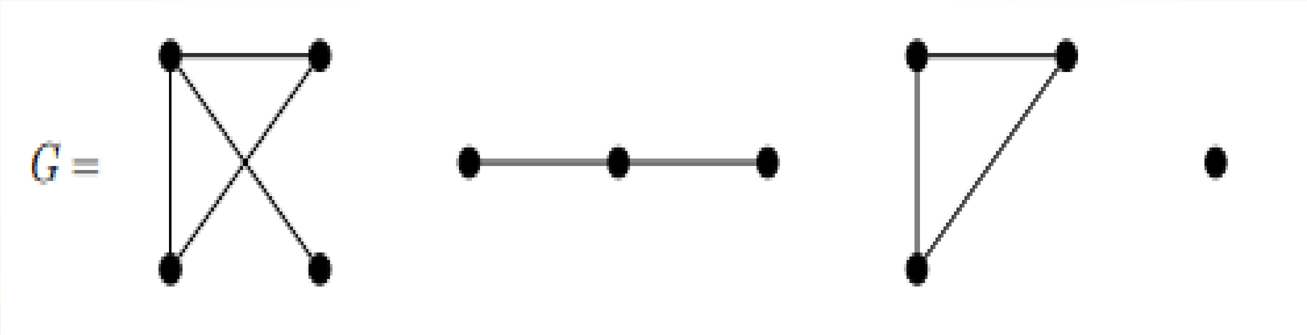
Maximal means that it is the largest possible subgraph you could not find another vertex anywhere in the graph such that it could be added to the subgraph and all the vertices in the subgraph would still be connected.



A graph that is itself connected has exactly one component.

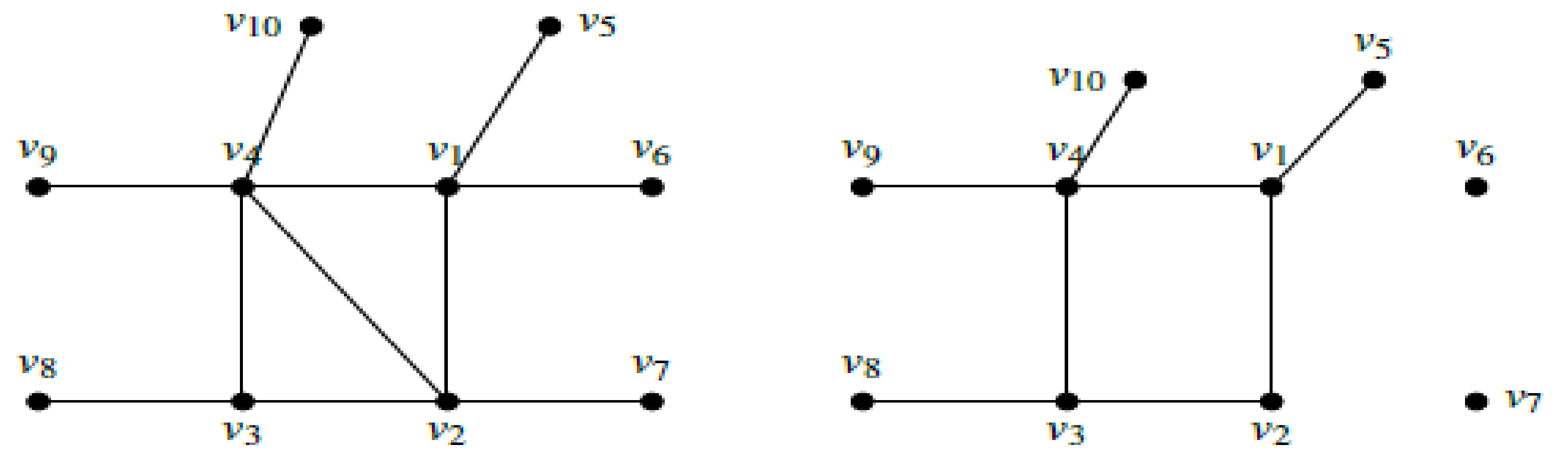
A vertex with no edges is itself a connected component.




G =

Edge Deleted and Vertex Deleted Subgraphs

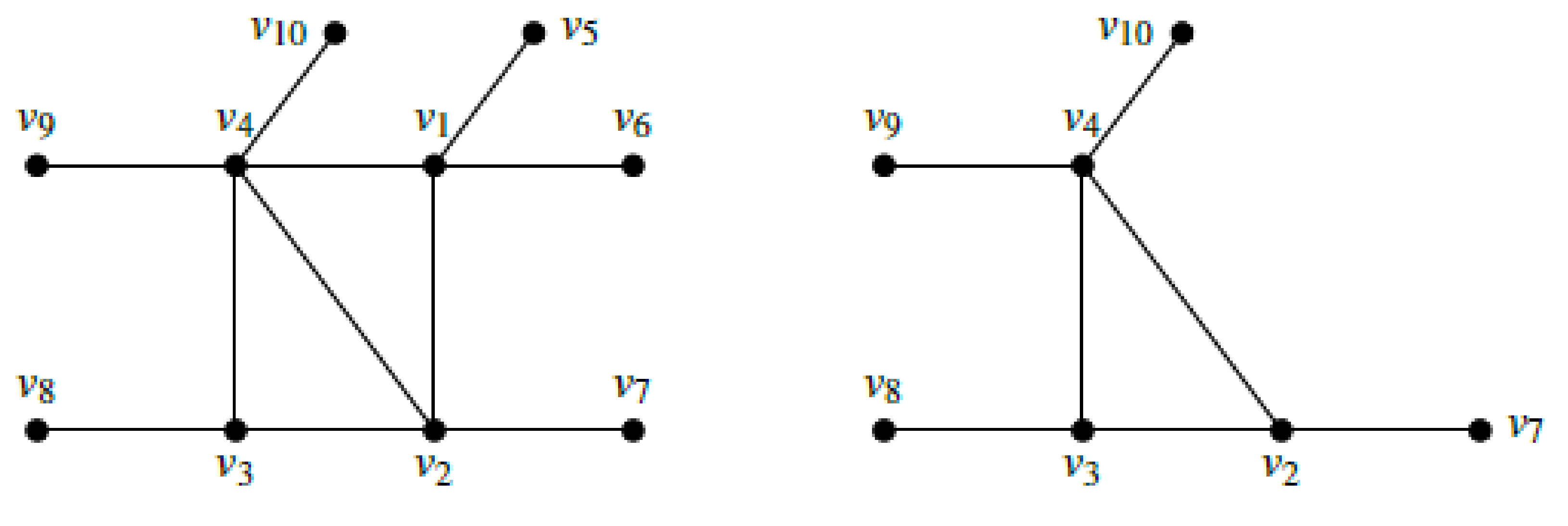
Definition: (Edge Deleted Subgraphs) Let $G(V, E)$ be a graph and $F \subseteq E$ be a set of edges of G. Then, the graph obtained by deleting F from G, denoted by $G - F$, is the subgraph of G obtained from G by removing all edges in F. Note that $V(G - F) = V(G)$.



(a) G (b) $G' = G - \{v_1v_6, v_2v_4, v_2v_7\}$.

Note that any edge deleted subgraph of a graph G is a spanning subgraph of G.

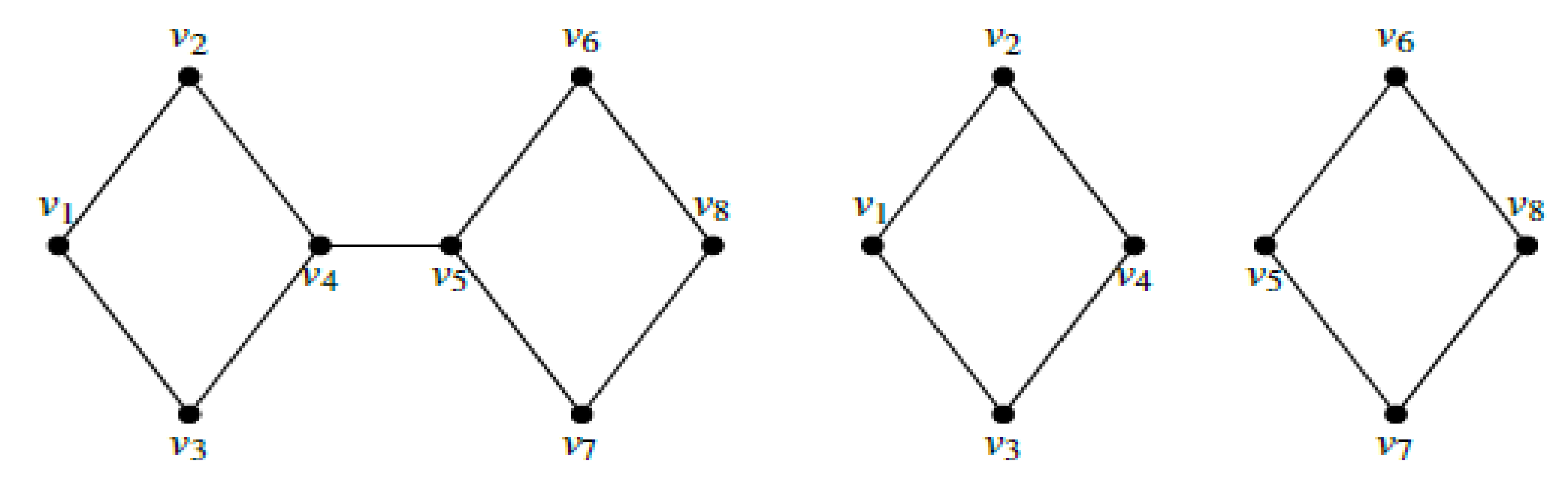
Definition: (Vertex Deleted Subgraphs) Let $W \subseteq V(G)$ be a set of vertices of G. Then the graph obtained by deleting W from G, denoted by $G - W$, is the subgraph of G obtained from G by removing all vertices in W and all edges incident to those vertices.



(a) G (b) $G' = G - \{v_1, v_5, v_6\}$.

Cut-Edges and Cut-Vertices

Definition: An edge e of a graph G is said to be a cut-edge or a bridge of G if $G - e$ is disconnected.

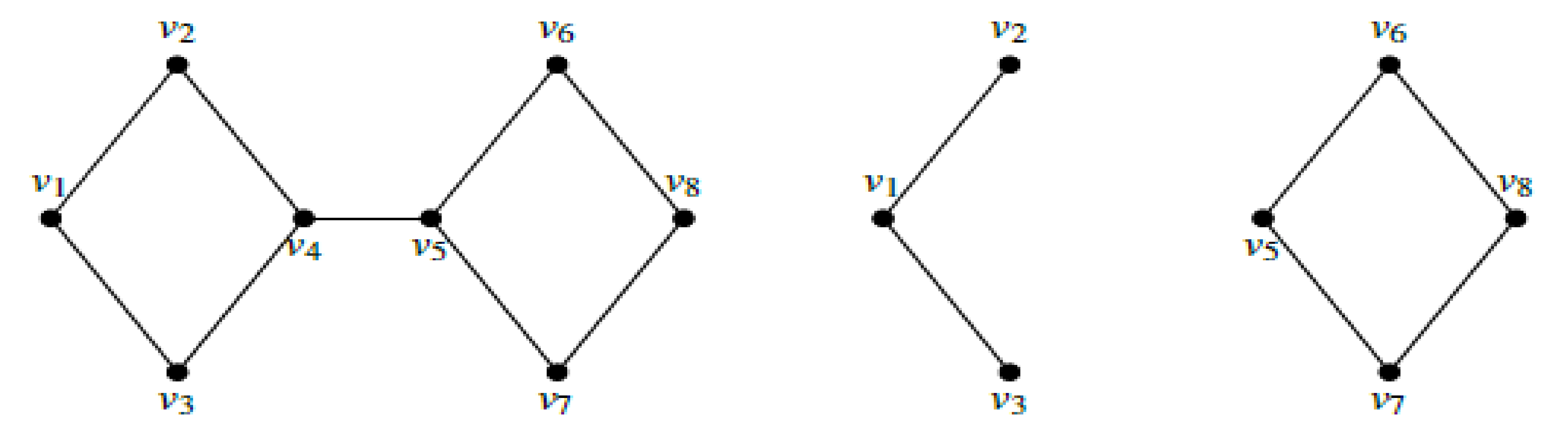


G $\qquad\qquad$ $G - v_4v_5$

The following is a necessary and sufficient condition for an edge of a graph G to be a cut-edge of G.

Theorem: An edge e of a graph G is a cut-edge of G if and only if it is not contained in any cycle of G.

Definition: A vertex v of a graph G is said to be a cut-vertex of G if $G - v$ is disconnected.



G $\qquad$ $G - v_4$

Note: Remove any pendent vertex will not disconnect a given graph, every cut-vertex will have degree greater than or equal to 2. But, note that every vertex v, with $d(v) \geq 2$ need not be a cut-vertex.

Thank You

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