2.5 Cartesian Vectors (Vectors in 3-Dimensions)

- **Right-Handed Coordinate System:** We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be *right-handed* if the thumb of the right hand points in the direction of the positive *z* axis when the right-hand fingers are curled about this axis and directed from the positive *x* towards the positive *y* axis, Fig. 2–21.

Fig. 2–21

- **Rectangular Components of a Vector:** A vector **A** may have one, two, or three rectangular components along the *x, y, z* coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when **A** is directed within an octant of the *x, y, z* frame, Fig. 2–22 , then by two successive applications of the parallelogram law, we may resolve the vector into components as: $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$ and then $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$. Combining these equations, to eliminate **A'**, **A** is represented by the vector sum of its *three* rectangular components,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \qquad (2\text{–}2)$$

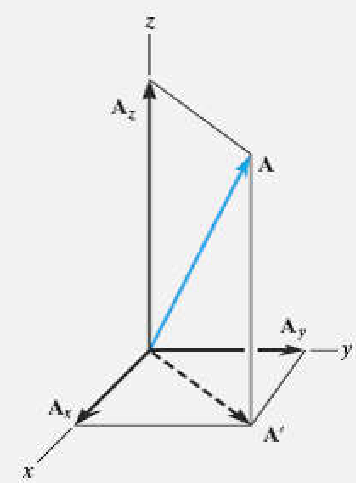


Fig. 2–22

- **Cartesian Unit Vectors:** In three dimensions, the set of Cartesian unit vectors, **i** , **j** , **k** , is used to designate the directions of the *x, y, z* axes, respectively.

The positive Cartesian unit vectors are shown in Fig. 2–23.

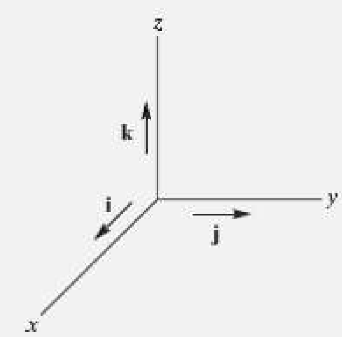


Fig. 2–23

- Since the three components of **A** in Eq. 2–2 act in the positive **i** , **j** , and **k** directions, Fig. 2–24 , we can write **A** in Cartesian vector form as:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad (2\text{–}3)$$

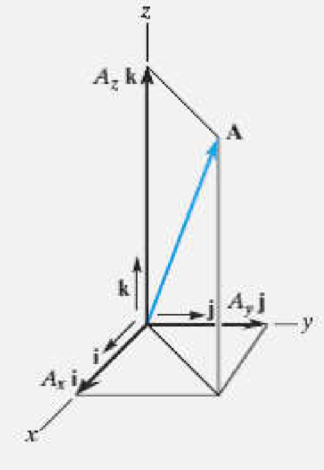


Fig. 2–24

- **Magnitude and direction of a Cartesian Vector:**

As shown in Fig. 2–25 , from the blue right triangle,

$A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle,

$A' = \sqrt{A_x^2 + A_y^2}$.

Combining these equations to eliminate A' yields:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad (2\text{–}4)$$

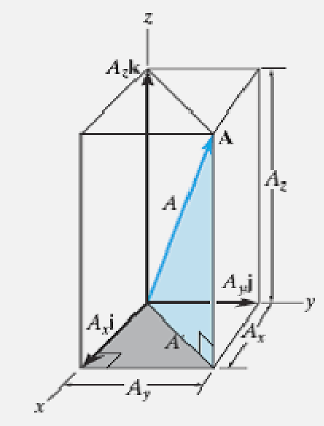


Fig. 2–25

- Now the *direction* of **A** will be defined by the *coordinate direction angles* α (alpha), β (beta), and γ (gamma), measured between the *tail* of **A** and the *positive x, y, z* axes provided they are located at the tail of **A** , Fig. 2-26 and Fig. 2-27 .

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A} \qquad (2\text{–}5)$$

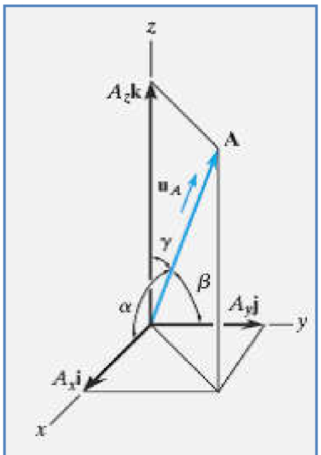


Fig. 2–26

These numbers are known as the ***direction cosines of A***. Once they have been obtained, the coordinate direction angles α, β, and γ can then be determined from the inverse cosines.

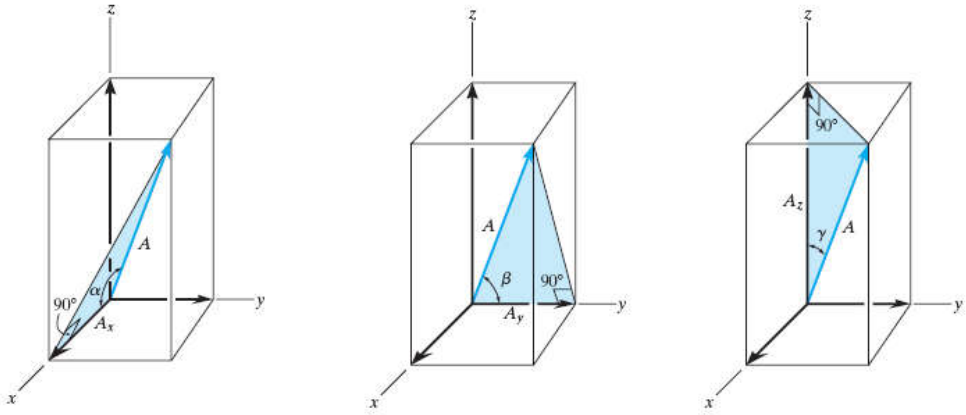


Fig. 2–27

- An easy way of obtaining these direction cosines is to form a unit vector u *A* in the direction of *A* , Fig. 2–26 . If A is expressed in Cartesian vector form, $\mathbf{A} = A_x\mathbf{i} + A_y\,\mathbf{j} + A_z\mathbf{k}$, then $\mathbf{u}_A$ will have a magnitude of one and be dimensionless provided **A** is divided by its magnitude, i.e. :

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \qquad (2\text{–}6)$$

Where: $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

By comparison with Eqs. 2–5 , it is seen that *the* **i** , **j** , **k** *components* of $\mathbf{u}_A$ *represent the direction cosines of* **A** , i.e.,

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \qquad (2\text{–}7)$$

Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\mathbf{u}_A$ has a magnitude of one, then from the above equation an important relation among the direction cosines can be formulated as:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (2\text{–}8)$$

Finally, if the magnitude and coordinate direction angles of **A** are known, then **A** may be expressed in Cartesian vector form as:

$$\begin{aligned}\mathbf{A} &= A\mathbf{u}_A \\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k} \\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned} \qquad (2\text{–}9)$$

2.6 Addition of Cartesian Vectors

Let $\mathbf{A} = A_x\mathbf{i} + A_y\,\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\,\mathbf{j} + B_z\mathbf{k}$, Fig. 2–29 , then the resultant vector, **R** , has components which are the scalar sums of the **i** , **j** and **k** components of **A** and **B** , i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (Ax + Bx)\mathbf{i} + (Ay + By)\mathbf{j} + (Az + Bz)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as:

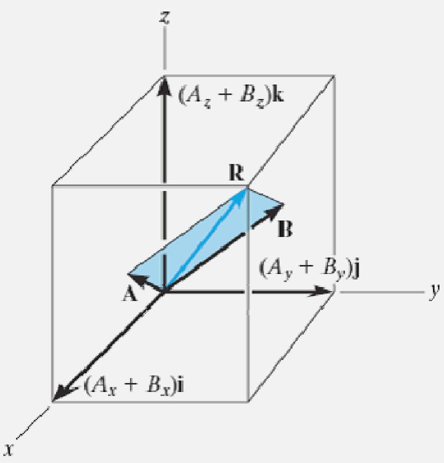


Fig. 2–29

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k} \qquad (2\text{–}10)$$

EXAMPLE 2.8

Express the force **F** shown in Fig. 2–30 as a Cartesian vector.

SOLUTION

Since only two coordinate direction angles are specified, the third angle α must be determined from Eq. 2–8; i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$
$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60^\circ \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120^\circ$$

By inspection it is necessary that $\alpha = 60^\circ$, since $\mathbf{F}_x$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\mathbf{F} = F\cos\alpha\,\mathbf{i} + F\cos\beta\,\mathbf{j} + F\cos\gamma\,\mathbf{k}$$
$$= (200\cos 60^\circ\ \text{N})\mathbf{i} + (200\cos 60^\circ\ \text{N})\mathbf{j} + (200\cos 45^\circ\ \text{N})\mathbf{k}$$
$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$

Show that indeed the magnitude of $F = 200$ N.

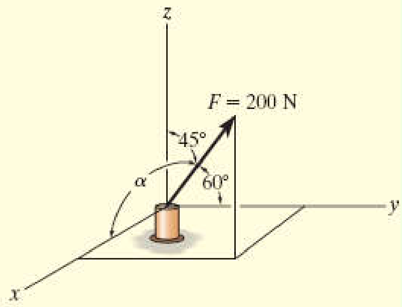


Fig. 2–30

EXAMPLE 2.9

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–31*a*.

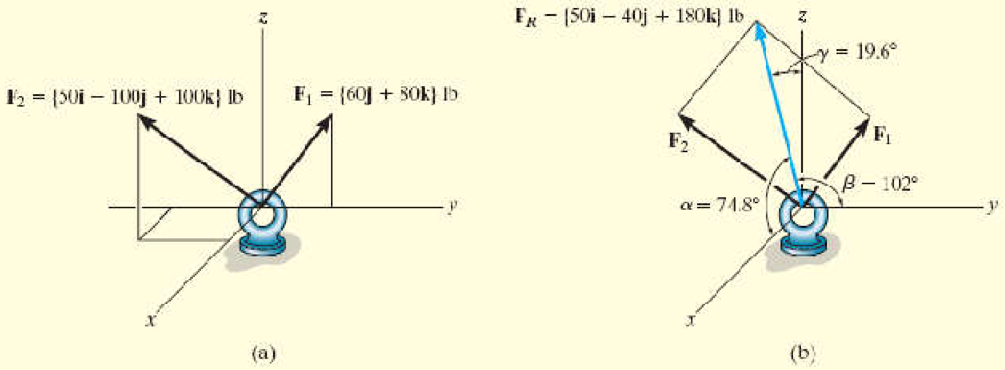


Fig. 2–31

SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in Fig. 2–31*b*, is

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\}\text{lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{lb}$$
$$= \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\}\text{lb}$$

The magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(50\ \text{lb})^2 + (-40\ \text{lb})^2 + (180\ \text{lb})^2} = 191.0\ \text{lb}$$
$$= 191\ \text{lb} \qquad \textit{Ans.}$$

The coordinate direction angles α, β, γ are determined from the components of the unit vector acting in the direction of $\mathbf{F}_R$.

$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{50}{191.0}\mathbf{i} - \frac{40}{191.0}\mathbf{j} + \frac{180}{191.0}\mathbf{k}$$
$$= 0.2617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$$

so that

$\cos\alpha = 0.2617 \qquad \alpha = 74.8°$ *Ans.*

$\cos\beta = -0.2094 \qquad \beta = 102°$ *Ans.*

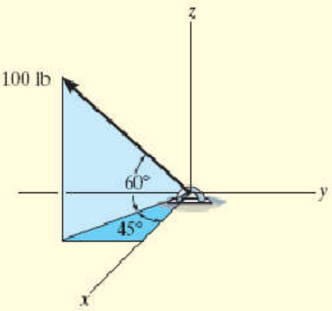
$\cos\gamma = 0.9422 \qquad \gamma = 19.6°$ *Ans.*

These angles are shown in Fig. 2–31*b*.

NOTE: In particular, notice that $\beta > 90°$ since the **j** component of $\mathbf{u}_{F_R}$ is negative. This seems reasonable considering how $\mathbf{F}_1$ and $\mathbf{F}_2$ add according to the parallelogram law.

EXAMPLE 2.10

Express the force **F** shown in Fig. 2–32*a* as a Cartesian vector.



(a)

SOLUTION

The angles of 60° and 45° defining the direction of **F** are *not* coordinate direction angles. Two successive applications of the parallelogram law are needed to resolve **F** into its *x, y, z* components. First $\mathbf{F} = \mathbf{F}' + \mathbf{F}_z$, then $\mathbf{F}' = \mathbf{F}_x + \mathbf{F}_y$, Fig. 2–32*b*. By trigonometry, the magnitudes of the components are

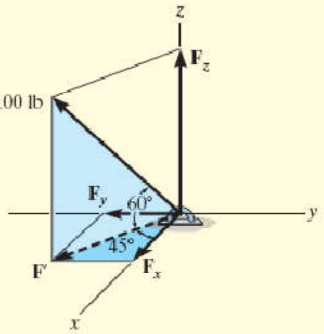
$$F_z = 100 \sin 60° \text{ lb} = 86.6 \text{ lb}$$
$$F' = 100 \cos 60° \text{ lb} = 50 \text{ lb}$$
$$F_x = F' \cos 45° = 50 \cos 45° \text{ lb} = 35.4 \text{ lb}$$
$$F_y = F' \sin 45° = 50 \sin 45° \text{ lb} = 35.4 \text{ lb}$$

Realizing that $\mathbf{F}_y$ has a direction defined by –**j**, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb}$$ *Ans.*

To show that the magnitude of this vector is indeed 100 lb, apply Eq. 2–4,

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$
$$= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb}$$



(b)

EXAMPLE 2.11

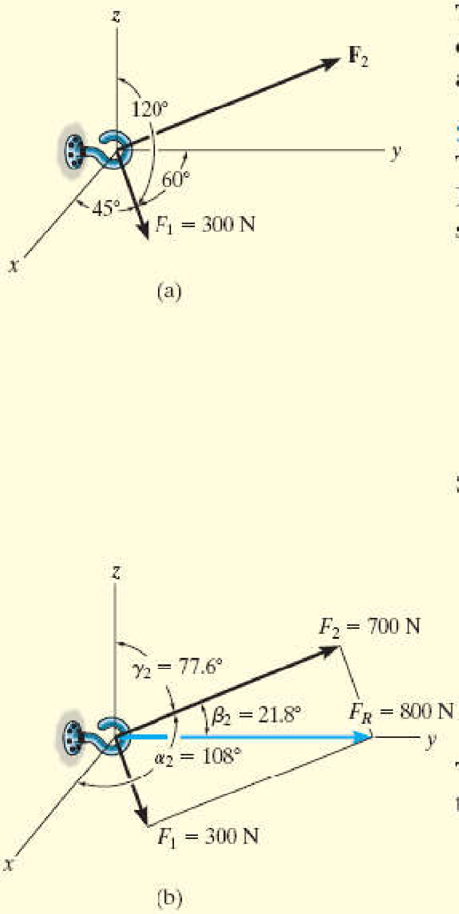


Fig. 2–33

Two forces act on the hook shown in Fig. 2–33*a*. Specify the magnitude of $\mathbf{F}_2$ and its coordinate direction angles so that the resultant force $\mathbf{F}_R$ acts along the positive *y* axis and has a magnitude of 800 N.

SOLUTION

To solve this problem, the resultant force $\mathbf{F}_R$ and its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, will each be expressed in Cartesian vector form. Then, as shown in Fig. 2–33*b*, it is necessary that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$.

Applying Eq. 2–9,

$$\mathbf{F}_1 = F_1 \cos\alpha_1 \mathbf{i} + F_1 \cos\beta_1 \mathbf{j} + F_1 \cos\gamma_1 \mathbf{k}$$
$$= 300 \cos 45° \,\mathbf{i} + 300 \cos 60° \,\mathbf{j} + 300 \cos 120° \,\mathbf{k}$$
$$= \{212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k}\} \text{ N}$$
$$\mathbf{F}_2 = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$

Since $\mathbf{F}_R$ has a magnitude of 800 N and acts in the +**j** direction,

$$\mathbf{F}_R = (800 \text{ N})(+\mathbf{j}) = \{800\mathbf{j}\} \text{ N}$$

We require

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$
$$800\mathbf{j} = 212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$
$$800\mathbf{j} = (212.1 + F_{2x})\mathbf{i} + (150 + F_{2y})\mathbf{j} + (-150 + F_{2z})\mathbf{k}$$

To satisfy this equation the **i**, **j**, **k** components of $\mathbf{F}_R$ must be equal to the corresponding **i**, **j**, **k** components of $(\mathbf{F}_1 + \mathbf{F}_2)$. Hence,

$$0 = 212.1 + F_{2x} \qquad F_{2x} = -212.1 \text{ N}$$
$$800 = 150 + F_{2y} \qquad F_{2y} = 650 \text{ N}$$
$$0 = -150 + F_{2z} \qquad F_{2z} = 150 \text{ N}$$

The magnitude of $\mathbf{F}_2$ is thus

$$F_2 = \sqrt{(-212.1 \text{ N})^2 + (650 \text{ N})^2 + (150 \text{ N})^2}$$
$$= 700 \text{ N} \qquad \textit{Ans.}$$

We can use Eq. 2–9 to determine $\alpha_2, \beta_2, \gamma_2$.

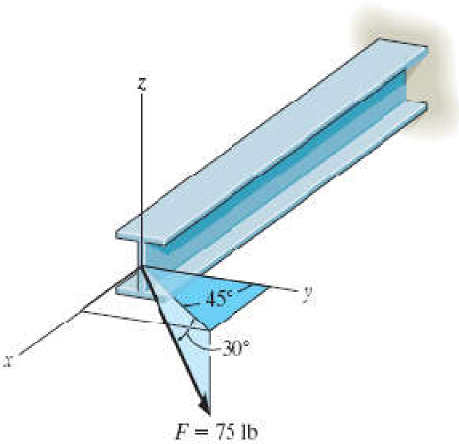
$$\cos\alpha_2 = \frac{-212.1}{700}; \qquad \alpha_2 = 108° \qquad \textit{Ans.}$$
$$\cos\beta_2 = \frac{650}{700}; \qquad \beta_2 = 21.8° \qquad \textit{Ans.}$$
$$\cos\gamma_2 = \frac{150}{700}; \qquad \gamma_2 = 77.6° \qquad \textit{Ans.}$$

These results are shown in Fig. 2–33*b*.

-

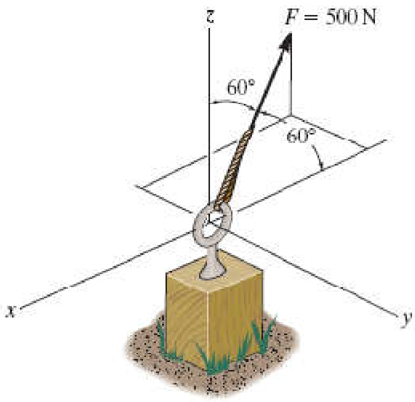
FUNDAMENTAL PROBLEMS

F2–13. Determine the coordinate direction angles of the force.



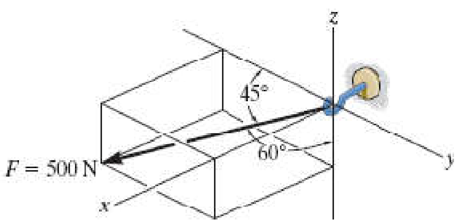
F2–13

F2–14. Express the force as a Cartesian vector.



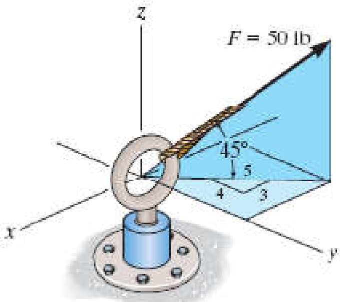
F2–14

F2–15. Express the force as a Cartesian vector.



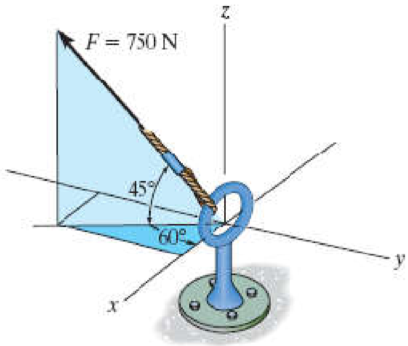
F2–15

F2–16. Express the force as a Cartesian vector.



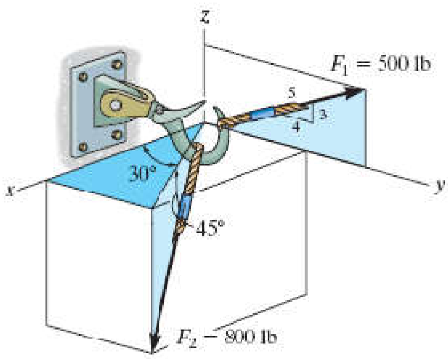
F2–16

F2–17. Express the force as a Cartesian vector.



F2–17

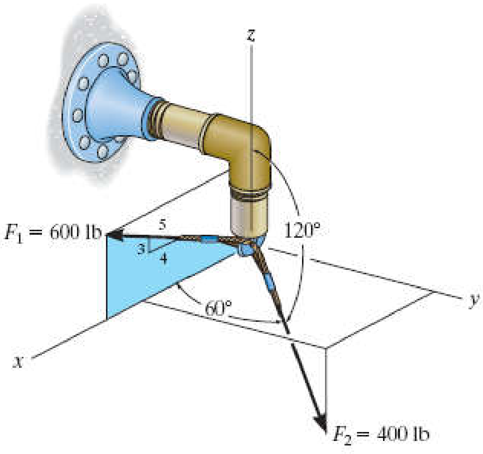
F2–18. Determine the resultant force acting on the hook.



F2–18

2–66. Express each force acting on the pipe assembly in Cartesian vector form.

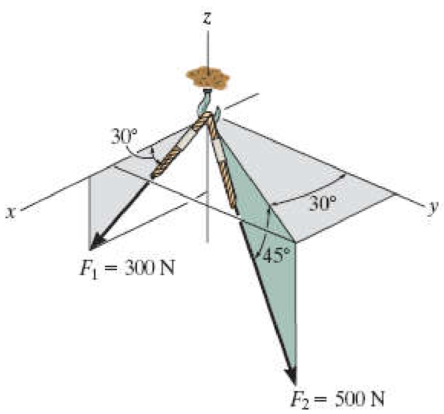
2–67. Determine the magnitude and the direction of the resultant force acting on the pipe assembly.



Probs. 2–66/67

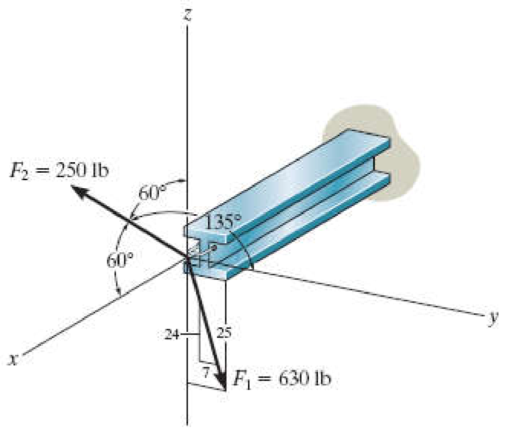
***2–68.** Express each force as a Cartesian vector.

2–69. Determine the magnitude and coordinate direction angles of the resultant force acting on the hook.



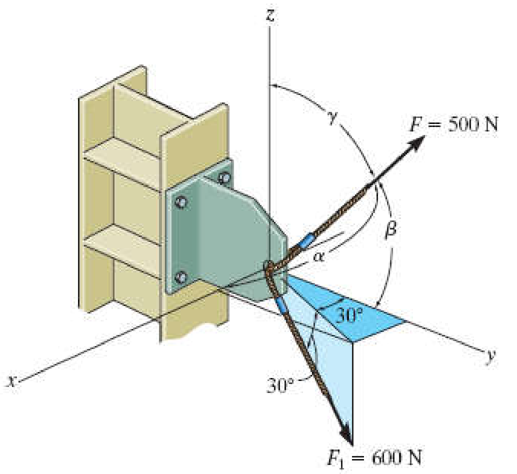
Probs. 2–68/69

2–70. The beam is subjected to the two forces shown. Express each force in Cartesian vector form and determine the magnitude and coordinate direction angles of the resultant force.



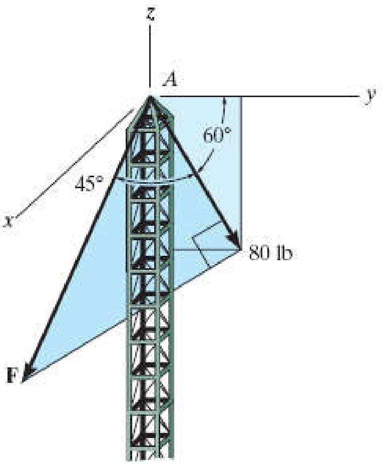
Prob. 2–70

2–71. If the resultant force acting on the bracket is directed along the positive y axis, determine the magnitude of the resultant force and the coordinate direction angles of **F** so that $\beta < 90°$.



Prob. 2–71

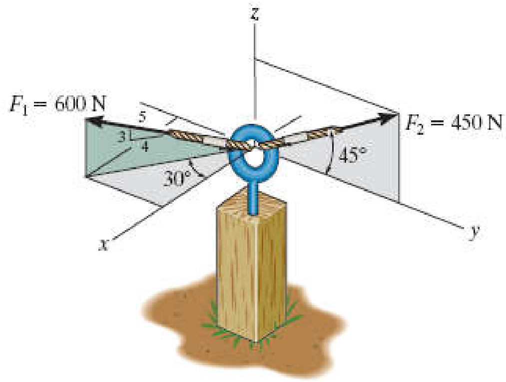
*2–72. A force **F** is applied at the top of the tower at A. If it acts in the direction shown such that one of its components lying in the shaded y-z plane has a magnitude of 80 lb, determine its magnitude F and coordinate direction angles α, β, γ.



Prob. 2–72

2–75. Determine the coordinate direction angles of force $\mathbf{F}_1$.

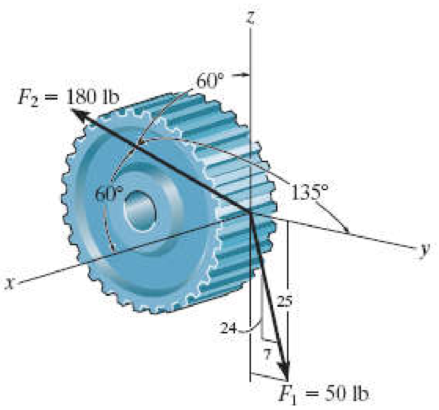
*2–76. Determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.



Probs. 2–75/76

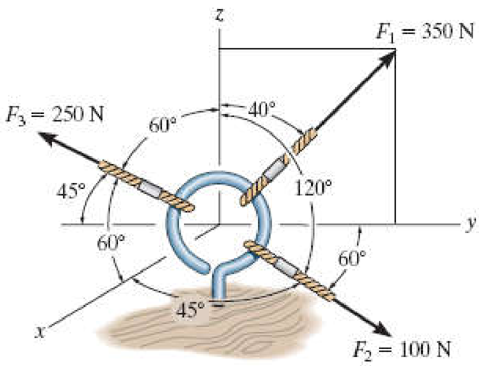
2–73. The spur gear is subjected to the two forces caused by contact with other gears. Express each force as a Cartesian vector.

2–74. The spur gear is subjected to the two forces caused by contact with other gears. Determine the resultant of the two forces and express the result as a Cartesian vector.



Probs. 2–73/74

2–77. The cables attached to the screw eye are subjected to the three forces shown. Express each force in Cartesian vector form and determine the magnitude and coordinate direction angles of the resultant force.



Prob. 2–77