جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات / الفصل الاول

Directed Graphs

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Lecture (8)

Directed Graphs

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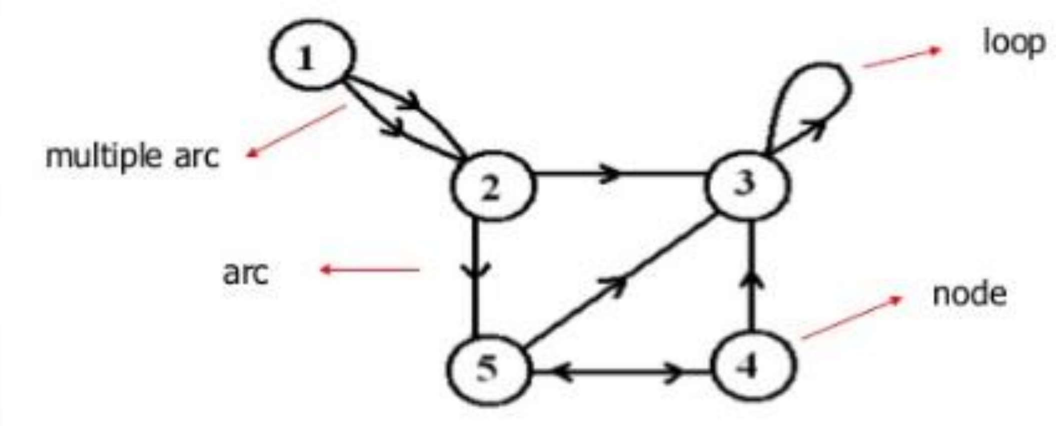
Directed Graphs

Definition: A graph is called **directed graph or digraph** D if each arc (edge) of graph has a direction.



Note:

1) Let $e = uv$ in the digraph D:

i) If $u = v$, then e is loop.

ii) u is called a **tail** of e, v the **head** of e and u, v are the ends of e.

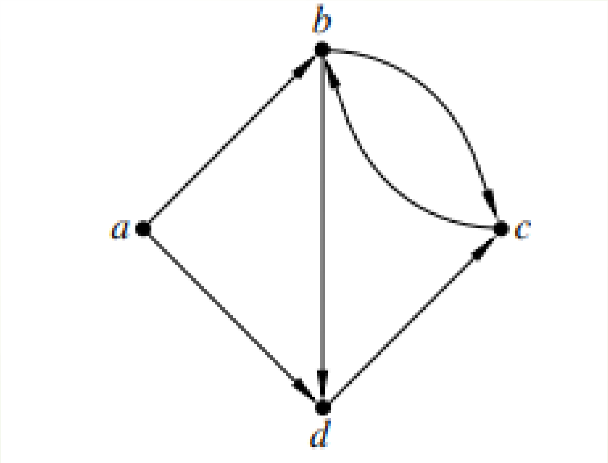
2) Two edges e, f are **parallel** if they have the **same tails** and the **same heads**.

3) If D has no loops or parallel edges, then we say that D is simple.

Definition: (Degree in Digraph) The **in-degree** of vertex v in a directed graph D is the number of edges which are **coming into** the vertex v and is denoted by $d^-(v)$. The **out-degree** vertex v in a directed graph D is the number of edges which are **going out** from the vertex v and is denoted by $d^+(v)$.

Note: In any digraph $D \rightarrow \sum_{v\in V(D)} d^-(v) = \sum_{v\in V(D)} d^+(v)$.

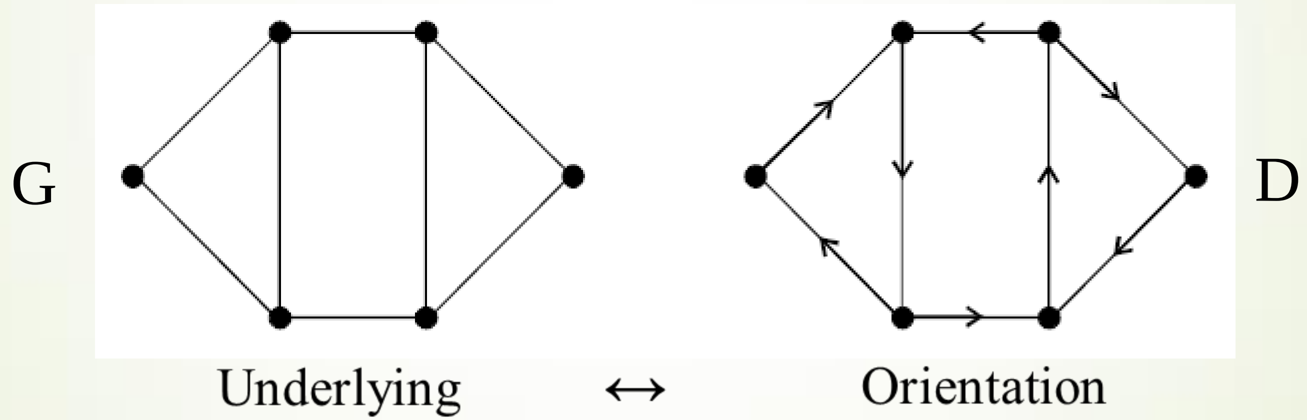
Theorem: In a digraph D, the sum of in-degrees and the sum of out-degrees of the vertices is equal to twice the number of edges. That is,
$\sum_{v\in V(D)} d^-(v) = \sum_{v\in V(D)} d^+(v) = \varepsilon$.



$$\sum_{v\in V(D)} d^-(v) = \sum_{v\in V(D)} d^+(v) = 6$$

Definition: (Orientation of Graph) If we assign directions to the edges of a given graph G, then the new directed graph D is called an **orientation** of G.

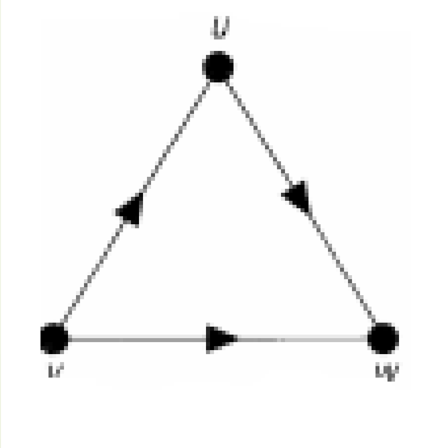
Definition: (Underlying Graph of Directed Graph) If remove the directions of the edges of a directed graph D, then the reduced graph G is called the **underlying** graph of D.



Note: The orientation of a graph G is **not unique**. Every edge of G can take any one of the two possible directions. Therefore, a graph $G = (V, E)$ can have at most $2^{|E|}$ different orientations. But, a directed graph can have a **unique** underlying graph.

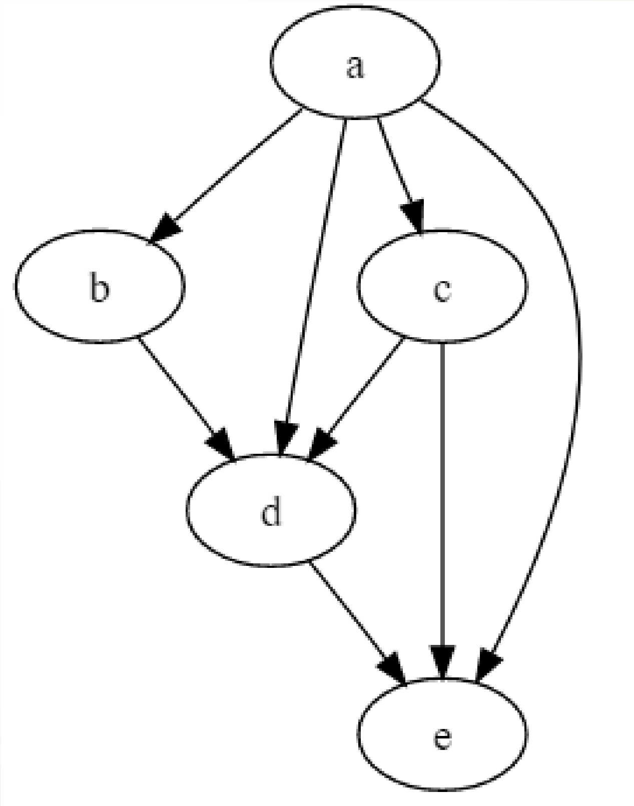
Definition: (Sources and Sinks) A vertex with **zero in-degree** is called a **source** and a vertex with **zero out-degree** is called a **sink**.

Source



Sink

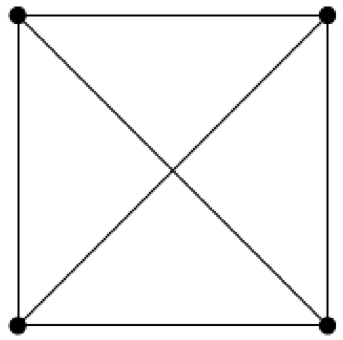
Source

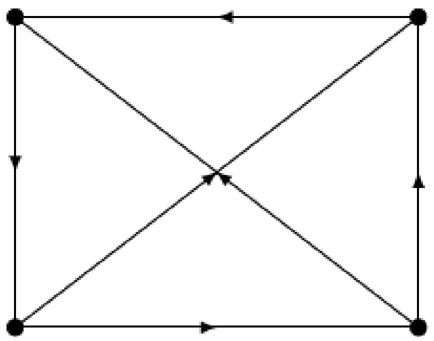


Sink

Definition: If the edges of a complete graph are each given an orientation, the resulting directed graph is called a **tournament**.(Clearly, a tournament is an orientation of K_n)

Complete Graph K_4

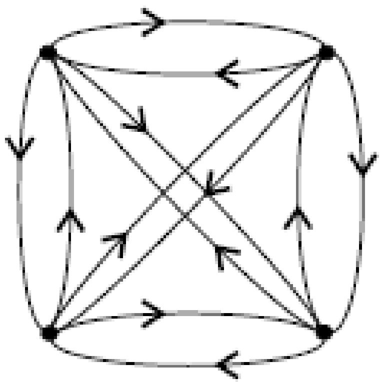




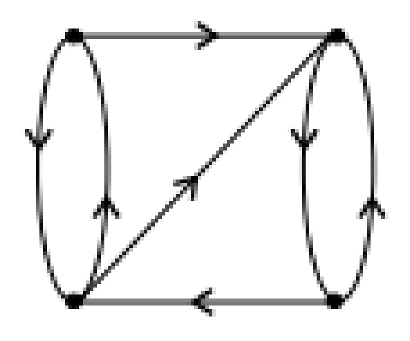
Tournament Orientation of K_4

Definition: (Complete Digraph) A complete digraph is a digraph in which every pair of distinct vertices is connected by a pair of unique edges (**one in each direction**).

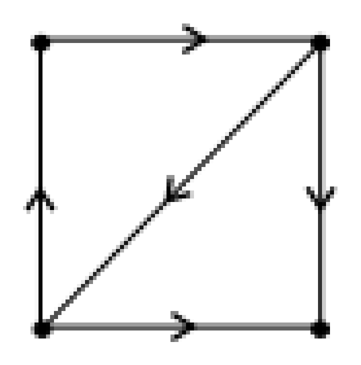
Complete Digraph



Oriented graph: A digraph containing no symmetric pair of arcs (edges) is called an oriented graph

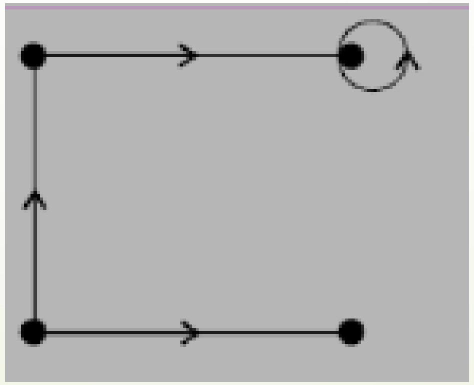


Not Oriented graph



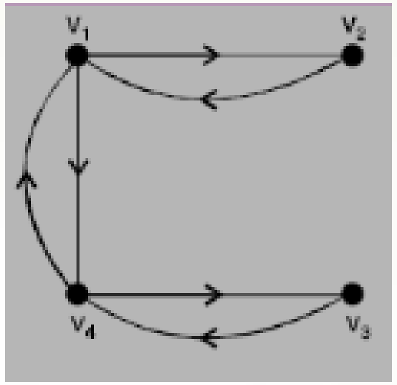
Oriented graph

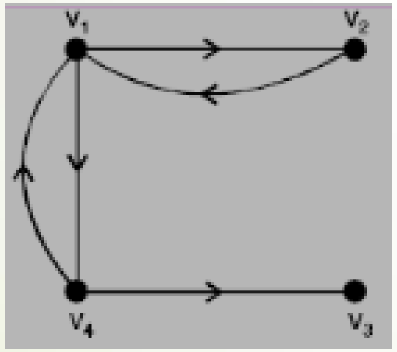
Asymmetric Digraph: Digraph that have at most one directed edge between a pair of vertices, but are allowed to have self-loop are called asymmetric or anti- symmetric digraph.



Symmetric Digraph are directed graph where all edges are bidirected (that is, for every arrow that belongs to the digraph, the corresponding inversed arrow also belongs to it).

Symmetric Digraph

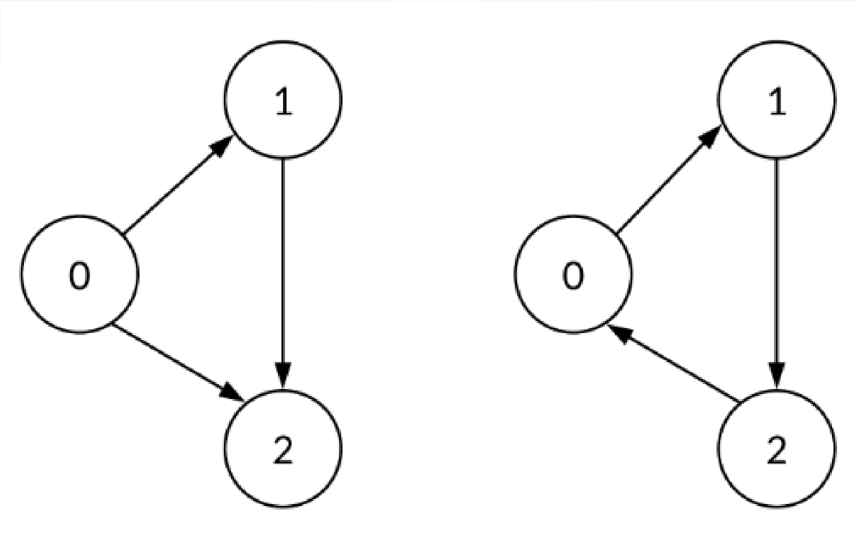




Neither Symmetric Nor Asymmetric Digraph

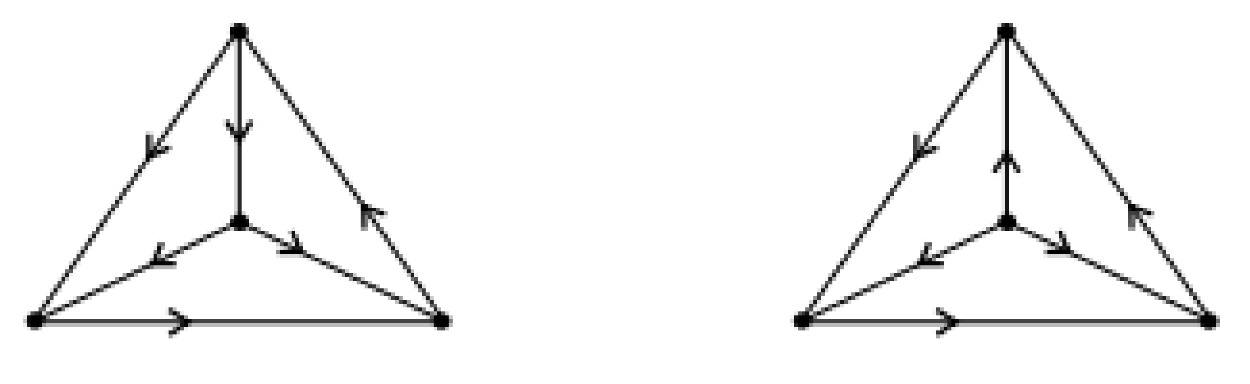
Directed Acyclic Graphs are directed graphs with no directed cycles.

DAG



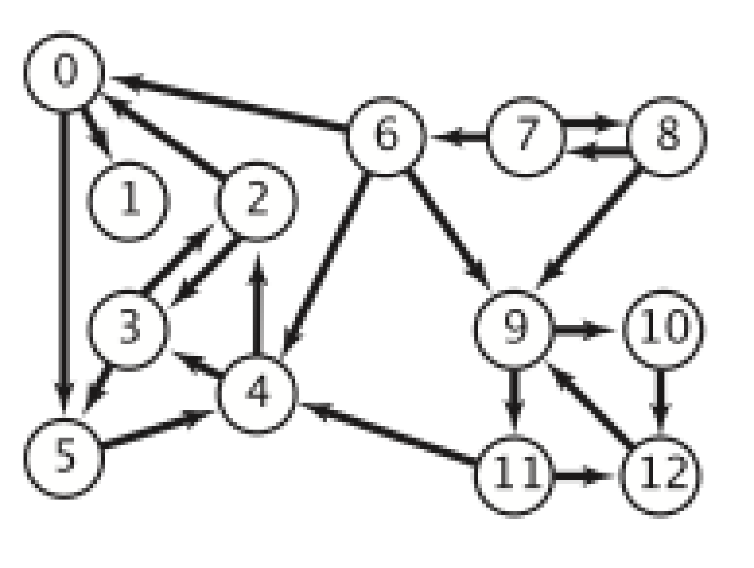
Not DAG
i.e DCG

Isomorphic digraphs: Two digraphs are said to be isomorphic if their underlying graphs are isomorphic and the direction of the corresponding arcs are same.



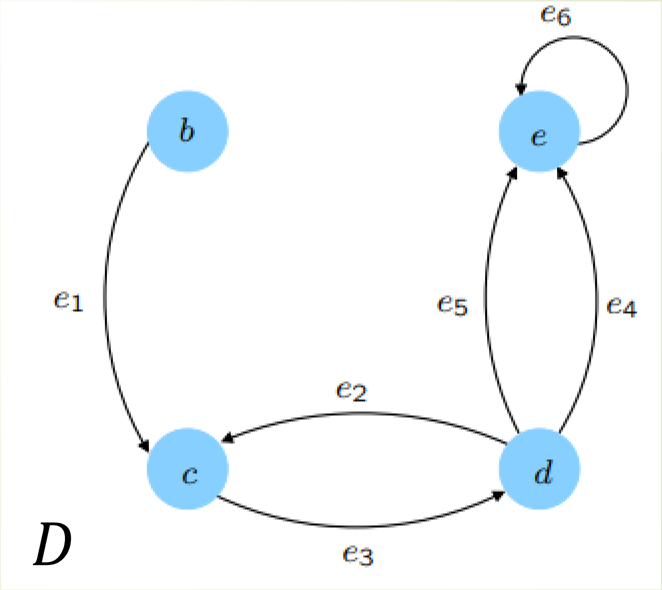
Two non-isomorphic digraphs

Connectedness Digraph



In **an undirected graph** G, a walk is a list $v_1 e_1 v_2 e_2 \cdots v_{k-1} e_k v_k$ such that $v_1, v_2, \cdots, v_k$ are vertices; $e_1, e_2, \cdots, e_k$ are edges; and $\forall i = 1, \ldots, k-1$, $e_i = v_i v_{i+1}$. The length of a walk is the number of edges. A trail is a walk without repeated edges. A path is a trail without repeated vertices. A trail whose first and last vertices are the same is called a closed trail, or a circuit. A closed path is called a cycle.

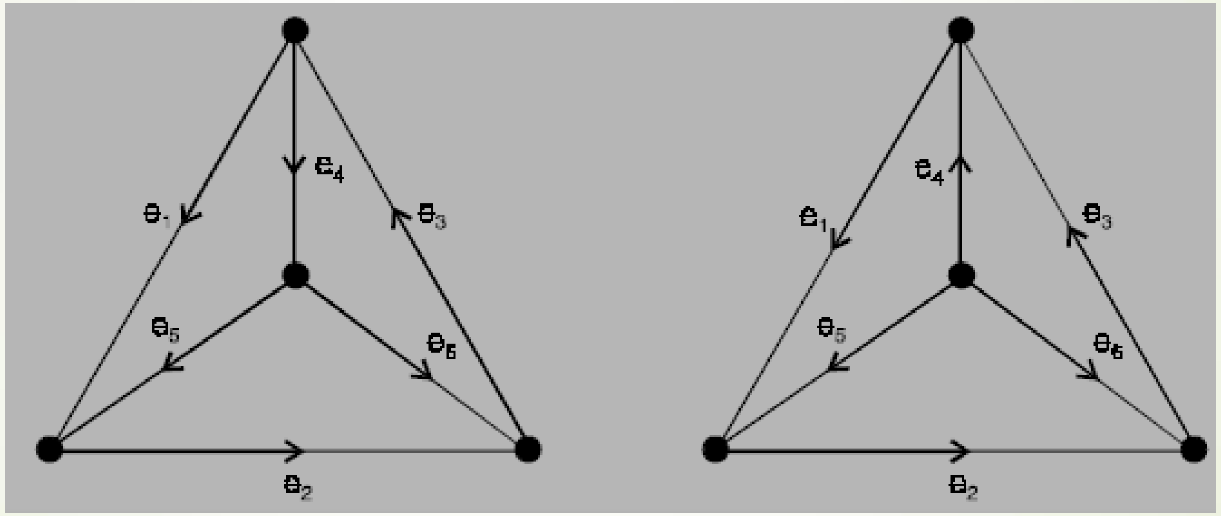
In **a directed graph** walk, trail, and path are defined similarly satisfying the follow the arrows rule: the head of an edge is the tail of the next edge in the sequence.



The sequence $b\, e_1\, c\, e_3\, d\, e_4\, e$ is a $b-e$ directed walk, a $b-e$ directed trail, and a $b-e$ directed path in D.

A digraph D is said to be **weakly connected** if its underlying graph is connected.

A digraph D is **strong or strongly connected** if for every pair u, v of vertices, D contains both a $u - v$ path and a $v - u$ path



strongly connected　　weakly connected

Thank You

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