Ex. if $g(x) = \sqrt{x+2}$ and $x = f(t) = t^3 - 1$ find $\frac{d}{dx}(g \circ f)$ when $t = 2$

Sol.

$$\underset{\text{at } t=2}{(g \circ f)'} = \underset{\text{at } x=f(2)}{g'} \cdot \underset{\text{at } t=2}{f'}$$

$$\frac{d}{dx}(g \circ f)\Big|_{t=2} = g'(f(2)) \,.\, f'(2)$$

$$= \frac{1}{2\sqrt{x+2}}\Big|_{x=7} \quad . \quad 3t^2\Big|_{t=2}$$

$$= \frac{1}{6} \cdot \ 12 = 2$$

Ex.

If $y = t^2 - 1$ and $x = 2t + 3$ find $\frac{dy}{dx}$

Sol.

$$\frac{dy}{dt} = 2t \quad , \frac{dx}{dt} = 2 \quad \Rightarrow \quad \frac{dt}{dx} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2t \cdot \frac{1}{2} = t$$

But $x = 2t + 3 \quad \Rightarrow t = \frac{x-3}{2}$

Now we substitute t in the $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{x-3}{2}$$

Exercise:-

1) Find $\frac{dy}{dx}$

a) $y = \frac{(x^2+x)(x^2-x+1)}{x^4}$ b) $y = (\frac{x+1}{x-1})^2$

b) c) $y = (x+1)^2(x^2+1)^{-3}$

2) $\frac{d^2y}{dx^2}$ if $y = (3-2x)^{-1}$

3) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by implicit defferentiation

a) $y^2 + 2y = 2x + 1$

b) $y + 2\sqrt{y} = x$

c) $y^2 + xy = 1$

4) Find $\frac{dy}{dt}$ and $\frac{d^2y}{dx^2}$ by Chain Rule. Expressing the results in terms of t

$y = x^4$, $x = \sqrt[3]{t}$

Applications of Derivatives

Rolle's Theorem مبرهنة رول

Let f be differentiable function on (a, b) and continuous on $[a, b]$, if $f(a) = f(b) = 0$ then there is at least one number c in (a, b) such that $f^-(c) = 0$.

Ex. $f(x) = x^2 - 2x - 3$ in $[-1,3]$

Sol. Clear that $f(x)$ is differentiable on (-1,3) and $f(x)$ is continuous on [-1,3] since (polynomial).

$$f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$$

$$f(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$$

$$\therefore f(-1) = f(3) = 0$$

Apply Rolle's Theorem

$f'(x) = 2x - 2$

$2x - 2 = 0 \implies x = 1$

$\therefore f'(1) = 0$ and $-1 < 1 < 3$

Ex.

Show that $f(x) = \frac{x^2-x-6}{x-1}$ satisfies the hypothesis of the Rolle's Theorem on [-2, 3] and find all values of *c* in the interval (-2, 3).

Sol.

$f(x)$ is discontinuous at $x = 1$ and 1 is point of discontinuity since $\lim_{x\to 1} f(x) =$ not exist therefore we can't apply Rolle's Theorem.

Ex.

$f(x) = x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$ in [0 , 8]

Sol.

$f'(x) = \frac{2}{3}x^{\frac{-1}{3}} - \frac{2}{3}x^{\frac{-2}{3}}$

HW.

Mean-Value Theorem مبرهنة القيمة الوسطى

Let f be differentiable function on (a, b) and continuous on $[a, b]$, then there is at least one number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Ex.

Show that $f(x) = \frac{x^3}{4} + 1$ satisfies the hypothesis of the Mean-Value Theorem on [0, 2] and find all values of c in the interval (0, 2).

Sol.

The function $f(x)$ is continuous and differentiable every where because it is a polynomial.

In particular, $f(x)$ is continuous on [0,2] and differentiable on (0,2), so the hypothesis is satisfied with $a = 0$, $b = 2$

$$f(a) = f(0) = 1 \quad , \quad f(ab) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4} \quad , \quad f'(c) = \frac{3c^2}{4}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{3c^2}{4} = \frac{3-1}{2-0} = 1 \quad \Rightarrow 3c^2 = 4 \Rightarrow c = \mp\frac{2}{\sqrt{3}}$$

Only the positive solution lies in the interval (0,2) therefor $c = \frac{2}{\sqrt{3}}$

Derivatives of Trigonometric Function

$\sin x$ جيب الزاوية , $\cos x$ جيب تمام الزاوية

$$\tan x = \frac{\sin x}{\cos x}$$ ظل الزاوية

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$ ظل تمام الزاوية

$\sec x = \frac{1}{\cos x}$ قاطع الزاوية , $\csc x = \frac{1}{\sin x}$ قاطع تمام الزاوية

$$\sin(-x) = -\sin x \quad , \quad \cos(-x) = \cos x$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \, , \, \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \, , \quad \csc^2\theta = 1 + \cot^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad , \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Ex.

Find $\frac{dy}{dx}$ if $y = x^2 \tan x$

Sol.

Using the product rule , we obtain

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\tan(x)) + \tan(x) \frac{d}{dx}(x^2)$$

$$= x^2\sec^2 x + 2x \tan x$$

Ex.

Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

Sol.

Using the quotient rule , we obtain

$$\frac{dy}{dx} = \frac{(1+\cos x)\cdot\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

Ex.

Find $y''(\pi/4)$ if $y(x) = \sec x$ $\quad (\pi/4 = 45^\circ)$

Sol.

$y'(x) = \sec x \tan x$

$y''(x) = \sec x \sec^2 x + \tan x \cdot \sec x \tan x$

Thus

$$y''\left(\pi/4\right) = \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)$$

$$= (\sqrt{2})^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$