Derivatives Logarithmic and Exponential Functions

Def.

The natural Logarithm of x is denoted by $\mathbf{ln}\ \boldsymbol{x}$ and is defined by the integral.

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad , \quad x > 0$$

Theorem:-

For any positive numbers a and c and any Rational number :-

a) $\ln ac = \ln a + \ln c$
b) $\ln \frac{1}{c} = -\ln c$
c) $\ln \frac{a}{c} = \ln a - \ln c$
d) $\ln a^r = r \ln a$

Rational no. is a number which can be expressed in the form $\frac{p}{q}$ wherein $q \neq 0$ and both p and q are integers.

Theorem:-

1- The domain of $\ln x$ is $(0, +\infty)$.
2- $\lim_{x\to 0^+} \ln x = +\infty$ and $\lim_{x\to +\infty} \ln x = +\infty$
3- The rang of $\ln x$ is $(-\infty, +\infty)$.

Derivatives Logarithmic:-

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$ wherein u is a differentiable function of x

Ex.

Find $\frac{d}{dx}[\ln(x^2+1)]$

Sol.

$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) = \frac{1}{x^2+1} \cdot (2x) = \frac{2x}{x^2+1}$

Ex.

Find $\frac{d}{dx}\left[\ln(\frac{x^2\sin x}{\sqrt{1+x}})\right]$

Sol.

$\frac{d}{dx}\left[\ln(\frac{x^2\sin x}{\sqrt{1+x}})\right] = \frac{d}{dx}[\ln(x^2\sin x) - \ln\sqrt{1+x}$

$= \frac{d}{dx}\left[\ln x^2 + \ln\sin x - \frac{1}{2}\ln(1+x)\right]$

$$= \frac{d}{dx}\left[2\ln x + \ln\sin x - \frac{1}{2}\ln(1+x)\right] = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}$

Def.

The Inverse of the natural logarithm function $\boldsymbol{\ln x}$ is denoted by $\boldsymbol{e^x}$ and is called the natural exponential function.

Theorem:-

The natural exponential function $\boldsymbol{e^x}$ is differentiable on $(-\infty, +\infty)$ and it has derivative :-

$$\frac{d}{dx}[e^x] = e^x$$

Note:-

If u is a differentiable function of x ,then

$$\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

Ex. Find

1) $\frac{d}{dx}\left[e^{x^3}\right] = e^{x^3} \cdot \frac{d}{dx}(x^3) = 3x^2 e^{x^3}$
2) $\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}(\cos x) = -\sin x \, e^{\cos x}$

Theorem:-

1) $\lim_{x\to+\infty} e^x = +\infty$ 2) $\lim_{x\to+\infty} e^{-x} = 0$

3) $\lim_{x\to-\infty} e^x = 0$ 4) $\lim_{x\to-\infty} e^{-x} = +\infty$

Notes:-

$\ln 1 = 0$, $\ln e = 1$, $\ln\frac{1}{e} = -1$, $\ln e^2 = 2$

$\boldsymbol{\ln(e^x) = x}$ for all real values of x , $\boldsymbol{e^{\ln x} = x}$ for $x > 0$

Ex.

Solve the equation $e^{2x-6} = 4$ for x.

Sol.

we take the natural logarithm of both sides of the equation and use the rule $\boldsymbol{\ln(e^x) = x}$

$\ln(e^{2x-6}) = \ln 4$

$2x - 6 = \ln 4$

$2x = 6 + \ln 4$

$x = 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{\frac{1}{2}}$

$\ln a^r = r \ln a$

$x = 3 + \ln 2$

Theorem:-

L'Hôpital's rule for form $\frac{0}{0}$ (قاعدة لوبيتال)

Suppose that f and g are differentiable function on an open interval containing $x = a$, except possibly at $x = a$ and that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ has a finite limit or if this limit is $-\infty$ or $+\infty$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$ or as $x \to +\infty$.

Ex.

$\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$

$= \lim_{x\to 0} \frac{\sin x}{1+2x}$ by L'Hôpital's rule

$= \frac{0}{1} = 0$

Ex.

$\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$

$=\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}$ (still $\frac{0}{0}$)

$=\lim\limits_{x\to 0}\frac{\sin x}{6x}$ (still $\frac{0}{0}$)

$=\lim\limits_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$

Theorem:-

L'Hôpital's rule for form $\frac{\infty}{\infty}$ (قاعدة لوبيتال)

Suppose that f and g are differentiable function on an open interval containing $x = a$, except possibly at $x = a$ and that

$$\lim_{x\to a} f(x) = \infty \quad \text{and} \quad \lim_{x\to a} g(x) = \infty$$

If $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit or if this limit is $-\infty$ or $+\infty$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$ or as $x\to +\infty$.

Exercises:

1) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$
2) $\lim\limits_{x\to 0}\frac{\sin 2x}{x}$
3) $\lim\limits_{x\to 0}(\frac{1}{x}-\frac{1}{\sqrt{x}})$

4) $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1}$

5) $\frac{d}{dx}[\ln|\sin x|]$

6) $\frac{d}{dx}[x^3 e^x]$

7) $\frac{d}{dx}[\frac{e^x}{\ln x}]$

8) $\frac{d}{dx}[\sin^2(\ln x)]$

Integration (التكامل)

A special symbol is used to denote the collection of all antiderivatives of a function f.

Def. The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)dx$$

The symbol 1 is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Note.

1) $\int dx = x + c$
2) $\int a\, dx = a \int dx$
3) $\int (dx + dy) = \int dx + \int dy$
4) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$
5) $\int \frac{1}{x} dx = \ln|x| + c = \ln x + c \quad , x > 0$
6) $\int e^x dx = e^x + c$
7) $\int a^x dx = \frac{a^x}{\ln a} + c$ **note:** $\frac{d}{dx}(a^x) = a^x \ln a$

Ex.

$$\int 2x dx = x^2 + c$$

$$\int (x^2 - 2x + 5)dx = \frac{x^3}{3} - x^2 + 5x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int (sec^2 x + \frac{1}{2\sqrt{x}})dx = \tan x + \sqrt{x} + c$$

Integration by parts (تكامل بالتجزئة)

The integration by parts formula

$$\int \boldsymbol{u(x)v'(x)dx = u(x)v(x)} - \int \boldsymbol{v(x)u'(x)dx}$$

Ex.Find $\int x \cos x \, dx$

Sol.

There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula.

$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. In this case we factor the expression $x \cos x$ into
$u(x) = x$ and $v'(x) = \cos x$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$u'(x) = 1$ and $v(x) = \sin x$

$$\int \underbrace{x}_{u(x)} \underbrace{\cos x}_{v'(x)} \, dx = \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)} \underbrace{(1)}_{u'(x)} \, dx$$

$= x \sin x + \cos x + c$

There are four apparent choices available for $u(x)$ and $v'(x)$ in Example.

1. Let $u(x) = 1$ and $v'(x) = x\cos x$. **2.** Let $u(x) = x$ and $v'(x) = \cos x$
3. Let $u(x) = x\cos x$ and $v'(x) = 1$ **4.** Let $u(x) = \cos x$ and $v'(x) = x$.

Choice 2 was used in Ex. The other three choices lead to integrals we don’t know how to integrate. For instance, Choice 3, with $u'(x) = \cos x - x\sin x$, leads to the integral.

$$\int (x\cos x - x^2 \sin x)dx$$

ملحوظة:

الهدف من التكامل بالتجزئة هو الأنتقال من الصيغة المعطاة في السؤال التي لا نمتلك رؤية واضحة لحلها إلى صيغة أبسط وأوضح . وبصورة عامة، نختار أولاً $v'(x)$ بحيث نستطيع أن نجري عليها التكامل بسهولة والجزء المتبقي نختاره $u(x)$ بحيث نستطيع ايجاد $v(x)$ من $v'(x)$.

Ex.

Find $\int \ln x \, dx$

Sol.

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Let $u(x) = \ln x$ and $v'(x) = 1$

$$\int \underset{u(x)}{\ln x} \cdot \underset{v'(x)}{1}\, dx = \underset{u(x)}{(\ln x)}\, \underset{v(x)}{x} - \int \underset{v(x)}{x}\, \underset{u'(x)}{\frac{1}{x}}\, dx$$

$$= x\ln x - x + c$$

The formula is often given in differential form. With

$v'(x)dx = dv$ and $u'(x)dx = du$, the integration by parts formula becomes

$$\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{vdu}$$

Ex.

Find $\int x^2 e^x dx$

Sol.

Let $u(x) = x^2$ and $v'(x) = e^x$

$\therefore \quad u'(x) = 2x$ and $v(x) = e^x$

Or if we use second formula $\int u\,dv = uv - \int vdu$

$u = x^2 \implies du = 2xdx$

$v = e^x \implies dv = e^x dx$

$$\int \underset{u}{x^2} \underbrace{e^x\,dx}_{dv} = \underset{u}{x^2}\,\underset{v}{e^x} - \int \underset{v}{e^x} \underbrace{2x\,dx}_{du}.$$

Integration by parts formula

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int \underset{u}{x} \underbrace{e^x\,dx}_{dv} = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x} \underbrace{dx}_{du} = xe^x - e^x + C.$$

Integration by parts Equation (2)
$u = x,\ dv = e^x\,dx$
$v = e^x, \quad du = dx$

Using this last evaluation, we then obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2\int xe^x\,dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C,$$