Cartesian vector formulation:

If we establish *x, y, z* coordinate axes, then the position vector **r** and force **F** can be expressed as Cartesian vectors (Fig 4-12-a) then we can write:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \qquad (4\text{–}7)$$

Where r_x, r_y, r_z represent the *x, y, z* components of the position vector drawn from point O to any point on the line of action of the force. F_x, F_y, F_z represent the *x, y, z* ***components*** of the force vector. If the determinant is expanded, then like Eq. 4–4 we have:

$$\mathbf{M}_O = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k} \qquad (4\text{–}8)$$

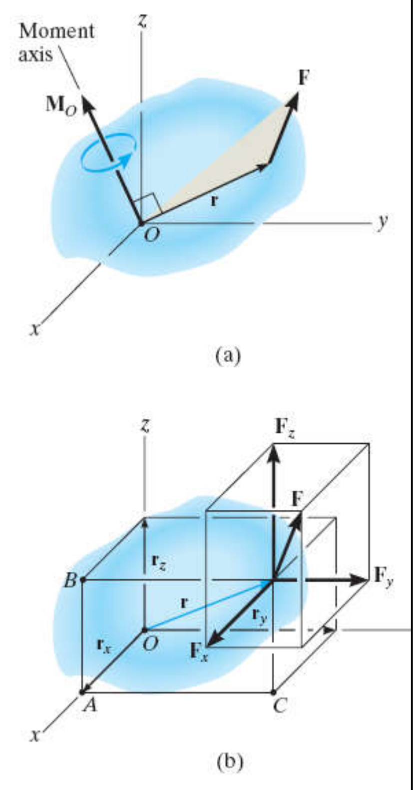


Fig. 4–12

Resultant Moment of a system of forces:

If a body is acted upon by a system of forces (Fig 4-13), the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as:

$$(\mathbf{M_R})_o = \Sigma\, (\mathbf{r} \times \mathbf{F})$$

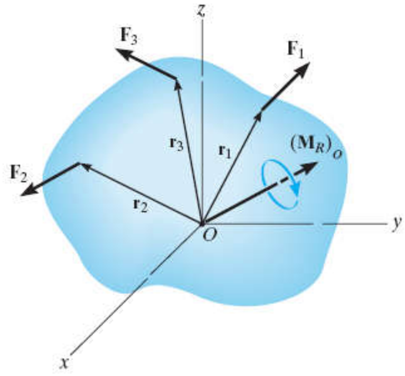
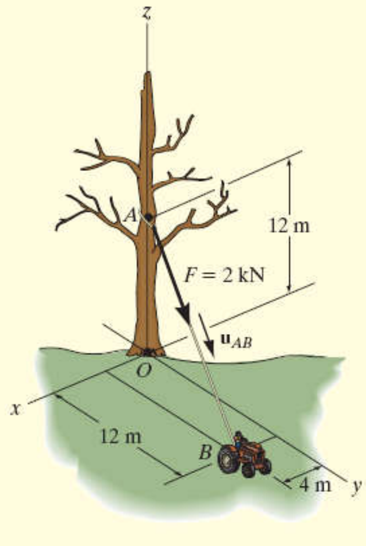


Fig. 4–13

EXAMPLE 4.3



(a)

Determine the moment produced by the force **F** in Fig. 4–14*a* about point *O*. Express the result as a Cartesian vector.

SOLUTION

As shown in Fig. 4–14*b*, either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point *O*. These position vectors are

$$\mathbf{r}_A = \{12\mathbf{k}\}\text{ m} \quad \text{and} \quad \mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}\text{ m}$$

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\text{ kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\text{ m}}{\sqrt{(4\text{ m})^2 + (12\text{ m})^2 + (-12\text{ m})^2}}\right]$$

$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\text{ kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$$

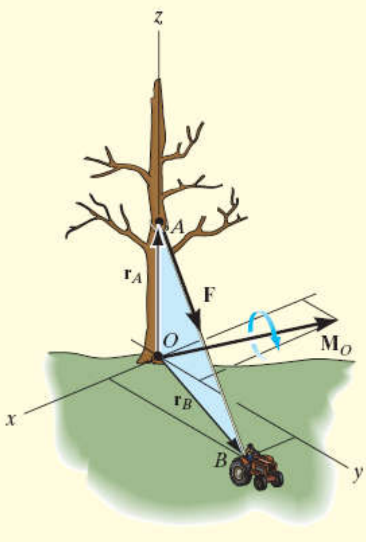
$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

or

$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



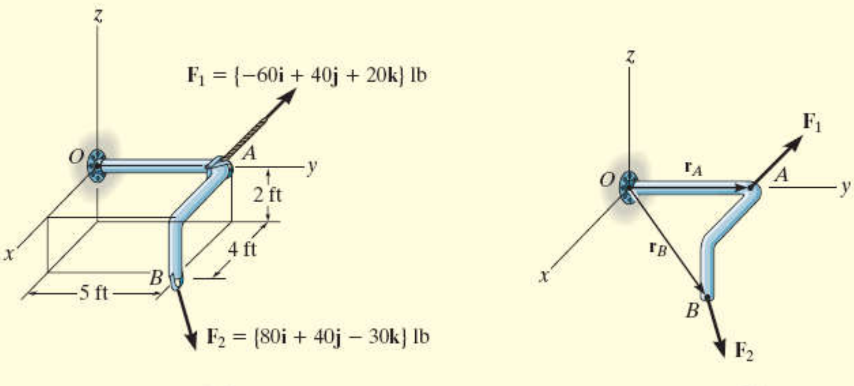
(b)

Fig. 4–14

NOTE: As shown in Fig. 4–14*b*, $\mathbf{M}_O$ acts perpendicular to the plane that contains $\mathbf{F}$, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm *d*.

EXAMPLE 4.4

Two forces act on the rod shown in Fig. 4–15*a*. Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.



SOLUTION

Position vectors are directed from point *O* to each force as shown in Fig. 4–15*b*. These vectors are

$$\mathbf{r}_A = \{5\mathbf{j}\}\text{ ft}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}\text{ ft}$$

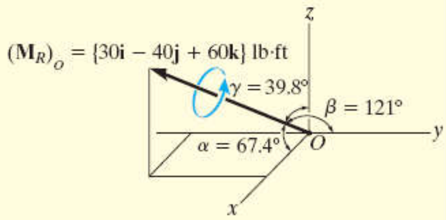


Fig. 4–15

The resultant moment about *O* is therefore

$$(\mathbf{M}_R)_O = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k}$$

$$+ [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k}$$

$$= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\text{ lb}\cdot\text{ft}$$

Ans.

NOTE: This result is shown in Fig. 4–15*c*. The coordinate direction angles were determined from the unit vector for $(\mathbf{M}_R)_O$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

Principle of Moments:

A concept often used in mechanics is the *principle of moments,* which is sometimes referred to as *Varignon's theorem*. It states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point.* For example, consider the moments of the force **F** and two of its components about point *O*. Fig. 4–16 . Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ we have:

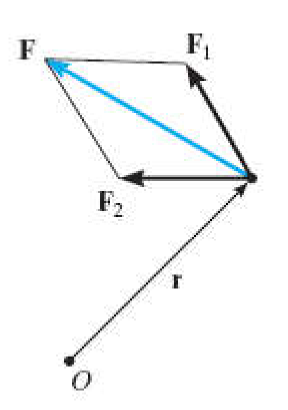


Fig. 4–16

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

For two-dimensional problems, Fig. 4–17 , we can use the principle of moments by resolving the force into its rectangular components and then determine the moment using a scalar analysis. Thus,

$$M_O = F_x y - F_y x$$

This method is generally easier than finding the same moment using $M_O = Fd$.

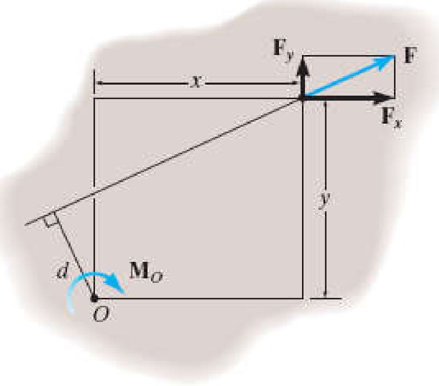


Fig. 4–17

Important Points

- The moment of a force creates the tendency of a body to turn about an axis passing through a specific point *O*.
- Using the right-hand rule, the sense of rotation is indicated by the curl of the fingers, and the thumb is directed along the moment axis, or line of action of the moment.
- The magnitude of the moment is determined from $M_O = Fd$, where *d* is called the moment arm, which represents the perpendicular or shortest distance from point *O* to the line of action of the force.
- In three dimensions the vector cross product is used to determine the moment, i.e., $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. Remember that **r** is directed *from* point *O to any point* on the line of action of **F**.
- The principle of moments states that the moment of a force about a point is equal to the sum of the moments of the force's components about the point. This is a very convenient method to use in two dimensions.

EXAMPLE 4.6

Force **F** acts at the end of the angle bracket in Fig. 4–19*a*. Determine the moment of the force about point *O*.

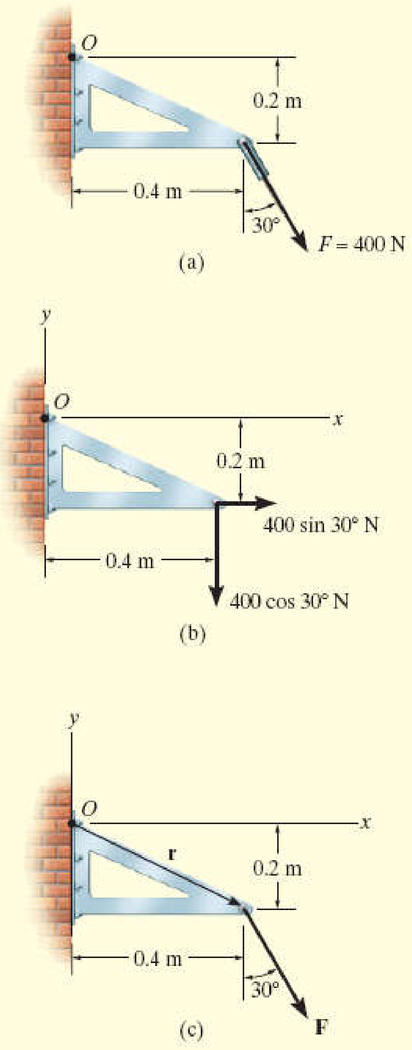


Fig. 4–19

SOLUTION I *(SCALAR ANALYSIS)*

The force is resolved into its *x* and *y* components, Fig. 4–19*b*, then

$$\circlearrowleft + M_O = 400 \sin 30° \text{ N}(0.2 \text{ m}) - 400 \cos 30° \text{ N}(0.4 \text{ m})$$

$$= -98.6 \text{ N}\cdot\text{m} = 98.6 \text{ N}\cdot\text{m} \circlearrowright$$

or

$$\mathbf{M}_O = \{-98.6\mathbf{k}\} \text{ N}\cdot\text{m}$$ *Ans.*

SOLUTION II *(VECTOR ANALYSIS)*

Using a Cartesian vector approach, the force and position vectors, Fig. 4–19*c*, are

$$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\} \text{ m}$$

$$\mathbf{F} = \{400 \sin 30°\mathbf{i} - 400 \cos 30°\mathbf{j}\} \text{ N}$$

$$= \{200.0\mathbf{i} - 346.4\mathbf{j}\} \text{ N}$$

The moment is therefore

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$

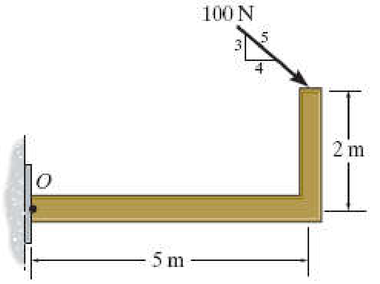
$$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$$

$$= \{-98.6\mathbf{k}\} \text{ N}\cdot\text{m}$$ *Ans.*

NOTE: It is seen that the scalar analysis (Solution I) provides a more *convenient method* for analysis than Solution II since the direction of the moment and the moment arm for each component force are easy to establish. Hence, this method is generally recommended for solving problems displayed in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three-dimensional problems.

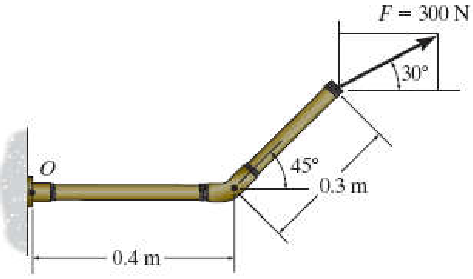
FUNDAMENTAL PROBLEMS

F4–1. Determine the moment of the force about point *O*.



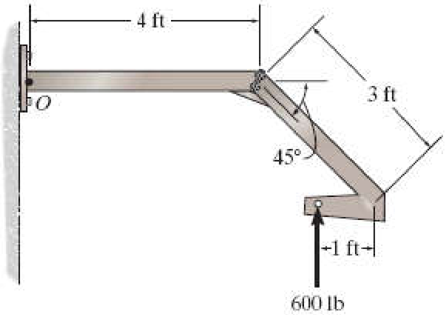
F4–1

F 4–2. Determine the moment of the force about point *O*.



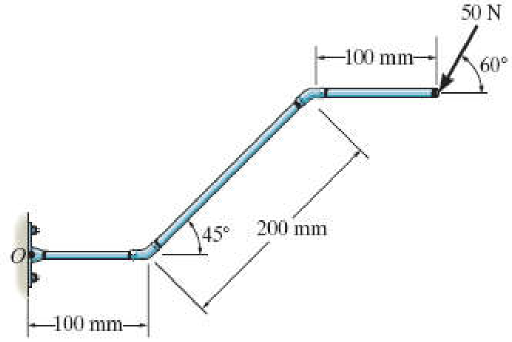
F4–2

F4–3. Determine the moment of the force about point *O*.



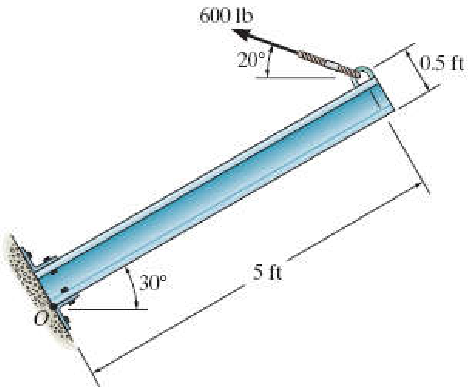
F4–3

F4–4. Determine the moment of the force about point *O*. Neglect the thickness of the member.



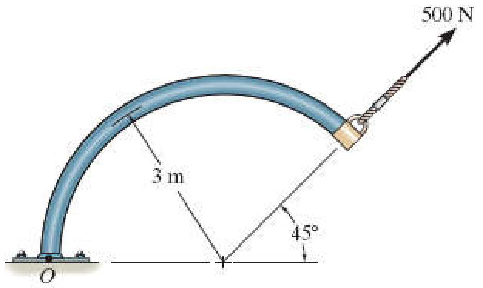
F4–4

F4–5. Determine the moment of the force about point *O*.



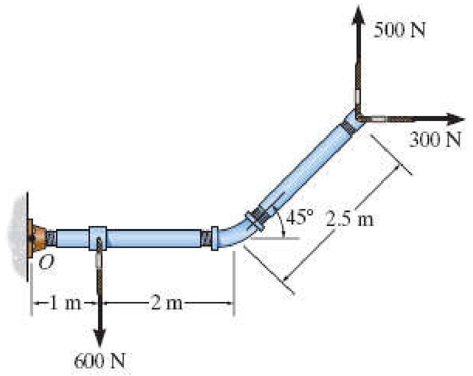
F4–5

F4–6. Determine the moment of the force about point *O*.



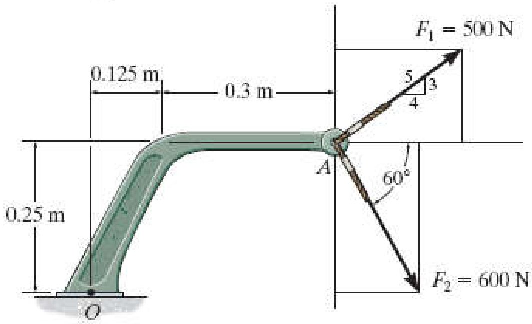
F4–6

F4–7. Determine the resultant moment produced by the forces about point O.



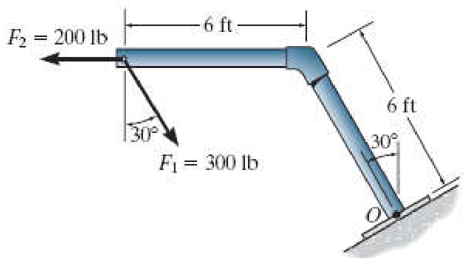
F4–7

F4–8. Determine the resultant moment produced by the forces about point O.



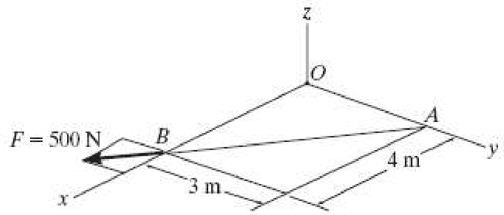
F4–8

F4–9. Determine the resultant moment produced by the forces about point O.



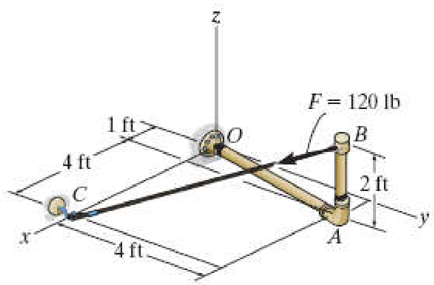
F4–9

F4–10. Determine the moment of force **F** about point O. Express the result as a Cartesian vector.



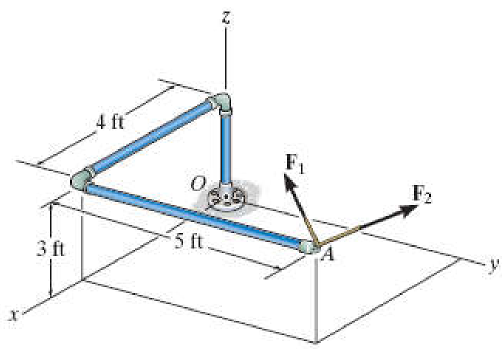
F4–10

F4–11. Determine the moment of force **F** about point O. Express the result as a Cartesian vector.



F4–11

F4–12. If $\mathbf{F}_1 = \{100\mathbf{i} - 120\mathbf{j} + 75\mathbf{k}\}$ lb and $\mathbf{F}_2 = \{-200\mathbf{i} + 250\mathbf{j} + 100\mathbf{k}\}$ lb, determine the resultant moment produced by these forces about point O. Express the result as a Cartesian vector.



F4–12

PROBLEMS

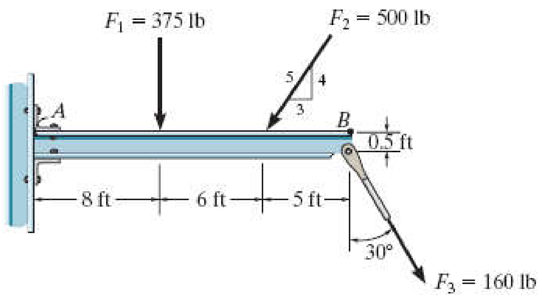
4–1. If **A**, **B**, and **D** are given vectors, prove the distributive law for the vector cross product, i.e., **A** × (**B** + **D**) = (**A** × **B**) + (**A** × **D**).

4–2. Prove the triple scalar product identity **A** · (**B** × **C**) = (**A** × **B**) · **C**.

4–3. Given the three nonzero vectors **A**, **B**, and **C**, show that if **A** · (**B** × **C**) = 0, the three vectors *must* lie in the same plane.

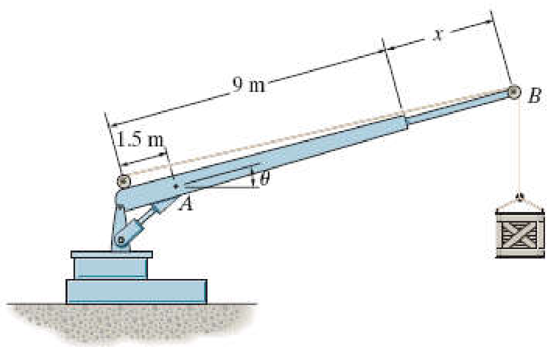
***4–4.** Determine the moment about point A of each of the three forces acting on the beam.

4–5. Determine the moment about point B of each of the three forces acting on the beam.



Probs. 4–4/5

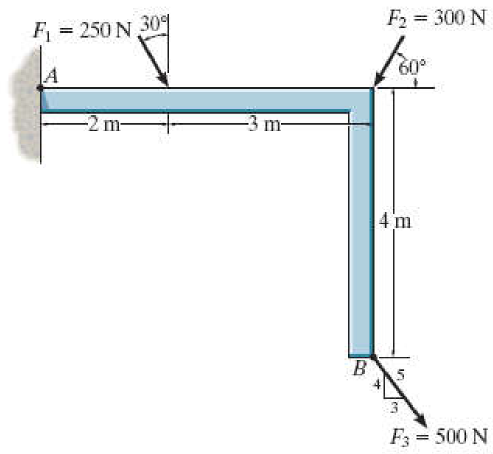
4–6. The crane can be adjusted for any angle $0° \leq \theta \leq 90°$ and any extension $0 \leq x \leq 5$ m. For a suspended mass of 120 kg, determine the moment developed at A as a function of x and θ. What values of both x and θ develop the maximum possible moment at A? Compute this moment. Neglect the size of the pulley at B.



Prob. 4–6

4–7. Determine the moment of each of the three forces about point A.

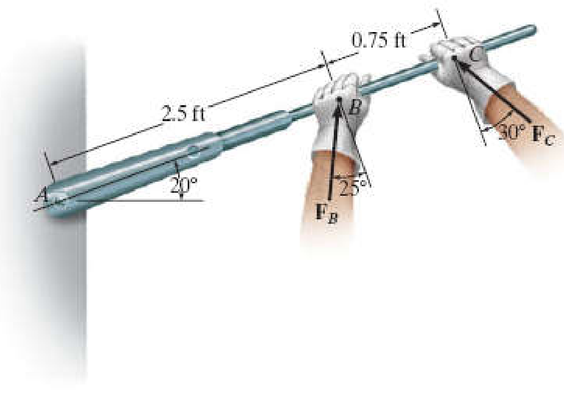
***4–8.** Determine the moment of each of the three forces about point B.



Probs. 4–7/8

4–9. Determine the moment of each force about the bolt located at A. Take $F_B = 40$ lb, $F_C = 50$ lb.

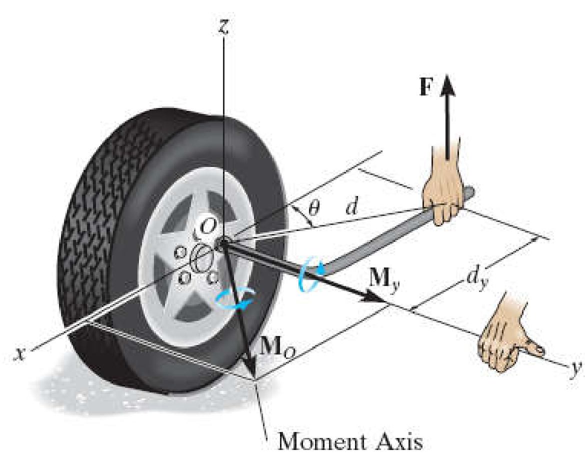
4–10. If $F_B = 30$ lb and $F_C = 45$ lb, determine the resultant moment about the bolt located at A.



Probs. 4–9/10

Moment of a Force about a Specified Axis

Sometimes, the moment produced by a force about a *specified axis* must be determined. For example, suppose the lug nut at *O* on the car tire in Fig. 4–20 *a* needs to be loosened. The force applied to the wrench will create a tendency for the wrench and the nut to rotate about the *moment axis* passing through *O;* however, the nut can only rotate about the *y* axis. Therefore, to determine the turning effect, only the *y* component of the moment is needed, and the total moment produced is not important. To determine this component, we can use either a scalar or vector analysis.



(a)

Fig. 4-20

Scalar Analysis: To use a scalar analysis in the case of the lug nut in Fig. 4–20 *a* , the moment arm perpendicular distance from the axis to the line of action of the force is $d_y = d \cos \theta$. Thus, the moment of **F** about the ***y***- axis is:

$$M_y = F\,d_y = F(d \cos \theta)$$

According to the right-hand rule, $\mathbf{M}_y$ is directed along the positive ***y***- axis as shown in the figure. In general, for any axis *a* , the moment is:

$$M_a = Fd_a$$

Vector Analysis: To find the moment of force **F** in Fig. 4–20 b about the ***y*** axis using a vector analysis, we must determine the moment of **F** about a point **O** on the axis: $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, and the projection of this moment onto the a axis is $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$. This combination is referred to as the scalar triple product. We have:

$$M_a = [u_{a_x}\mathbf{i} + u_{a_y}\mathbf{j} + u_{a_z}\mathbf{k}] \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= u_{a_x}(r_yF_z - r_zF_y) - u_{a_y}(r_xF_z - r_zF_x) + u_{a_z}(r_xF_y - r_yF_x)$$

This result can also be written in the form of a determinant, making it easier to memorize.

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

When *Ma* is evaluated from above equation, it will yield a positive or negative scalar. The sign of this scalar indicates the sense of direction of $\mathbf{M}_a$ along the ***a*** axis. If it is positive, then $\mathbf{M}_a$ will have the same sense as $\mathbf{u}_a$, whereas if it is negative, then $\mathbf{M}_a$ will act opposite to $\mathbf{u}_a$.
Once M_a is determined, we can then express $\mathbf{M}_a$ as a Cartesian vector, namely,

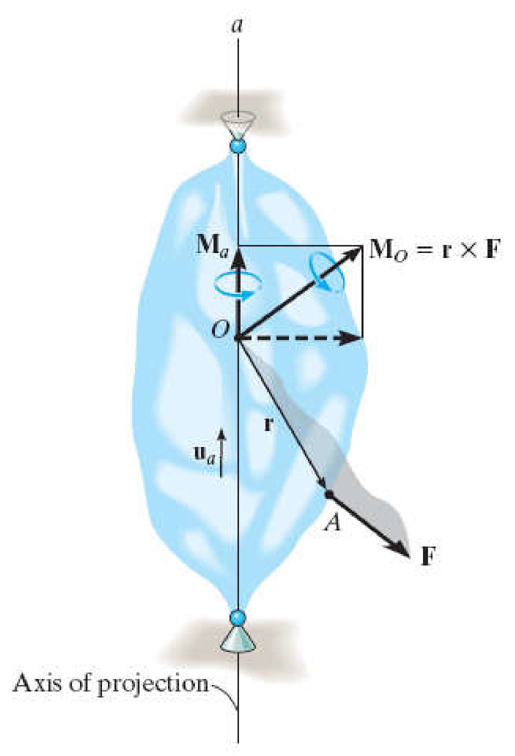


Fig. 4–21

$$\mathbf{M}_a = M_a\mathbf{u}_a$$

Important Points

- The moment of a force about a specified axis can be determined provided the perpendicular distance d_a from the force line of action to the axis can be determined. $M_a = Fd_a$.
- If vector analysis is used, $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$, where $\mathbf{u}_a$ defines the direction of the axis and $\mathbf{r}$ is extended from *any point* on the axis to *any point* on the line of action of the force.
- If M_a is calculated as a negative scalar, then the sense of direction of $\mathbf{M}_a$ is opposite to $\mathbf{u}_a$.
- The moment $\mathbf{M}_a$ expressed as a Cartesian vector is determined from $\mathbf{M}_a = M_a\mathbf{u}_a$.

EXAMPLE 4.7

Determine the resultant moment of the three forces in Fig. 4–22 about the x axis, the y axis, and the z axis.

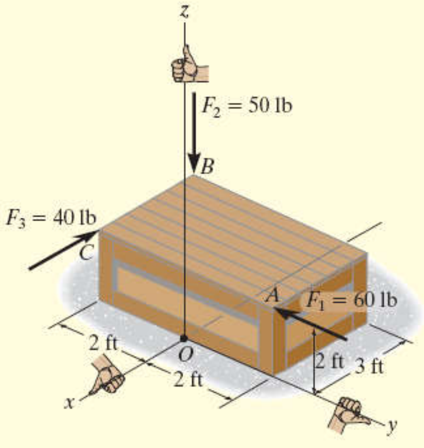


Fig. 4–22

SOLUTION

A force that is *parallel* to a coordinate axis or has a line of action that passes through the axis does *not* produce any moment or tendency for turning about that axis. Therefore, defining the positive direction of the moment of a force according to the right-hand rule, as shown in the figure, we have

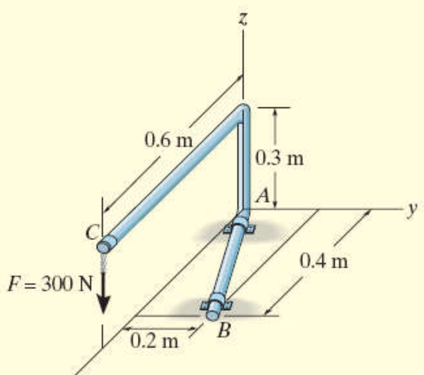
$$M_x = (60\text{ lb})(2\text{ ft}) + (50\text{ lb})(2\text{ ft}) + 0 = 220\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_y = 0 - (50\text{ lb})(3\text{ ft}) - (40\text{ lb})(2\text{ ft}) = -230\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_z = 0 + 0 - (40\text{ lb})(2\text{ ft}) = -80\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

The negative signs indicate that $\mathbf{M}_y$ and $\mathbf{M}_z$ act in the $-y$ and $-z$ directions, respectively.

EXAMPLE 4.8



(a)

Determine the moment $\mathbf{M}_{AB}$ produced by the force $\mathbf{F}$ in Fig. 4–23a, which tends to rotate the rod about the AB axis.

SOLUTION

A vector analysis using $M_{AB} = \mathbf{u}_B \cdot (\mathbf{r} \times \mathbf{F})$ will be considered for the solution rather than trying to find the moment arm or perpendicular distance from the line of action of $\mathbf{F}$ to the AB axis. Each of the terms in the equation will now be identified.

Unit vector $\mathbf{u}_B$ defines the direction of the AB axis of the rod, Fig. 4–23b, where

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{\{0.4\mathbf{i} + 0.2\mathbf{j}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (0.2\text{ m})^2}} = 0.8944\mathbf{i} + 0.4472\mathbf{j}$$

Vector $\mathbf{r}$ is directed from *any point* on the AB axis to *any point* on the line of action of the force. For example, position vectors $\mathbf{r}_C$ and $\mathbf{r}_D$ are suitable, Fig. 4–23b. (Although not shown, $\mathbf{r}_{BC}$ or $\mathbf{r}_{BD}$ can also be used.) For simplicity, we choose $\mathbf{r}_D$, where

$$\mathbf{r}_D = \{0.6\mathbf{i}\}\text{ m}$$

The force is

$$\mathbf{F} = \{-300\mathbf{k}\}\text{ N}$$

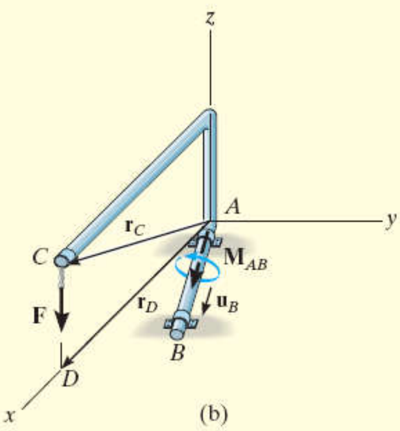


Fig. 4–23

Substituting these vectors into the determinant form and expanding, we have

$$M_{AB} = \mathbf{u}_B \cdot (\mathbf{r}_D \times \mathbf{F}) = \begin{vmatrix} 0.8944 & 0.4472 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & -300 \end{vmatrix}$$

$$= 0.8944[0(-300) - 0(0)] - 0.4472[0.6(-300) - 0(0)] + 0[0.6(0) - 0(0)]$$

$$= 80.50\ \text{N}\cdot\text{m}$$

This positive result indicates that the sense of $\mathbf{M}_{AB}$ is in the same direction as $\mathbf{u}_B$.

Expressing $\mathbf{M}_{AB}$ in Fig. 4-23*b* as a Cartesian vector yields

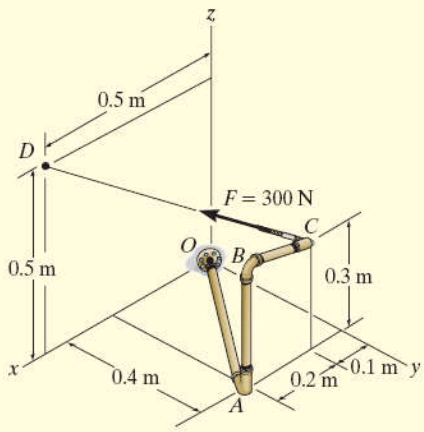
$$\mathbf{M}_{AB} = M_{AB}\mathbf{u}_B = (80.50\ \text{N}\cdot\text{m})(0.8944\mathbf{i} + 0.4472\mathbf{j})$$

$$= \{72.0\mathbf{i} + 36.0\mathbf{j}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: If axis AB is defined using a unit vector directed from B toward A, then in the above formulation $-\mathbf{u}_B$ would have to be used. This would lead to $M_{AB} = -80.50\ \text{N}\cdot\text{m}$. Consequently, $\mathbf{M}_{AB} = M_{AB}(-\mathbf{u}_B)$, and the same result would be obtained.

EXAMPLE 4.9

Determine the magnitude of the moment of force **F** about segment *OA* of the pipe assembly in Fig. 4–24*a*.



(a)

SOLUTION

The moment of **F** about the *OA* axis is determined from $M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r} \times \mathbf{F})$, where **r** is a position vector extending from any point on the *OA* axis to any point on the line of action of **F**. As indicated in Fig. 4–24*b*, either $\mathbf{r}_{OD}$, $\mathbf{r}_{OC}$, $\mathbf{r}_{AD}$, or $\mathbf{r}_{AC}$ can be used; however, $\mathbf{r}_{OD}$ will be considered since it will simplify the calculation.

The unit vector $\mathbf{u}_{OA}$, which specifies the direction of the *OA* axis, is

$$\mathbf{u}_{OA} = \frac{\mathbf{r}_{OA}}{r_{OA}} = \frac{\{0.3\mathbf{i} + 0.4\mathbf{j}\}\text{ m}}{\sqrt{(0.3\text{ m})^2 + (0.4\text{ m})^2}} = 0.6\mathbf{i} + 0.8\mathbf{j}$$

and the position vector $\mathbf{r}_{OD}$ is

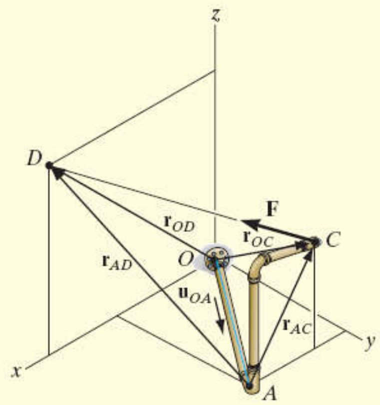
$$\mathbf{r}_{OD} = \{0.5\mathbf{i} + 0.5\mathbf{k}\}\text{ m}$$

The force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\left(\frac{\mathbf{r}_{CD}}{r_{CD}}\right)$$

$$= (300\text{ N})\left[\frac{\{0.4\mathbf{i} - 0.4\mathbf{j} + 0.2\mathbf{k}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (-0.4\text{ m})^2 + (0.2\text{ m})^2}}\right]$$

$$= \{200\mathbf{i} - 200\mathbf{j} + 100\mathbf{k}\}\text{ N}$$



(b)

Fig. 4–24

Therefore,

$$M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r}_{OD} \times \mathbf{F})$$

$$= \begin{vmatrix} 0.6 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 200 & -200 & 100 \end{vmatrix}$$

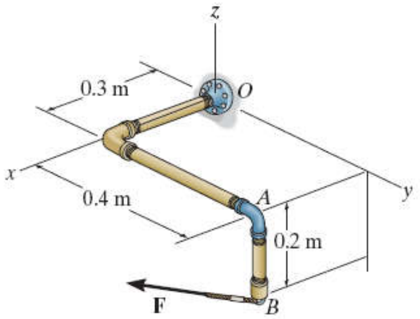
$$= 0.6[0(100) - (0.5)(-200)] - 0.8[0.5(100) - (0.5)(200)] + 0$$

$$= 100\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

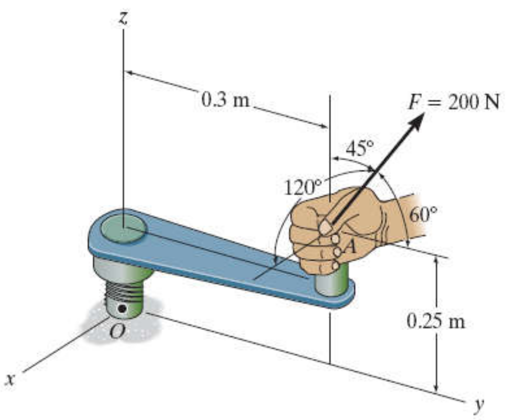
F4–13. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the x axis.

F4–14. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the OA axis.



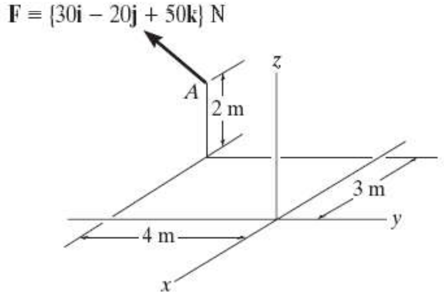
F4–13/14

F4–15. Determine the magnitude of the moment of the 200-N force about the x axis. Solve the problem using both a scalar and a vector analysis.



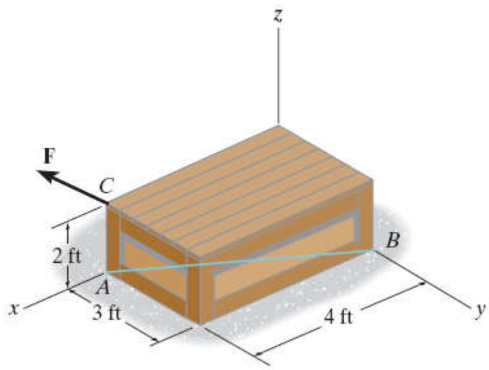
F4–15

F4–16. Determine the magnitude of the moment of the force about the y axis.



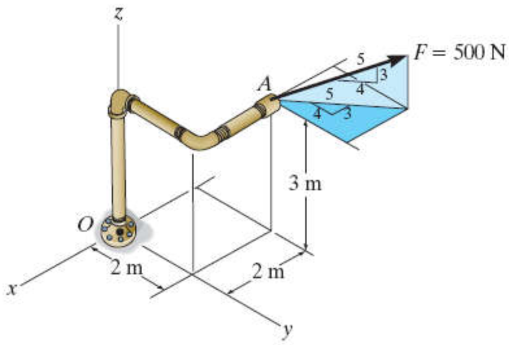
F4–16

F4–17. Determine the moment of the force $\mathbf{F} = \{50\mathbf{i} - 40\mathbf{j} + 20\mathbf{k}\}$ lb about the AB axis. Express the result as a Cartesian vector.



F4–17

F4–18. Determine the moment of force **F** about the x, the y, and the z axes. Solve the problem using both a scalar and a vector analysis.



F4–18