## 2.7 Position Vectors:

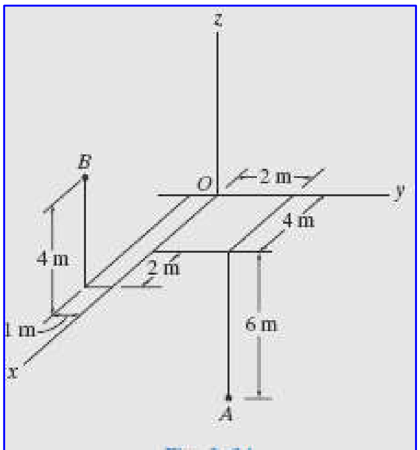


Fig. 2–34

In this section we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

***x, y, z* Coordinates,** we will use a *right-handed* coordinate system to reference the location of points in space, Fig. 2–34. Points in space are located relative to the origin of coordinates, *O* , by successive measurements along the *x, y, z* axes. For example, the coordinates of point *A* are obtained by starting at *O* and measuring $x_A = +4$ m along the *x* axis, then $y_A = +2$ m along the *y* axis, and finally $z_A = -6$ m along the *z* axis. Thus, *A* (4 m, 2 m, - 6 m). In a similar manner, measurements along the *x, y, z* axes from *O* to *B* yield the coordinates of *B* , i.e., *B* (6 m, -1 m, 4 m).

**Position Vector,** A *position vector* **r** is defined as a fixed vector which locates a point in space relative to another point. For example, if **r** extends from the origin of coordinates, *O* , to point *P* ( *x, y, z* ), Fig. 2–35 *a* , then **r** can be expressed in Cartesian vector form as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Note how the head-to-tail vector addition of the three components yields vector **r** , Fig. 2–35 *b* . Starting at the origin *O* , one "travels" *x* in the +**i** direction, then *y* in the +**j** direction, and finally *z* in the +**k** direction to arrive at point ***P*** ( *x, y, z* ).

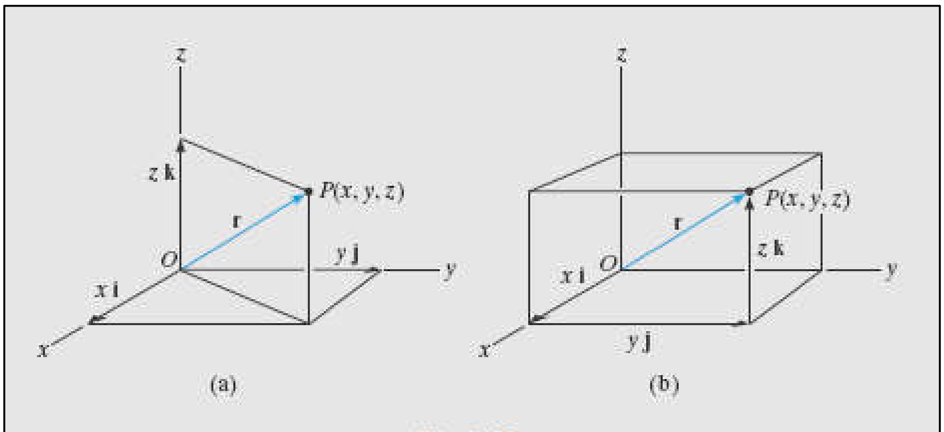
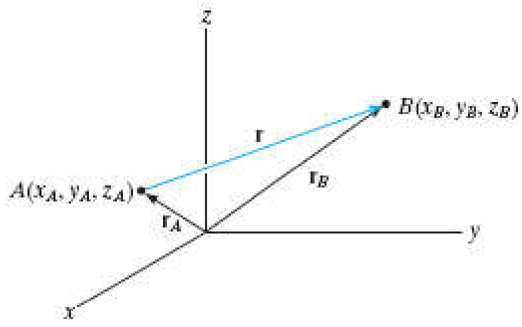


Fig. 2–35

In the more general case, the position vector may be directed from point *A* to point *B* in space, Fig. 2–36 *a* . From Fig. 2–36 *a* , by the head-to-tail vector addition, using the triangle rule, we require:

$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$



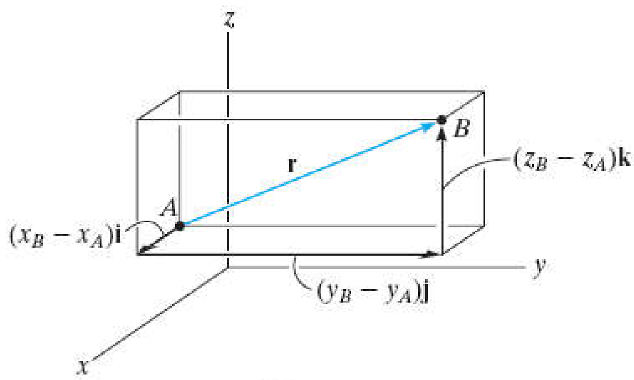
(a)

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

or

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} \qquad (2\text{–}11)$$

We can also form these components *directly* , Fig. 2–36 *b* , by starting at *A* and moving through a distance of $(x_B - x_A)$ along the positive *x* axis (+**i** ), then $(y_B - y_A)$ along the positive *y* axis (+**j** ), and finally $(z_B - z_A)$ along the positive *z* axis (+**k** ) to get to *B* .



(b)

**Fig. 2–36**

## EXAMPLE 2.12

An elastic rubber band is attached to points $A$ and $B$ as shown in Fig. 2–37*a*. Determine its length and its direction measured from $A$ toward $B$.

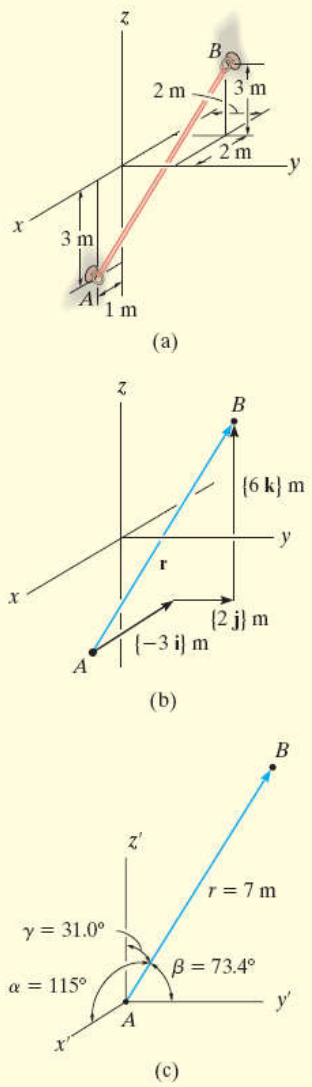


**Fig. 2–37**

### SOLUTION

We first establish a position vector from $A$ to $B$, Fig. 2–37*b*. In accordance with Eq. 2–11, the coordinates of the tail $A(1\text{ m}, 0, -3\text{ m})$ are subtracted from the coordinates of the head $B(-2\text{ m}, 2\text{ m}, 3\text{ m})$, which yields

$$\mathbf{r} = [-2\text{ m} - 1\text{ m}]\mathbf{i} + [2\text{ m} - 0]\mathbf{j} + [3\text{ m} - (-3\text{ m})]\mathbf{k}$$
$$= \{-3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\}\text{ m}$$

These components of $\mathbf{r}$ can also be determined *directly* by realizing that they represent the direction and distance one must travel along each axis in order to move from $A$ to $B$, i.e., along the $x$ axis $\{-3\mathbf{i}\}$ m, along the $y$ axis $\{2\mathbf{j}\}$ m, and finally along the $z$ axis $\{6\mathbf{k}\}$ m.

The length of the rubber band is therefore

$$r = \sqrt{(-3\text{ m})^2 + (2\text{ m})^2 + (6\text{ m})^2} = 7\text{ m} \qquad \textit{Ans.}$$

Formulating a unit vector in the direction of $\mathbf{r}$, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = -\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

The components of this unit vector give the coordinate direction angles

$$\alpha = \cos^{-1}\left(-\frac{3}{7}\right) = 115^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{2}{7}\right) = 73.4^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31.0^\circ \qquad \textit{Ans.}$$

**NOTE:** These angles are measured from the *positive axes* of a localized coordinate system placed at the tail of $\mathbf{r}$, as shown in Fig. 2–37*c*.

## 2.8 Force Vector Directed Along a Line

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Fig. 2–38, where the force **F** is directed along the cord *AB*. We can formulate **F** as a Cartesian vector by realizing that it has the *same direction* and *sense* as the position vector **r** directed from point *A* to point *B* on the cord. This common direction is specified by the *unit vector* **u** = **r**>*r*. Hence,

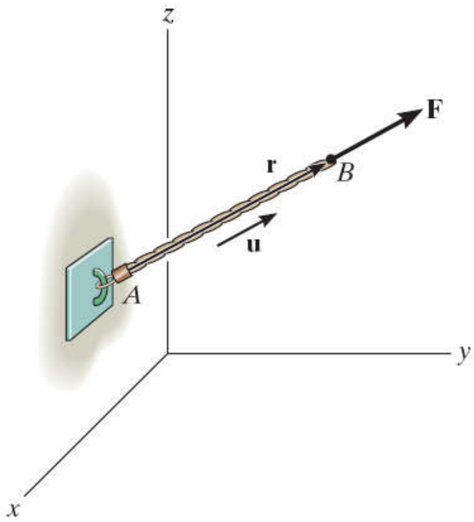



**Fig. 2–38**

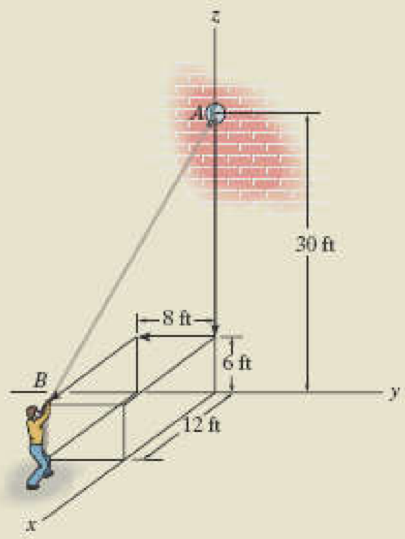
$$\mathbf{F} = F\,\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$


The force **F** acting along the rope can be represented as a Cartesian vector by establishing *x*, *y*, *z* axes and first forming a position vector **r** along the length of the rope. Then the corresponding unit vector **u** = **r**/*r* that defines the direction of both the rope and the force can be determined. Finally, the magnitude of the force is combined with its direction, **F** = *F***u**.

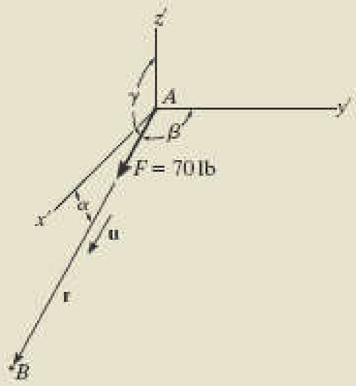
**Important Points**

- A position vector locates one point in space relative to another point.
- The easiest way to formulate the components of a position vector is to determine the distance and direction that must be traveled along the *x, y, z* directions—going from the tail to the head of the vector.
- A force **F** acting in the direction of a position vector **r** can be represented in Cartesian form if the unit vector **u** of the position vector is determined and it is multiplied by the magnitude of the force, i.e., **F** = *F***u** = *F* ( **r**>*r* ).

## EXAMPLE 2.13




(a)




(b)

**Fig. 2–39**

The man shown in Fig. 2–39*a* pulls on the cord with a force of 70 lb. Represent this force acting on the support *A* as a Cartesian vector and determine its direction.

### SOLUTION

Force **F** is shown in Fig. 2–39*b*. The *direction* of this vector, **u**, is determined from the position vector **r**, which extends from *A* to *B*. Rather than using the coordinates of the end points of the cord, **r** can be determined *directly* by noting in Fig. 2–39*a* that one must travel from *A* {−24**k**} ft, then {−8**j**} ft, and finally {12**i**} ft to get to *B*. Thus,

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\}\text{ ft}$$

The magnitude of **r**, which represents the *length* of cord *AB*, is

$$r = \sqrt{(12\text{ ft})^2 + (-8\text{ ft})^2 + (-24\text{ ft})^2} = 28\text{ ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F** has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70\text{ lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\}\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles are measured between **r** (or **F**) and the *positive axes* of a localized coordinate system with origin placed at *A*, Fig. 2–39*b*. From the components of the unit vector:

$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149^\circ \qquad \textit{Ans.}$$

**NOTE:** These results make sense when compared with the angles identified in Fig. 2–39*b*.

## EXAMPLE 2.14

The force in Fig. 2–40*a* acts on the hook. Express it as a Cartesian vector.

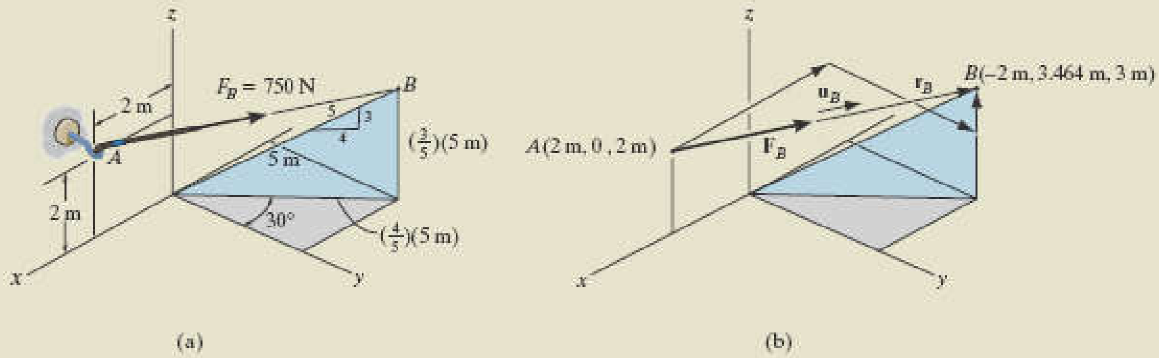



Fig. 2–40

### SOLUTION

As shown in Fig. 2–40*b*, the coordinates for points *A* and *B* are

$$A(2\text{ m}, 0, 2\text{ m})$$

and

$$B\left[-\left(\frac{4}{5}\right)5\sin 30^\circ\text{ m}, \left(\frac{4}{5}\right)5\cos 30^\circ\text{ m}, \left(\frac{3}{5}\right)5\text{ m}\right]$$

or

$$B(-2\text{ m}, 3.464\text{ m}, 3\text{ m})$$

Therefore, to go from *A* to *B*, one must travel {−4**i**} m, then {3.464**j**} m, and finally {1**k**} m. Thus,

$$\mathbf{u}_B = \left(\frac{\mathbf{r}_B}{r_B}\right) = \frac{\{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}\}\text{ m}}{\sqrt{(-4\text{ m})^2 + (3.464\text{ m})^2 + (1\text{ m})^2}}$$

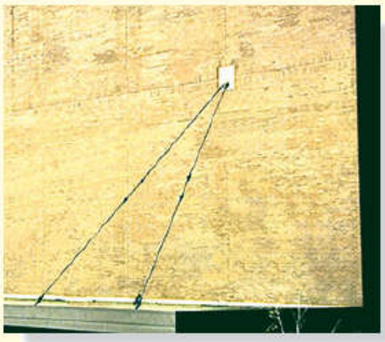
$$= -0.7428\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k}$$

Force $\mathbf{F}_B$ expressed as a Cartesian vector becomes

$$\mathbf{F}_B = F_B\mathbf{u}_B = (750\text{ N})(-0.74281\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k})$$

$$= \{-557\mathbf{i} + 482\mathbf{j} + 139\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

## EXAMPLE 2.15



The roof is supported by cables as shown in the photo. If the cables exert forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the wall hook at $A$ as shown in Fig. 2–41*a*, determine the resultant force acting at $A$. Express the result as a Cartesian vector.

### SOLUTION

The resultant force $\mathbf{F}_R$ is shown graphically in Fig. 2–41*b*. We can express this force as a Cartesian vector by first formulating $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ as Cartesian vectors and then adding their components. The directions of $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ are specified by forming unit vectors $\mathbf{u}_{AB}$ and $\mathbf{u}_{AC}$ along the cables. These unit vectors are obtained from the associated position vectors $\mathbf{r}_{AB}$ and $\mathbf{r}_{AC}$. With reference to Fig. 2–41*a*, to go from $A$ to $B$, we must travel $\{-4\mathbf{k}\}$ m, and then $\{4\mathbf{i}\}$ m. Thus,

$$\mathbf{r}_{AB} = \{4\mathbf{i} - 4\mathbf{k}\}\text{ m}$$

$$r_{AB} = \sqrt{(4\text{ m})^2 + (-4\text{ m})^2} = 5.66\text{ m}$$

$$\mathbf{F}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = (100\text{ N})\left(\frac{4}{5.66}\mathbf{i} - \frac{4}{5.66}\mathbf{k}\right)$$

$$\mathbf{F}_{AB} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\text{ N}$$

To go from $A$ to $C$, we must travel $\{-4\mathbf{k}\}$ m , then $\{2\mathbf{j}\}$ m, and finally $\{4\mathbf{i}\}$. Thus,

$$\mathbf{r}_{AC} = \{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\}\text{ m}$$

$$r_{AC} = \sqrt{(4\text{ m})^2 + (2\text{ m})^2 + (-4\text{ m})^2} = 6\text{ m}$$

$$\mathbf{F}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = (120\text{ N})\left(\frac{4}{6}\mathbf{i} + \frac{2}{6}\mathbf{j} - \frac{4}{6}\mathbf{k}\right)$$

$$= \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\text{ N}$$

The resultant force is therefore

$$\mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\text{ N} + \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\text{ N}$$

$$= \{151\mathbf{i} + 40\mathbf{j} - 151\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

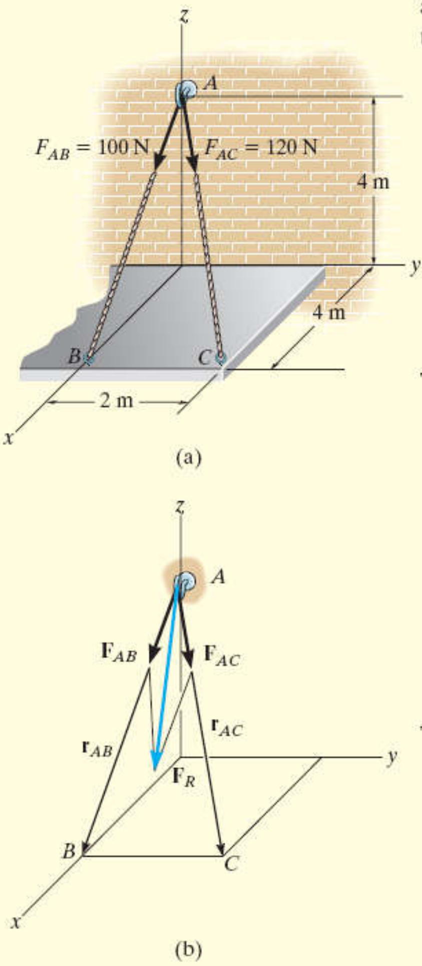
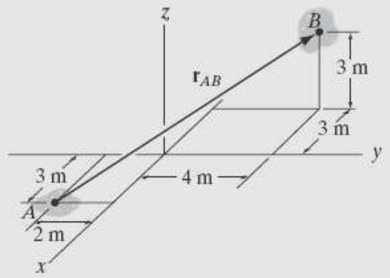



Fig. 2–41

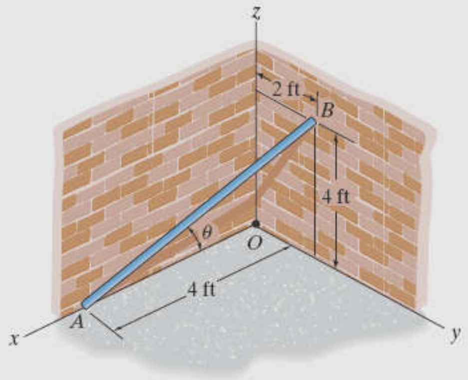
## FUNDAMENTAL PROBLEMS

**F2–19.** Express the position vector $\mathbf{r}_{AB}$ in Cartesian vector form, then determine its magnitude and coordinate direction angles.



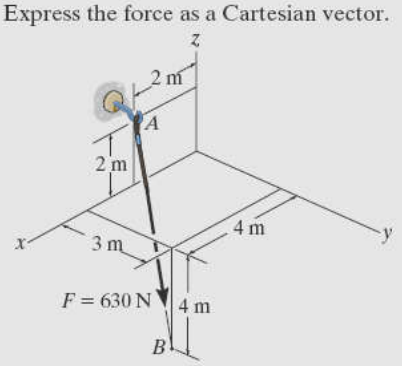

**F2–19**

**F2–20.** Determine the length of the rod and the position vector directed from $A$ to $B$. What is the angle $\theta$?



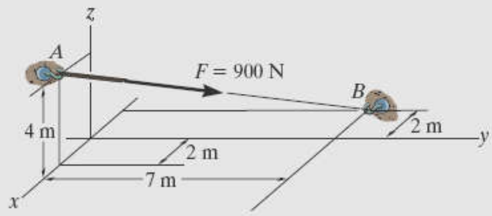

**F2–20**

**F2–21.** Express the force as a Cartesian vector.



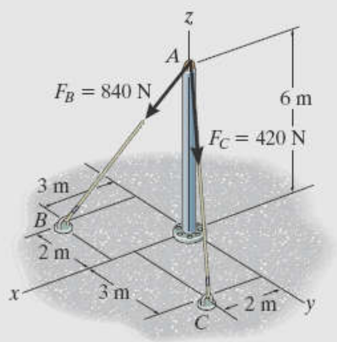

**F2–21**

**F2–22.** Express the force as a Cartesian vector.



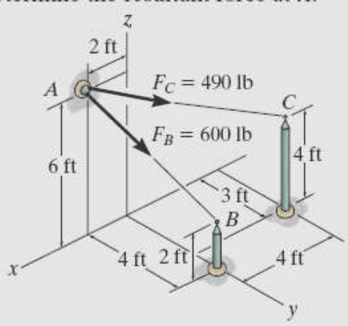

**F2–22**

**F2–23.** Determine the magnitude of the resultant force at $A$.




**F2–23**

**F2–24.** Determine the resultant force at $A$.




**F2–24**