Limit (الغاية)

We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

نستخدم الغايات لوصف الطريقة التي تختلف بها الدالة. تختلف بعض الدوال باستمرار ؛ التغييرات الصغيرة في x تنتج فقط تغييرات صغيرة في $f(x)$. ودوال أخرى يمكن أن يكون القيم التي تقفز أو تتغير بشكل متقطع أو تميل إلى الزيادة أو النقصان بدون حدود. مفهوم الغاية تعطي طريقة دقيقة للتمييز بين تلك السلوكيات.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance d of c (Fig. 3).

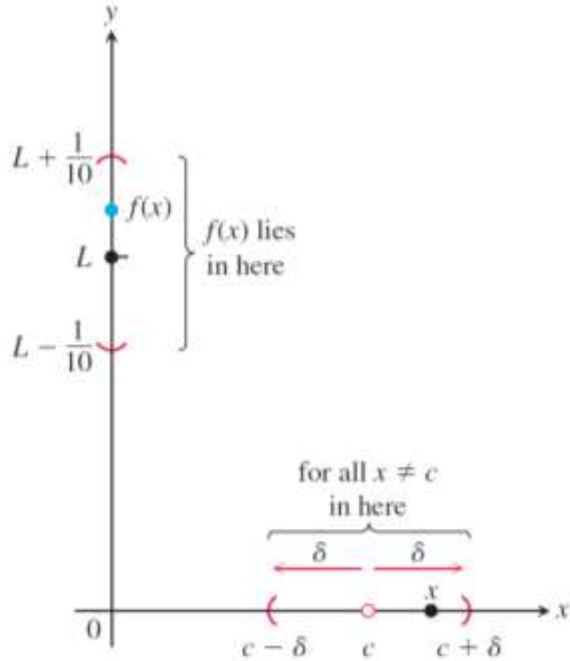


Fig.(3) How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval (L-1/10),(L+1/10)?

But that in itself is not enough, because as x continues on its course toward c, what is to prevent $f(x)$ from jumping around within the interval from L - (1/10) to L + (1/10) without tending toward L? We can be told that the error can be no more than 1/100 or 1/1000 or 1/100,000. Each time, we find a new δ -interval about c so that keeping x within that interval satisfies the new error tolerance.

And each time the possibility exists that $f(x)$ might jump away from L at some later stage.
We can present a matching distance d that keeps x "close enough" to c to keep $f(x)$ within that ε-tolerance of L (Fig.4). This leads us to the precise definition of a limit.

Def: - Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L$$

if, for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ a such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance .The diameter of the shaft is determined by turning a dial to a setting measured by a variable x. We try for diameter L, but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L.

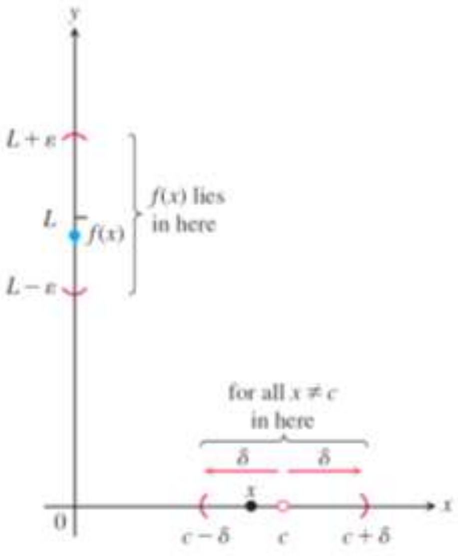


Fig.(4) The relation of δ and ε in the definition of limit.

As the tolerance for error becomes stricter, we may have to adjust δ. The value of δ, how tight our control setting must be, depends on the value of ε, the error tolerance.The definition of limit extends to functions on more general domains. It is only required that each open interval around c contains points in the domain of the function other than c.

Theorem:-

If L, M, c, and k are real no.s and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c} (f(x) + g(x)) = L + M$
2. *Difference Rule*: $\lim_{x\to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule*: $\lim_{x\to c} \left(k.f(x)\right) = k.L$
4. *Product Rule*: $\lim_{x\to c} \left(f(x).g(x)\right) = L.M$
5. *Quotient Rule*: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule*: $\lim_{x\to c} [f(x)]^n = L^n, n$ a positive integer
7. *Root Rule*: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(if n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Ex. If $f(x) = 2x + 5$, Find: $\lim_{x\to 1} f(x)$.

Sol.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+5) = \lim_{x\to 1} 2x + \lim_{x\to 1} 5 = 2\lim_{x\to 1} x + 5$$
$$= 2(1) + 5 = 7$$

Ex. If $f(x) = \frac{x^2-3x+2}{x-2}$, Find : $\lim_{x\to 2} f(x)$.

Sol.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \frac{4-6+2}{2-2} = \frac{0}{0}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)} = \lim_{x\to 2}(x-1)$$
$$= 2 - 1 = 1$$

Limits of Polynomials (الغاية لمتعددات الحدود)

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots \ldots \ldots \ldots + a_\circ$ is any polynomial fun. ,Then

$$\lim_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots \ldots \ldots \ldots + a_\circ$$

Limits of Quotients of Polynomials

If $f(x)$ and $g(x)$ are polynomials fun. , Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \quad , \quad g(c) \neq 0$$

Ex. If $f(x) = (x^2 + 3x - 1)$, Find : $\lim_{x\to -1} f(x)$.

Sol.

$$\lim_{x\to -1} f(x) = (-1)^2 + 3(-1) - 1 = -3$$

Ex. Find $\lim_{x\to 2} \frac{x^2+2x+4}{x+2}$

Sol.

$$\lim_{x\to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{x + 2} = \frac{12}{4} = 3$$

Ex. Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$, $x \neq 1$

Sol.

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{x^3-1}{x-1} \qquad \boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1}(x^2 + x + 1) = 3$$

Ex. Find $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$, $h \neq 0$

Sol.

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$ (الضرب بمرافق البسط)

$$=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

One-Sided Limits (الغايات من جهة واحدة)

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only. These allow us to describe functions that have different limits at a point,depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**. If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left.**

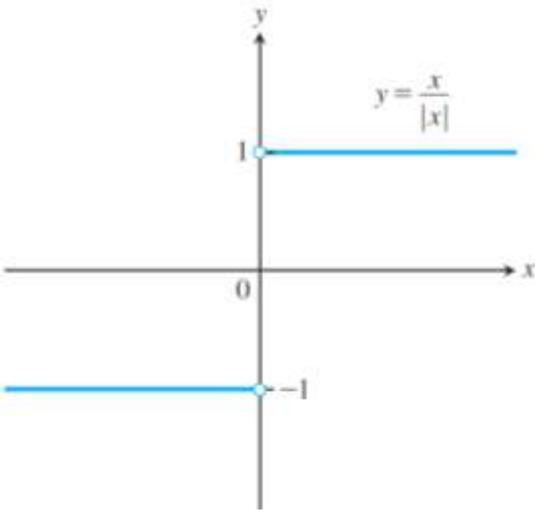


Fig.(5) Different right-hand and left-hand limits at the origin.

The function $f(x) = \frac{x}{|x|}$ (Fig. 5) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we only consider the values of $f(x)$ on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. In this case we write

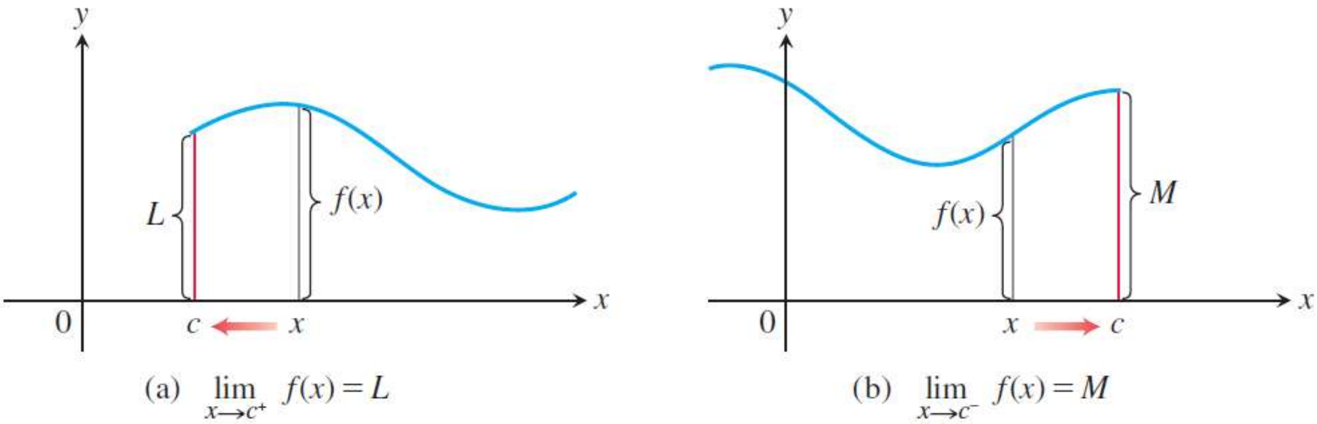
$$\lim_{x \to c^+} f(x) = L$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \leq c$
Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M$$

The symbol "$x \to c^-$" means that we consider the values of f only at x-values less than c. These informal definitions of one-sided limits are illustrated in (Fig. 6). For the function $f(x) = \frac{x}{|x|}$ in (Fig.5) we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$



Fig(6) (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c