Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

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Example 2:

Use the revised simplex method to solve the following problem.

$$\begin{aligned} \text{minimize:} \quad & z = 3x_1 + 2x_2 + 4x_3 + 6x_4 \\ \text{subject to:} \quad & x_1 + 2x_2 + x_3 + x_4 \geq 1000 \\ & 2x_1 + x_2 + 3x_3 + 7x_4 \geq 1500 \\ \text{with:} \quad & \text{all variables nonnegative} \end{aligned}$$

Sol:

This program is put in standard form by introducing the surplus variables x_5 and x_7, and the artificial variables x_6 and x_8.

$$\begin{aligned} \text{minimize:} \quad & z = 3x_1 + 2x_2 + 4x_3 + 6x_4 + 0x_5 + Mx_6 + 0x_7 + Mx_8 \\ \text{subject to:} \quad & x_1 + 2x_2 + x_3 + x_4 - x_5 + x_6 = 1000 \\ & 2x_1 + x_2 + 3x_3 + 7x_4 - x_7 + x_8 = 1500 \\ \text{with:} \quad & \text{all variables nonnegative} \end{aligned}$$

$$\mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{P}_4 = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \mathbf{P}_5 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{P}_6 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P}_7 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{P}_8 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1000 \\ 1500 \end{pmatrix}$$

Initialization:

$$\mathbf{X}_S = (x_6, x_8)^T; \qquad \mathbf{C}_S^T = (M, M)$$

$$\mathbf{S} = (\mathbf{P}_6, \mathbf{P}_8) = \mathbf{I} = \mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Iteration No. 1:

The nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (M, M)\mathbf{I} = (M, M)$$

$$(c_1 - z_1, c_2 - z_2, c_3 - z_3, c_4 - z_4, c_5 - z_5, c_7 - z_7) = (c_1, c_2, c_3, c_4, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_7)$$

$$= (3, 2, 4, 6, 0, 0) - (M, M)\begin{pmatrix} 1 & 2 & 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & 7 & 0 & -1 \end{pmatrix}$$

$$= (-3M + 3, -3M + 2, -4M + 4, -8M + 6, M, M)$$

Since the most negative coefficient corresponds to $\mathbf{P}_4$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \mathbf{IB} = \mathbf{B} = \begin{pmatrix} 1000 \\ 1500 \end{pmatrix}; \qquad t_4 = \mathbf{S}^{-1}\mathbf{P}_4 = \mathbf{IP}_4 = \mathbf{P}_4 = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\theta = \min\{1000, 1500/7\} = 1500/7$$

Since the minimum ratio corresponds to $\mathbf{P}_8$, it becomes the departing vector (**D.V.**).

(*c*) New Basis:

$$\boldsymbol{\eta} = \begin{pmatrix} \dfrac{-t_{64}}{t_{84}} \\ \dfrac{1}{t_{84}} \end{pmatrix} = \begin{pmatrix} -1/7 \\ 1/7 \end{pmatrix}; \qquad \mathbf{E} = (\mathbf{u}_1, \boldsymbol{\eta})$$

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{ES}^{-1} = \mathbf{EI} = \mathbf{E} = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}$$

Summary of Iteration No. 1:

$$\mathbf{X}_S = (x_6, x_4)^T; \qquad \mathbf{C}_S^T = (M, 6)$$

Iteration No. 2:
Now the nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_8$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (M, 6)\begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix} = (M, 6/7 - M/7)$$

$$\begin{aligned}(c_1 - z_1, c_2 - z_2, c_3 - z_3, c_8 - z_8, c_5 - z_5, c_7 - z_7) &= (c_1, c_2, c_3, c_8, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_8, \mathbf{P}_5, \mathbf{P}_7) \\ &= (3, 2, 4, M, 0, 0) - (M, 6/7 - M/7) \\ &\quad \times \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & -1 \end{pmatrix} \\ &= (-5M/7 + 9/7, -13M/7 + 8/7, -4M/7 + 10/7, \\ &\quad 8M/7 - 6/7, M, -M/7 + 6/7)\end{aligned}$$

Since the most negative coefficient corresponds to $\mathbf{P}_2$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \end{pmatrix} = (1000 - 1500/7, 1500/7) = (5500/7, 1500/7)$$

$$\mathbf{t}_2 = \mathbf{S}^{-1}\mathbf{P}_2 = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 13/7 \\ 1/7 \end{pmatrix}$$

$$\theta = \min\left\{\frac{5500/7}{13/7}, \frac{1500/7}{1/7}\right\} = \min\left\{\frac{5500}{3}, 1500\right\} = \frac{5500}{3}$$

Since the minimum ratio corresponds to $\mathbf{P}_6$, it becomes the departing vector (**D.V.**).

(*c*) New Basis:

$$\eta = \begin{pmatrix} \dfrac{1}{t_{62}} \\ \dfrac{-t_{42}}{t_{62}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{13/7} \\ -\dfrac{1/7}{13/7} \end{pmatrix} = \begin{pmatrix} 7/13 \\ -1/13 \end{pmatrix}; \qquad \mathbf{E} = (\eta, \mathbf{u}_2)$$

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{E}\mathbf{S}^{-1} = \begin{pmatrix} 7/13 & 0 \\ -1/13 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix} = \begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix}$$

Summary of Iteration No. 2:

$$\mathbf{X}_S = (x_2, x_4)^T; \qquad \mathbf{C}_S^T = (2, 6)$$

Iteration No. 3:

Now the nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_6$, $\mathbf{P}_3$, $\mathbf{P}_8$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (2, 6)\begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix} = (8/13, 10/13)$$

$$\begin{aligned}(c_1 - z_1, c_6 - z_6, c_3 - z_3, c_8 - z_8, c_5 - z_5, c_7 - z_7) &= (c_1, c_6, c_3, c_8, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_6, \mathbf{P}_3, \mathbf{P}_8, \mathbf{P}_5, \mathbf{P}_7) \\ &= (3, M, 4, M, 0, 0) - (8/13, 10/13) \\ &\quad \times \begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & 3 & 1 & 0 & -1 \end{pmatrix} \\ &= 11/13, -8/13 + M, 14/13, -10/13 + M, \\ &\quad 8/13, 10/13)\end{aligned}$$

Since all the coefficients are nonnegative, the above step gives the optimal basis. The optimal values of the variables and the objective function are as follows:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \end{pmatrix} = \begin{pmatrix} 5500/13 \\ 2000/13 \end{pmatrix}$$

$$z = \mathbf{C}_S^T\mathbf{X}_S = (2, 6)\begin{pmatrix} 5500/13 \\ 2000/13 \end{pmatrix} = 23000/13$$

Example:

An industry must have a water supply of at least $4x10^6$ liters/day of a quality such that total dissolved solids (TDS) is kept below 100 mg/l. The water can be obtained from two sources: (1) purchase from the city system at $100 per million liters, and (2) pump from a nearby stream at $50 per million liters. The concentration of TDS in the city source is 50 mg/l. TDS in the stream is 200 mg/l. Water from the two sources is completely mixed before it is used. The city can supply up to $3.5x10^6$ l/day, and water rights permit pumping up to $2x10^6$ l/day from the stream.

a. Formulate a linear program to optimize the amount of water used from each source. Define your decision variables and the meaning of the objective function and constraints.

b. Use the revised simplex method to determine the optimal solution.

Quiz 24-11-2015: Revised simplex

An aqueduct constructed to supply water to industrial users has an excess capacity in the months of June, July, and August of 14,000 acft, 18,000 acft, and 6,000 acft, respectively. It is proposed to develop not more than 10,000 acres of new land by utilizing the excess aqueduct capacity for irrigation water deliveries. Two crops, hay and grain, are to be grown. Their monthly water requirements and expected net returns are given in the following table:

	Monthly Water Requirement (acft/acre)			
	June	July	August	Return, $/acre
Hay	2	1	1	100
Grain	1	2	0	120

Formulate and solve a linear program to optimize the irrigation development. (use revised simplex method)