

# **CHAPTER FIVE**

# **Equilibrium of a Rigid Body**

# **CHAPTER OBJECTIVES**

- To develop the equations of equilibrium for a rigid body.
- To introduce the concept of the free-body diagram for a rigid body.
- To show how to solve rigid-body equilibrium problems using the equations of equilibrium.

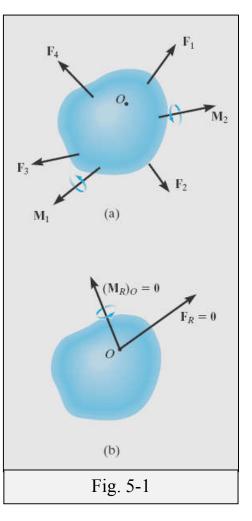
# 5.1 Conditions for Rigid-Body Equilibrium

The necessary and sufficient conditions for the equilibrium of the rigid body in Fig. 5–1 a, will be discussed. As shown, this body is subjected to an external force and couple moment system.

If the resultant force and couple moment resultant are both equal to zero, then the body is said to be in equilibrium .Mathematically, the equilibrium of a body is expressed as:

$F_R = \Sigma F = 0$	(5–1)
$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O = 0$	(5-1)

The first of these equations states that the sum of the forces acting on the body is equal to *zero*. The second equation states that the sum of the moments of all the forces in the system about point *O*, added to all the couple moments, is equal to *zero*. These two equations are not only necessary for equilibrium, they are also sufficient.





Note: When applying the equations of equilibrium, we will assume that the body remains *rigid*. In reality, however, all bodies *deform* when subjected to loads. Although this is the case, most engineering materials such as steel and concrete are *very rigid* and so their deformation is usually very small. Therefore, when applying the equations of equilibrium, we can generally assume that the body will remain *rigid* and *not deform* under the applied load without introducing any significant error.

# **EQUILIBRIUM IN TWO DIMENSIONS: (Coplanar Force System)**

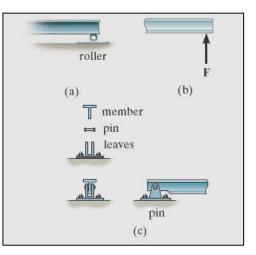
# 5.2 Free-Body Diagrams:

This *diagram* is a *sketch of the outlined shape of the body*, which is represented as being *isolated or "free"* from its surroundings, i.e. a "free body" on this sketch it is necessary to show all **the forces** and **couple moments** that the surroundings exert on the body so that these effects can be accounted for when *the equations of equilibrium* are applied. A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.

**Support Reactions:** Before presenting a formal procedure as to how to draw a freebody diagram, we will first consider the various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems. As a general rule,

- If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.
- If rotation is prevented, a couple moment is exerted on the body.

For example, let us consider three ways in which a horizontal member, such as a beam, is supported at its end. One method consists of a *roller* or cylinder, Fig. 5–3 a. Since this support only prevents the beam from *translating* in the vertical direction, the roller will only exert a *force* on the beam in this direction, Fig. 5–3 b. The beam can be supported in a more restrictive manner by using a *pin*, Fig. 5–3 c. The pin passes through a hole in the beam and two leaves which are fixed to the ground. Here the pin





can prevent *translation* of the beam in *any direction* f, Fig. 5–3 d, and so the pin must exert a *force* F on the beam in this direction. For purposes of analysis, it is generally easier to represent this resultant force F by its two rectangular components  $F_x$  and  $F_y$ , Fig. 5–3 e. If  $F_x$  and  $F_y$  are known, then F and  $\emptyset$  can be calculated. The most restrictive way to support the beam would be to use a *fixed support* as shown in Fig. 5–3 f. This support will prevent both *translation and rotation* of the beam. To do this a *force and couple moment* must be developed on the beam at its point of connection, Fig. 5–3 g. As in the case of the pin, the force is usually represented by its rectangular components  $F_x$  and  $F_y$ .

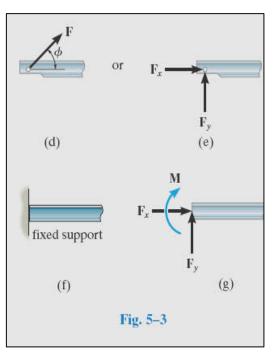
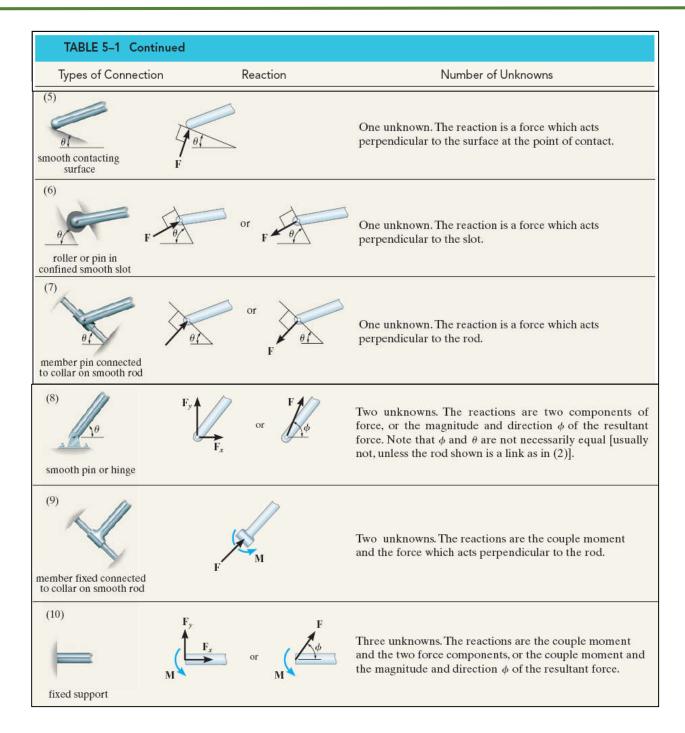


Table 5–1 lists other common types of supports for bodies subjected to coplanar force systems.

Types of Connection	Reaction	Number of Unknowns
1) b cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
2) weightless link	or F	One unknown. The reaction is a force which acts along the axis of the link.
3) $\theta$ roller F		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
	Hel.	One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.







Typical examples of actual supports are shown in the following sequence of photos. The numbers refer to the connection types in Table 5–1.



The cable exerts a force on the bracket in the direction of the cable. (1)





The rocker support for this bridge girder allows horizontal movement so the bridge is free to expand and contract due to a change in temperature. (4)

This utility building is pin supported at the top of the column. (8)

This concrete girder rests on the ledge that is assumed to act as a smooth contacting surface. (5)



The floor beams of this building are welded together and thus form fixed connections. (10)



To construct a free–body diagram for a rigid body or any group of bodies considered as a single system. The following steps should be performed:

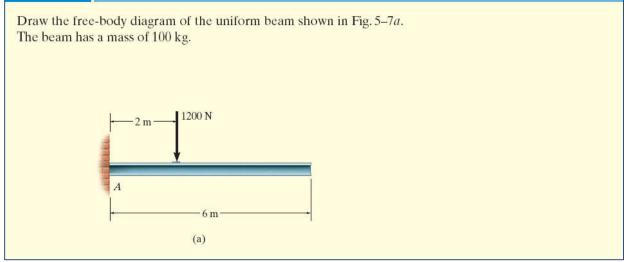
- **Draw outlined shape:** Imagine the body to be isolated or cut "free" from its constraints and connections and draw (sketch) its outlined shape.
- Show all Forces and Couple moments: Identify all the known and unknown external forces and couple moments act on the body. Those generally encountered are due to: (1) applied loadings (2) reactions developed at the supports or at points of contact with other bodies (see table 5-1) and (3) the weight of the body.
- Identify each loading and give dimensions: The forces and couple moments that are known should be labeled with their proper magnitudes and directions. Letters are used to represents the magnitudes and direction angles of forces and couple moments that are unknown. Establish an x, y coordinate system so that these unknowns,  $A_x$ ,  $A_y$ , etc can be identified. Finally indicate the dimensions of the body necessary for calculating the moments of forces.



# **Important Points**

- No equilibrium problem should be solved without *first drawing the free-body diagram*, so as to account for all the forces and couple moments that act on the body.
- If a support *prevents translation* of a body in a particular direction, then the support exerts a *force* on the body in that direction.
- If *rotation is prevented*, then the support exerts a *couple moment* on the body.
- Study Table 5–1.
- Internal forces are never shown on the free-body diagram since they occur in equal but opposite collinear pairs and therefore cancel out.
- The weight of a body is an external force, and its effect is represented by a single resultant force acting through the body's center of gravity *G*.
- Couple moments can be placed anywhere on the free-body diagram since they are *free vectors*. Forces can act at any point along their lines of action since they are *sliding vectors*.

### EXAMPLE 5.1

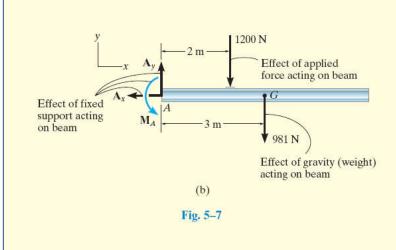


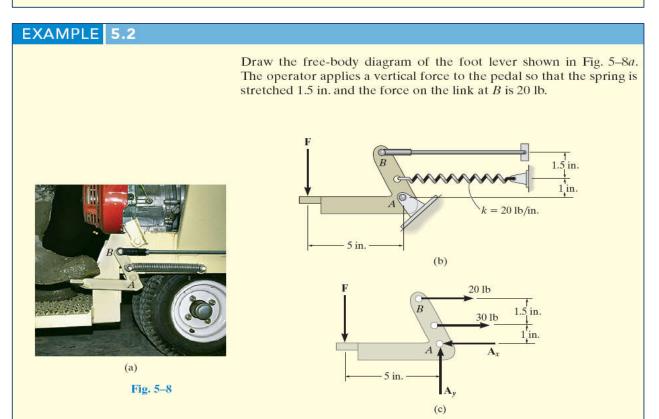


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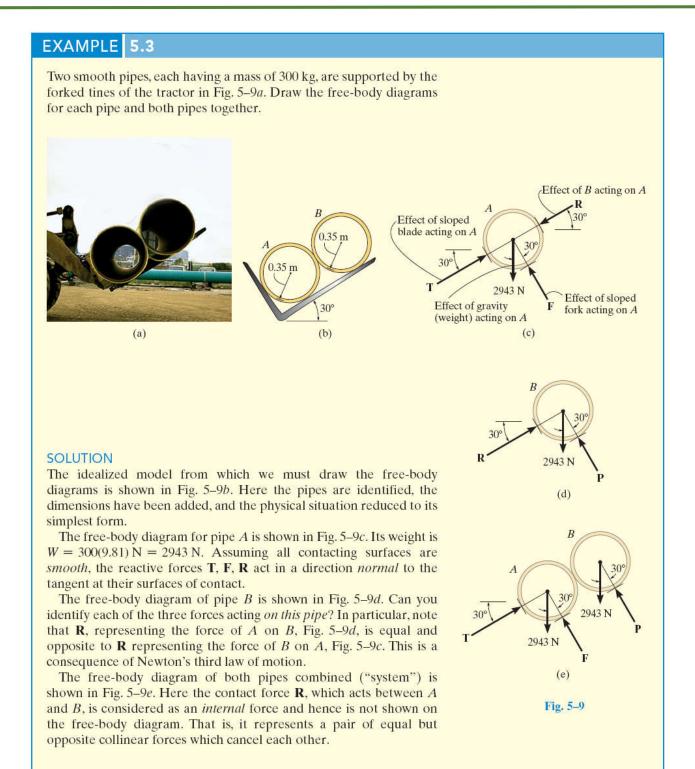
#### SOLUTION

The free-body diagram of the beam is shown in Fig. 5–7b. Since the support at A is fixed, the wall exerts three reactions on the beam, denoted as  $A_x$ ,  $A_y$ , and  $M_A$ . The magnitudes of these reactions are *unknown*, and their sense has been *assumed*. The weight of the beam, W = 100(9.81) N = 981 N, acts through the beam's center of gravity G, which is 3 m from A since the beam is uniform.









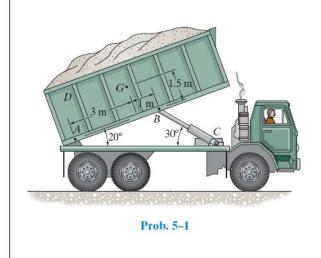


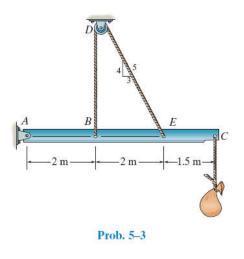
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## PROBLEMS

**5–1.** Draw the free-body diagram of the dumpster D of the truck, which has a mass of 2.5 Mg and a center of gravity at G. It is supported by a pin at A and a pin-connected hydraulic cylinder BC (short link). Explain the significance of each force on the diagram. (See Fig. 5–7b.)

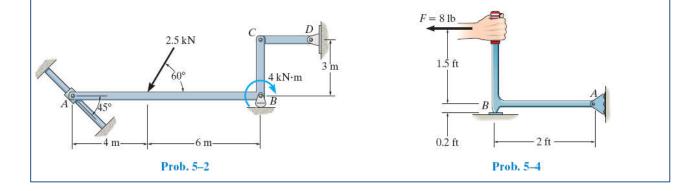
**5–3.** Draw the free-body diagram of the beam which supports the 80-kg load and is supported by the pin at A and a cable which wraps around the pulley at D. Explain the significance of each force on the diagram. (See Fig. 5–7b.)





5–2. Draw the free-body diagram of member ABC which is supported by a smooth collar at A, rocker at B, and short link CD. Explain the significance of each force acting on the diagram. (See Fig. 5–7*b*.)

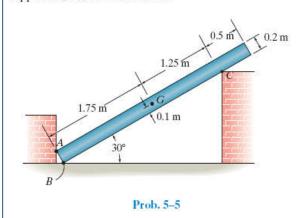
\*5-4. Draw the free-body diagram of the hand punch, which is pinned at A and bears down on the smooth surface at B.



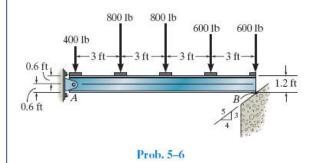


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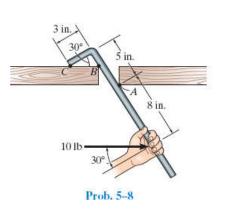
5-5. Draw the free-body diagram of the uniform bar, which has a mass of 100 kg and a center of mass at G. The supports A, B, and C are smooth.



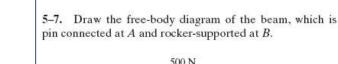
**5–6.** Draw the free-body diagram of the beam, which is pin-supported at A and rests on the smooth incline at B.

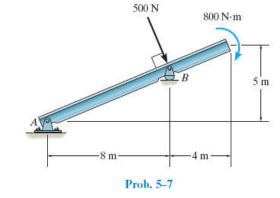


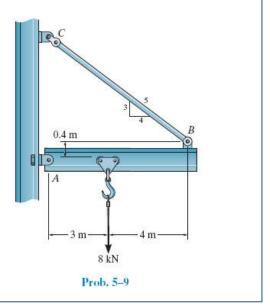
\*5–8. Draw the free-body diagram of the bar, which has a negligible thickness and smooth points of contact at A, B, and C. Explain the significance of each force on the diagram. (See Fig. 5–7b.)



5-9. Draw the free-body diagram of the jib crane AB, which is pin connected at A and supported by member (link) BC.









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# **5.3 Equations of Equilibrium**

In Sec. 5.1 we developed the two equations which are both necessary and sufficient for the equilibrium of a rigid body, namely,  $\Sigma \mathbf{F} = \mathbf{0}$  and  $\Sigma \mathbf{M}_0 = \mathbf{0}$ . When the body is subjected to a system of forces, which all lie in the x - yplane, then the forces can be resolved into their x and ycomponents. Consequently, the conditions for equilibrium in two dimensions are:

$$\Sigma F_x = 0$$
  

$$\Sigma F_y = 0$$
  

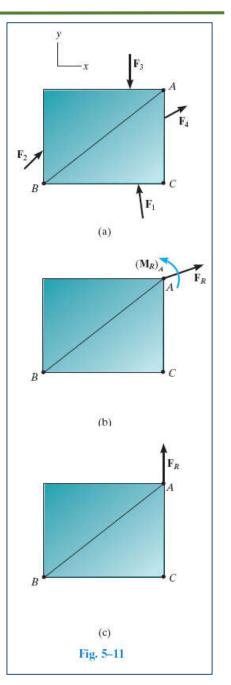
$$\Sigma M_O = 0$$
(5-2)

## Alternative Sets of Equilibrium Equations:

$\Sigma F_x = 0$	
$\Sigma M_A = 0$	(5–3)
$\Sigma M_B = 0$	1938 - W

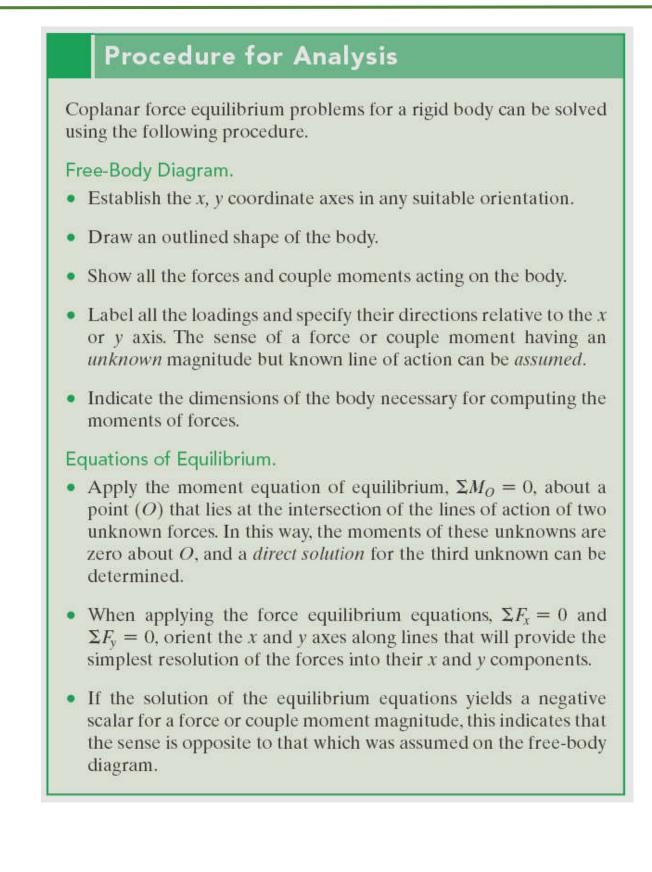
A second alternative set of equilibrium equations is:

$$\begin{split} \Sigma M_A &= 0\\ \Sigma M_B &= 0\\ \Sigma M_C &= 0 \end{split} \tag{5-4}$$

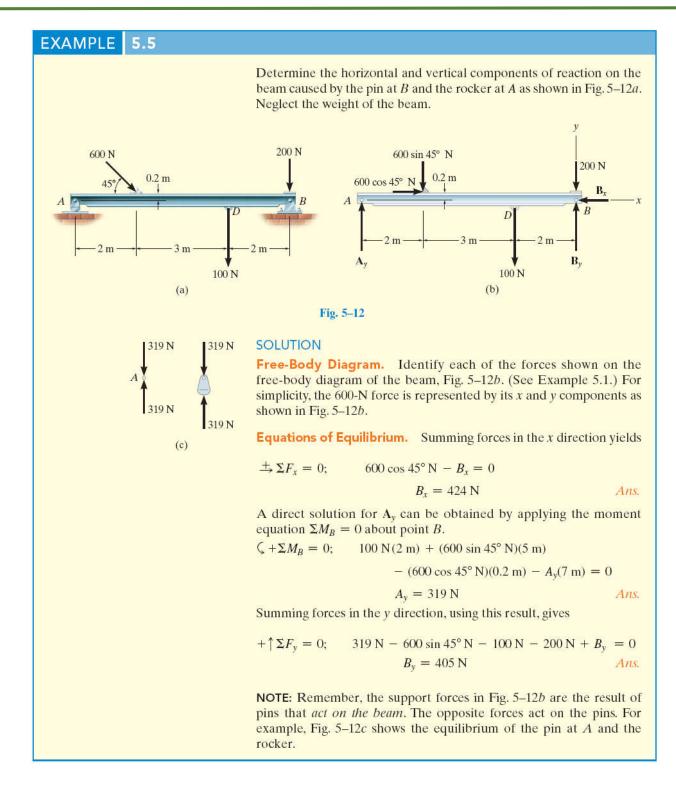


• Here it is necessary that points *A*, *B*, and *C* do not lie on the same line. To prove that these equations, when satisfied, ensure equilibrium, consider again the freebody diagram in Fig. 5–11 *b*. If  $\Sigma M_A = 0$  is to be satisfied, then  $(M_R)_A = 0$ .  $\Sigma M_C = 0$  is satisfied if the line of action of  $\mathbf{F}_R$  passes through point *C* as shown in Fig. 5–11 *c*. Finally, if we require  $\Sigma M_B = 0$ , it is necessary that  $\mathbf{F}_R = 0$ , and so the plate in Fig. 5–11 *a* must then be in equilibrium.

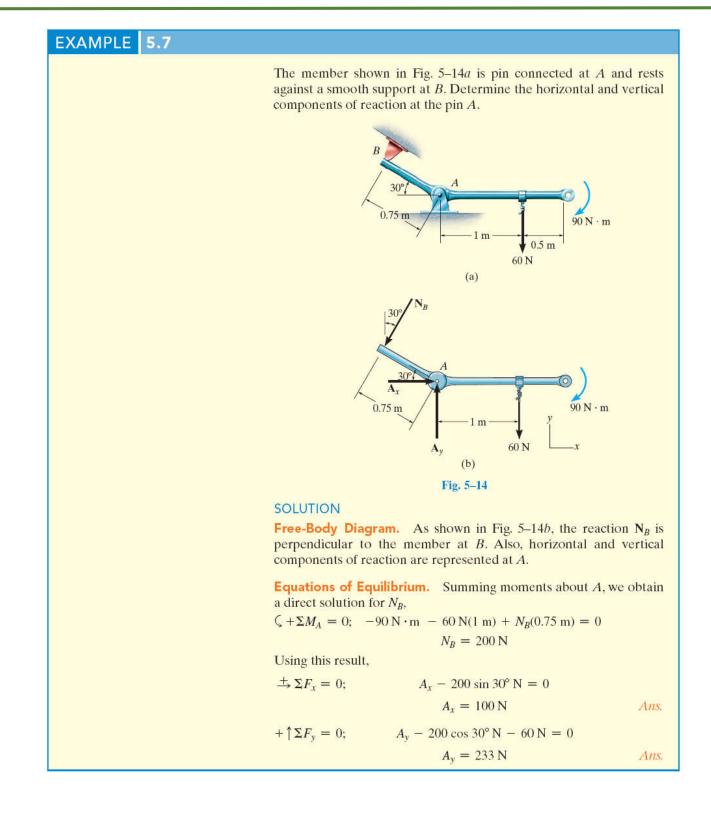














## EXAMPLE 5.8

The box wrench in Fig. 5-15a is used to tighten the bolt at A. If the wrench does not turn when the load is applied to the handle, determine the torque or moment applied to the bolt and the force of the wrench on the bolt.

### SOLUTION

**Free-Body Diagram.** The free-body diagram for the wrench is shown in Fig. 5–15*b*. Since the bolt acts as a "fixed support," it exerts force components  $A_x$  and  $A_y$  and a moment  $M_A$  on the wrench at *A*.

Equations of Equilibrium.

$$\stackrel{+}{\to} \Sigma F_x = 0; \qquad A_x - 52 \left(\frac{5}{13}\right) N + 30 \cos 60^\circ N = 0$$
$$A_x = 5.00 N \qquad Ans.$$

+↑
$$\Sigma F_y = 0;$$
  $A_y - 52(\frac{12}{13})$  N - 30 sin 60° N = 0  
 $A_y = 74.0$  N Ans

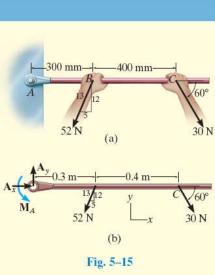
$$\zeta + \Sigma M_A = 0; \quad M_A - \left[ 52 \left( \frac{12}{13} \right) N \right] (0.3 \text{ m}) - (30 \sin 60^\circ \text{ N})(0.7 \text{ m}) = 0$$
  
 $M_A = 32.6 \text{ N} \cdot \text{m}$  Ans.

Note that 
$$\mathbf{M}_A$$
 must be *included* in this moment summation. This couple moment is a free vector and represents the twisting resistance of the bolt on the wrench. By Newton's third law, the wrench exerts an equal but opposite moment or torque on the bolt. Furthermore, the resultant force on the wrench is

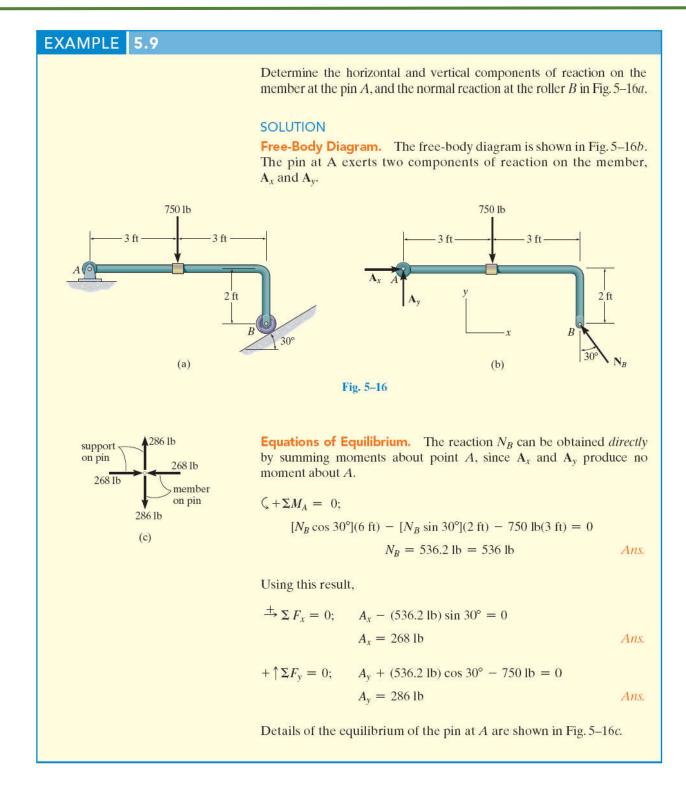
$$F_A = \sqrt{(5.00)^2 + (74.0)^2} = 74.1 \text{ N}$$
 Ans

**NOTE:** Although only *three* independent equilibrium equations can be written for a rigid body, it is a good practice to *check* the calculations using a fourth equilibrium equation. For example, the above computations may be verified in part by summing moments about point *C*:

$$\zeta + \Sigma M_C = 0;$$
  $\left[ 52 \left( \frac{12}{13} \right) N \right] (0.4 \text{ m}) + 32.6 \text{ N} \cdot \text{m} - 74.0 \text{ N}(0.7 \text{ m}) = 0$   
19.2 N · m + 32.6 N · m - 51.8 N · m = 0

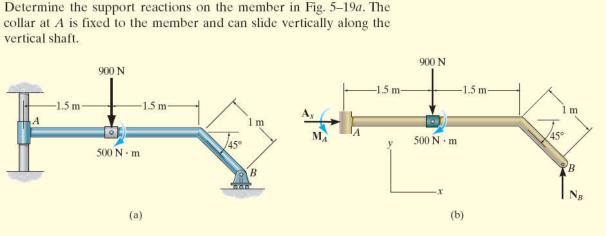








### EXAMPLE 5.12





#### SOLUTION

**Free-Body Diagram.** The free-body diagram of the member is shown in Fig. 5–19*b*. The collar exerts a horizontal force  $A_x$  and moment  $M_A$  on the member. The reaction  $N_B$  of the roller on the member is vertical.

**Equations of Equilibrium.** The forces  $A_x$  and  $N_B$  can be determined directly from the force equations of equilibrium.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x = 0 \qquad Ans.$$

$$+ \uparrow \Sigma F_y = 0; \qquad N_B - 900 \text{ N} = 0$$

$$N_B = 900 \text{ N} \qquad Ans.$$

The moment  $M_A$  can be determined by summing moments either about point A or point B.

 $\begin{aligned} \zeta + \Sigma M_A &= 0; \\ M_A - 900 \ \mathrm{N}(1.5 \ \mathrm{m}) - 500 \ \mathrm{N} \cdot \mathrm{m} + 900 \ \mathrm{N} \ [3 \ \mathrm{m} + (1 \ \mathrm{m}) \cos 45^\circ] &= 0 \\ M_A &= -1486 \ \mathrm{N} \cdot \mathrm{m} = 1.49 \ \mathrm{kN} \cdot \mathrm{m} \ \mathcal{A} ns. \end{aligned}$ or

$$\zeta + \Sigma M_B = 0; \quad M_A + 900 \text{ N} [1.5 \text{ m} + (1 \text{ m}) \cos 45^\circ] - 500 \text{ N} \cdot \text{m} = 0$$
  
 $M_A = -1486 \text{ N} \cdot \text{m} = 1.49 \text{ kN} \cdot \text{m} \mathcal{A}$  Ans

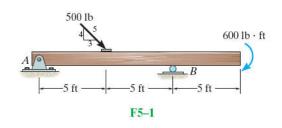
The negative sign indicates that  $M_A$  has the opposite sense of rotation to that shown on the free-body diagram.



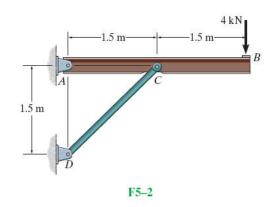
# FUNDAMENTAL PROBLEMS

#### All problem solutions must include an FBD.

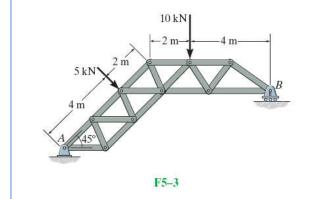
**F5–1.** Determine the horizontal and vertical components of reaction at the supports. Neglect the thickness of the beam.



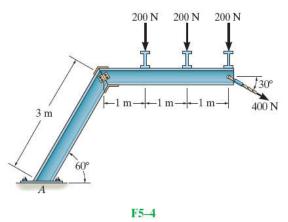
**F5–2.** Determine the horizontal and vertical components of reaction at the pin A and the reaction on the beam at C.



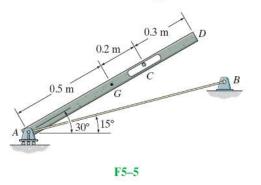
**F5–3.** The truss is supported by a pin at A and a roller at B. Determine the support reactions.



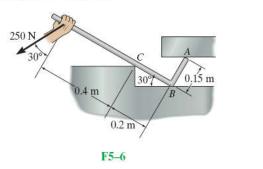
**F5-4.** Determine the components of reaction at the fixed support *A*. Neglect the thickness of the beam.



**F5–5.** The 25-kg bar has a center of mass at G. If it is supported by a smooth peg at C, a roller at A, and cord AB, determine the reactions at these supports.



**F5–6.** Determine the reactions at the smooth contact points A, B, and C on the bar.

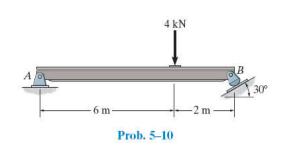




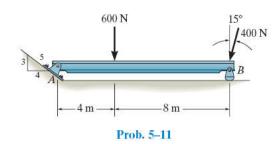
# PROBLEMS

#### All problem solutions must include an FBD.

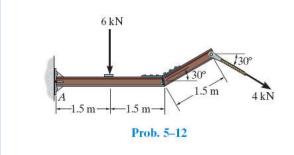
**5–10.** Determine the horizontal and vertical components of reaction at the pin A and the reaction of the rocker B on the beam.



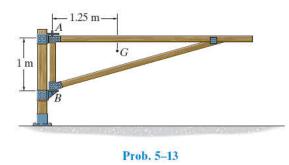
5–11. Determine the magnitude of the reactions on the beam at A and B. Neglect the thickness of the beam.

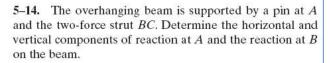


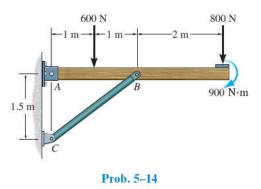
\*5–12. Determine the components of the support reactions at the fixed support A on the cantilevered beam.



**5–13.** The 75-kg gate has a center of mass located at G. If A supports only a horizontal force and B can be assumed as a pin, determine the components of reaction at these supports.







**5–15.** Determine the horizontal and vertical components of reaction at the pin at A and the reaction of the roller at B on the lever.

