Euler's Method for Ordinary Differential Equations

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \; y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \; y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \; y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^2y^2=2\sin(3x),\ \ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^2y^2}{e^y},\ \ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^2y^2}{e^y}$$

Derivation of Euler's method

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0)=y_0$.

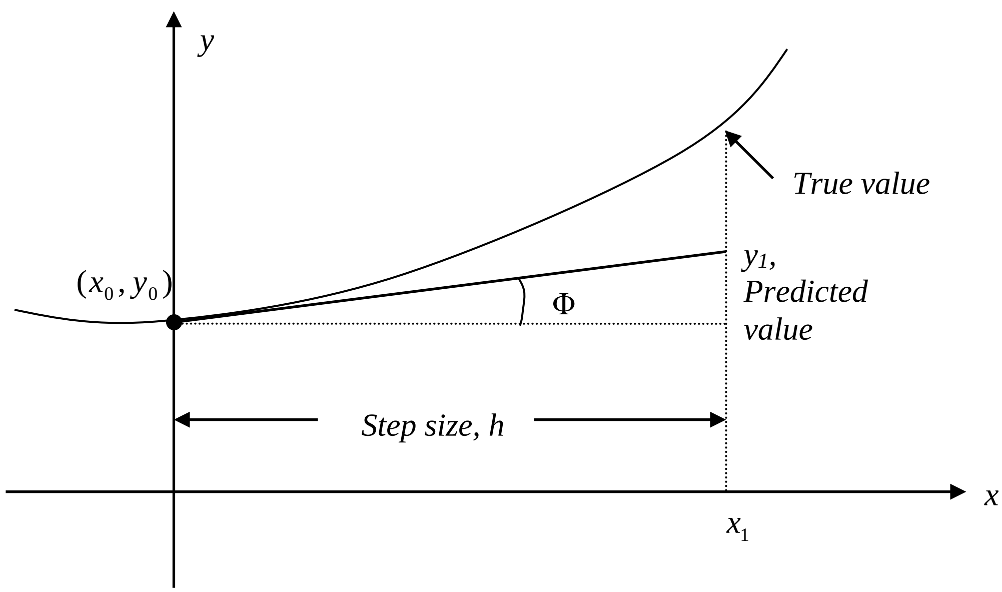

Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x=x_0$ as shown in Figure 1 is

$$\begin{aligned}\text{Slope} &= \frac{Rise}{Run}\\ &= \frac{y_1-y_0}{x_1-x_0}\\ &= f(x_0,y_0)\end{aligned}$$

From here

$$y_1=y_0+f(x_0,y_0)(x_1-x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \tag{2}$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{3}$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

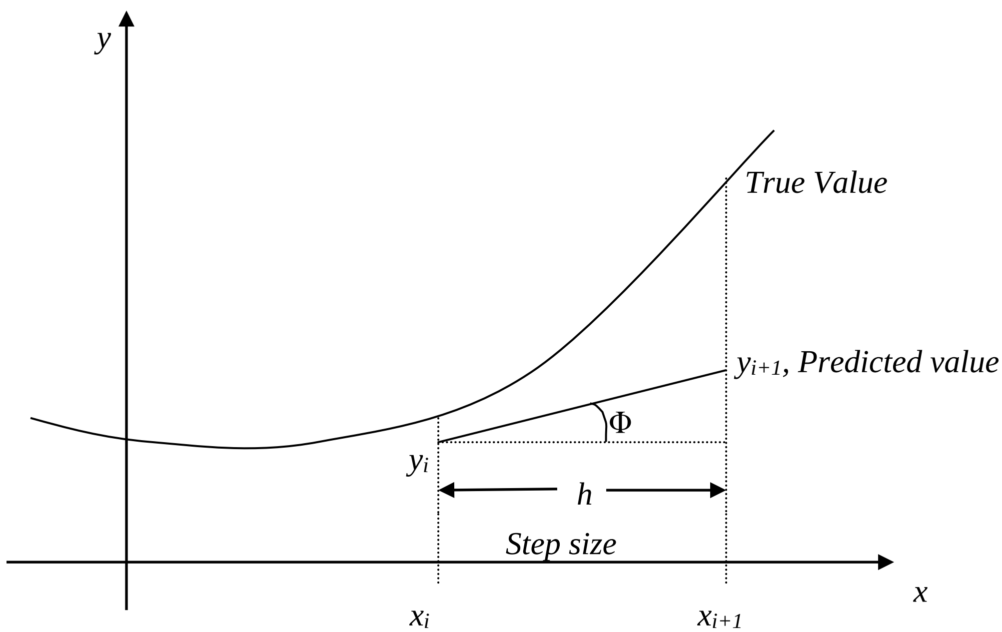

Figure 2 General graphical interpretation of Euler's method.

Example 3

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right),\ \theta(0) = 1200\text{K}$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

Per Equation (3), Euler's method reduces to

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200$

$$\begin{aligned}
\theta_1 &= \theta_0 + f(t_0,\theta_0)h \\
&= 1200 + f(0,1200)\times240 \\
&= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)\times240 \\
&= 1200 + (-4.5579)\times240 \\
&= 106.09\,\mathrm{K}
\end{aligned}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta_1 = \theta(240) \approx 106.09\,\mathrm{K}$$

For $i = 1$, $t_1 = 240$, $\theta_1 = 106.09$

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1,\theta_1)h \\
&= 106.09 + f(240,106.09)\times240 \\
&= 106.09 + \left(-2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right)\right)\times240 \\
&= 106.09 + (0.017595)\times240 \\
&= 110.32\,\mathrm{K}
\end{aligned}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta_2 = \theta(480) \approx 110.32\,\mathrm{K}$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 240$.

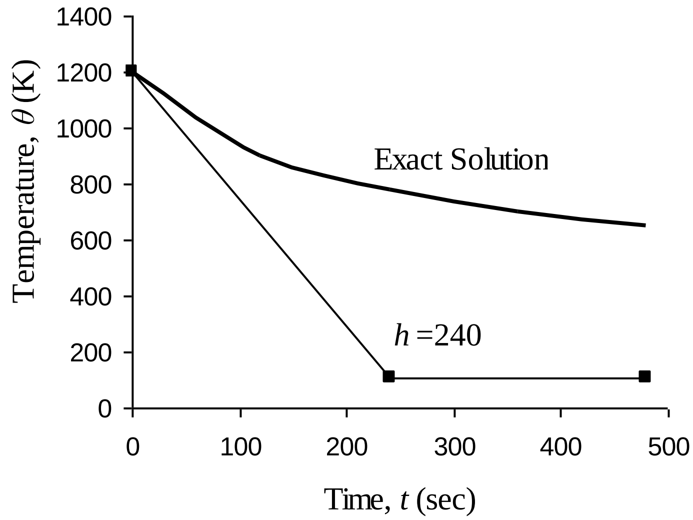

Figure 3 Comparing the exact solution and Euler's method.

The problem was solved again using a smaller step size. The results are given below in Table 1.

Table 1 Temperature at 480 seconds as a function of step size, h.

Step size, h	$\theta(480)$	E_t	$\lvert\in_t\rvert$ %
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

Figure 4 shows how the temperature varies as a function of time for different step sizes.

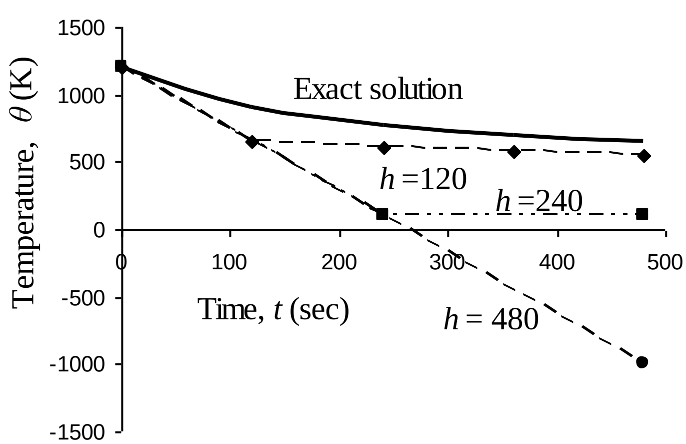

Figure 4 Comparison of Euler's method with the exact solution for different step sizes.

The values of the calculated temperature at $t = 480\text{s}$ as a function of step size are plotted in Figure 5.

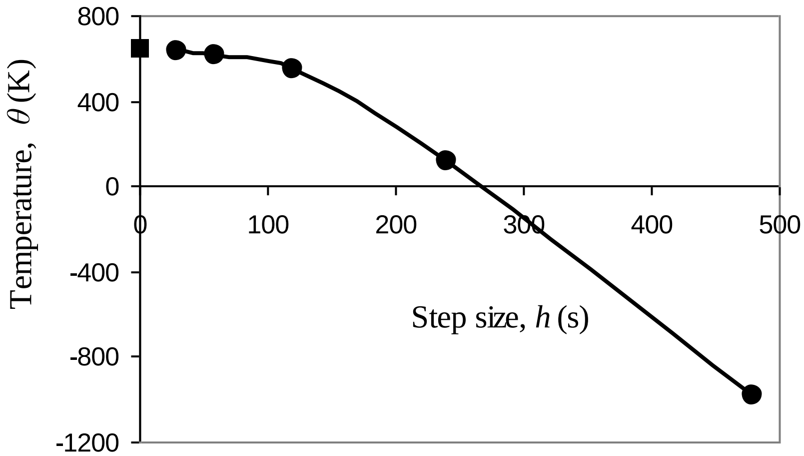

Figure 5 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta - 300}{\theta + 300} - 1.8519\tan^{-1}\left(0.333\times 10^{-2}\theta\right) = -0.22067\times 10^{-3}t - 2.9282 \qquad (4)$$

The solution to this nonlinear equation is

$$\theta = 647.57\text{K}$$

It can be seen that Euler's method has large errors. This can be illustrated using the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1} - x_i)^3 + \ldots \qquad (5)$$

$$= y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!}f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!}f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots \qquad (6)$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h$$

are Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots \qquad (7)$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error, being proportioned to the square of the step size, is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx .$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x), \quad y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

Example 4

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3, \; y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y), \; y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

<u>Step 1</u>

$$i = 0, \; x_0 = 5, \; y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + (6 \times 5^3) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

<u>Step 2</u>

$$i = 1, \; x_1 = 6.5, \; y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + (6 \times 6.5^3) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$