Ex. $(x) = \sqrt{x}$, Find $\lim_{x\to 0} f(x)$.

Sol.

Dom. of $f(x)$ is $x \geq 0$

Since $\sqrt{x}$ is not defined for –ve values x so

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$$

Ex. $(x) = |x|$, Find $\lim_{x\to 0^+} |x|$.

Sol.

$$\text{Since}|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Then $f(x) = x$ where $x \to 0^+$

So $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

Ex. $(x) = \sqrt{1-x}$, Find $\lim_{x\to 1} f(x)$.

Sol.

Dom. $1 - x \geq 0 \quad \Leftrightarrow x \leq 1$

Since $\sqrt{1-x}$ is not defined for $x > 1$, so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = \sqrt{1-1} = 0 = \lim_{x\to 1} f(x)$$

Theorem:

1) $\lim_{x\to 0} \sin x = 0$
2) $\lim_{x\to 0} \cos x = 1$
3) $\lim_{x\to 0} \frac{\sin x}{x} = 1$
4) $\lim_{x\to 0} \frac{\cos x}{x} = \infty$
5) $\lim_{x\to 0} \frac{1}{x} = \infty$

Ex. Find the following limits:-

a) $\lim_{x\to 0} \frac{\sin 3x}{x} = 3\lim_{x\to 0} \frac{\sin 3x}{3x}$

$\because$ as $x \to 0 \quad \Longrightarrow \quad 3x \to 0$

$$= 3\lim_{3x\to 0} \frac{\sin 3x}{3x} = 3 \times 1 = 3$$

b) $\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \frac{5}{3}\lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 3x}{3x}} = \frac{5}{3} \frac{\lim_{5x\to 0} \frac{\sin 5x}{5x}}{\lim_{3x\to 0} \frac{\sin 3x}{3x}} = \frac{5}{3}$

Limits At Infinity (الغاية عند اللانهاية)

Def:-

Limits as $x \to \infty$ or $x \to -\infty$

1. The limit of the fun. $f(x)$ as x approaches infinity is the no. L.

$\lim_{x\to\infty} f(x) = L$ if

Given any $\varepsilon > 0$ there exists a no. $M > 0$ such that for all x,

$$M < x \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

2. The limit of the fun. $f(x)$ as x approaches negative infinity is the no. L.

$\lim_{x \to -\infty} f(x) = L$ if

Given any $\varepsilon > 0$ there exists a no. $N < 0$ such that for all x,

$$x < N \implies |f(x) - L| < \varepsilon$$

Theorem:

If $f(x) = k$ (for any no. k)

1. $\lim_{x\to\infty} k = \lim_{x\to-\infty} k = k$
2. If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$

When L and M (are positive real no.s) then:-

a) $\lim_{x\to\infty}\left(f(x) \mp g(x)\right) = \lim_{x\to\infty} f(x) \mp \lim_{x\to\infty} g(x) = L \mp M$

b) $\lim_{x\to\infty}\left(f(x).g(x)\right) = LM$

c) $\lim_{x\to\infty} kf(x) = k\lim_{x\to\infty} f(x) = kL$ (for any no. k)

d) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

These results hold for $x \to -\infty$

Ex. Find the limit of :-

1. $\lim_{x\to\infty} \frac{x}{7x+4}$

عندما نعوض ونجد أن النتيجة $\frac{\infty}{\infty}$ نقسم على اكبر أس

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{7x+4}{x}} = \lim_{x\to\infty} \frac{1}{7+\frac{4}{x}} = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} 7 + \lim_{x\to\infty} \frac{4}{x}} = \frac{1}{7}$$

2. $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{x} = 0 \cdot 0 = 0$

Theorem:

1. $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
2. $\lim_{x\to\infty} \frac{\cos x}{x} = 0$
3. $\lim_{x\to\infty} \frac{1}{x} = 0$

Ex. Find

1. $\lim_{x\to\infty} (2 + \frac{\sin x}{x})$

$= \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{\sin x}{x} = 2 + 0 = 2$

2. $\lim_{x\to\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\infty} \frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}} = \frac{2-0+0}{3+0} = \frac{2}{3}$

Infinite Limits (الغايات اللانهاية)

Def:-

1. The limit of the fun. $f(x)$ as $x \to a$ is the infinity such that $a \notin D_f$
   $\lim_{x\to a} f(x) = \infty$
   if $\forall\, M > 0$ There exists $\delta > 0$ such that
   $\forall x \in D_f ,\quad 0 < |x-a| < \delta \implies f(x) > M$
2. The limit of the fun. $f(x)$ as $x \to a$ is the negative infinity such that $a \notin D_f$
   $\lim_{x\to a} f(x) = -\infty$
   if $\forall\, M < 0$ There exists $\delta > 0$ such that
   $\forall x \in D_f ,\quad 0 < |x-a| < \delta \implies f(x) < M$

Ex.

1. $\lim_{x\to 0^+} \frac{1}{x} = \infty$
2. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$
3. $\lim_{x\to 0} \frac{1}{x^2} = \infty$
4. $\lim_{x\to 2^+} \frac{1}{x^2-4} = \frac{1}{0} = \infty$
5. $\lim_{x\to 2^-} \frac{1}{x^2-4} = \frac{1}{0} = -\infty$
6. $\lim_{x\to \infty} \frac{2x^3+2x-1}{x^2-5x+2} = \lim_{x\to\infty} \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}} = \frac{2+0-0}{0+0+0} = \infty$

Continuity (الأستمرارية)

Def:-

The fun. $y = f(x)$ is continuous at $x = c$ iff all three of the following statements are true:-

1. $f(c)$ is exists (c is in the domain of f).
2. $\lim_{x\to c} f(x)$ is exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the fun. value).

Continuous function

Def:-

A fun. is continuous if it is continuous at each point of its domain.

Discontinuity at a point

Def:-

If a fun. f is not cont. at a point c, we say that f is discontinuous at c and call c a point of discontinuity of f .