False-Position Method of Solving a Nonlinear Equation

Introduction

In the previous lecture, the bisection method was described as one of the simple bracketing methods of solving a nonlinear equation of the general form

$$f(x) = 0 \quad (1)$$

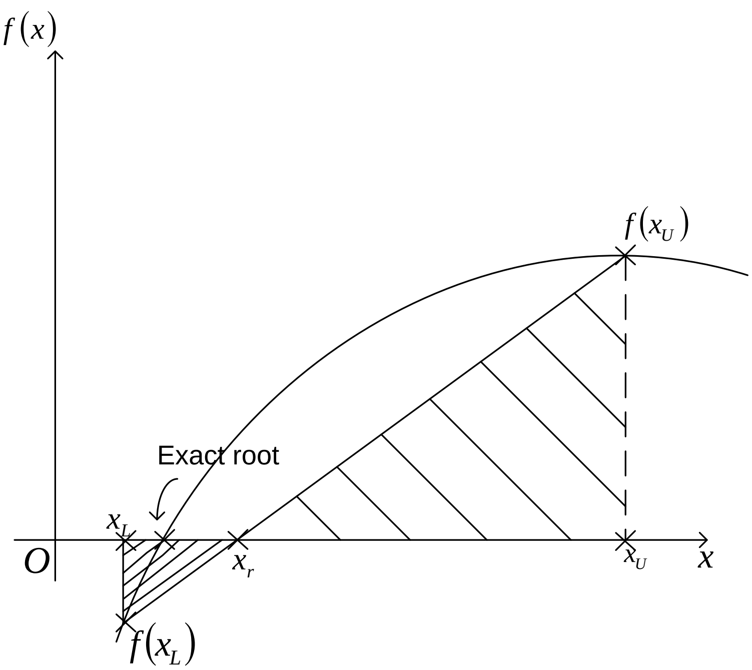

Figure 1 False-Position Method

The above nonlinear equation can be stated as finding the value of *x* such that Equation (1) is satisfied.
In the bisection method, we identify proper values of x_L (lower bound value) and x_U (upper bound value) for the current bracket, such that

$$f(x_L)\, f(x_U) < 0 \,. \quad (2)$$

The next predicted/improved root x_r can be computed as the midpoint between x_L and x_U as

$$x_r = \frac{x_L + x_U}{2} \quad (3)$$

The new upper and lower bounds are then established, and the procedure is repeated until the convergence is achieved (such that the new lower and upper bounds are sufficiently close to each other).

However, in the example shown in Figure 1, the bisection method may not be efficient because it does not take into consideration that $f(x_L)$ is much closer to the zero of the function $f(x)$ as compared to $f(x_U)$. In other words, the next predicted root x_r would be closer to x_L (in the example as shown in Figure 1), than the mid-point between x_L and x_U. The false-position method takes advantage of this observation mathematically by drawing a secant from the function value at x_L to the function value at x_U, and estimates the root as where it crosses the *x*-axis.

False-Position Method

Based on two similar triangles, shown in Figure 1, one gets

$$\frac{0-f(x_L)}{x_r-x_L}=\frac{0-f(x_U)}{x_r-x_U} \tag{4}$$

From Equation (4), one obtains

$$(x_r-x_L)f(x_U)=(x_r-x_U)f(x_L)$$

$$x_U f(x_L)-x_L f(x_U)=x_r\{f(x_L)-f(x_U)\}$$

The above equation can be solved to obtain the next predicted root x_m as

$$x_r=\frac{x_U f(x_L)-x_L f(x_U)}{f(x_L)-f(x_U)} \tag{5}$$

The above equation, through simple algebraic manipulations, can also be expressed as

$$x_r=x_U-\frac{f(x_U)}{\left\{\frac{f(x_L)-f(x_U)}{x_L-x_U}\right\}} \tag{6}$$

or

$$x_r=x_L-\frac{f(x_L)}{\left\{\frac{f(x_U)-f(x_L)}{x_U-x_L}\right\}} \tag{7}$$

Observe the resemblance of Equations (6) and (7) to the secant method.

False-Position Algorithm

The steps to apply the false-position method to find the root of the equation $f(x)=0$ are as follows.

1. Choose x_L and x_U as two guesses for the root such that $f(x_L)f(x_U)<0$, or in other words, $f(x)$ changes sign between x_L and x_U.
2. Estimate the root, x_r of the equation $f(x)=0$ as

$$x_r=\frac{x_U f(x_L)-x_L f(x_U)}{f(x_L)-f(x_U)}$$

3. Now check the following

If $f(x_L)f(x_r)<0$, then the root lies between x_L and x_r; then $x_L=x_L$ and $x_U=x_r$.

If $f(x_L)f(x_r) > 0$, then the root lies between x_r and x_U; then $x_L = x_r$ and $x_U = x_U$.
If $f(x_L)f(x_r) = 0$, then the root is x_r. Stop the algorithm.

4. Find the new estimate of the root

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

Find the absolute relative approximate error as

$$|\in_a| = \left|\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right| \times 100$$

where

x_r^{new} = estimated root from present iteration

x_r^{old} = estimated root from previous iteration

5. Compare the absolute relative approximate error $|\in_a|$ with the pre-specified relative error tolerance $\in_s$. If $|\in_a| > \in_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.
Note that the false-position and bisection algorithms are quite similar. The only difference is the formula used to calculate the new estimate of the root x_r as shown in steps #2 and #4!

Example 1

You are working for "DOWN THE TOILET COMPANY" that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5cm. You are asked to find the depth to which the ball is submerged when floating in water. The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the false-position method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of third iteration.

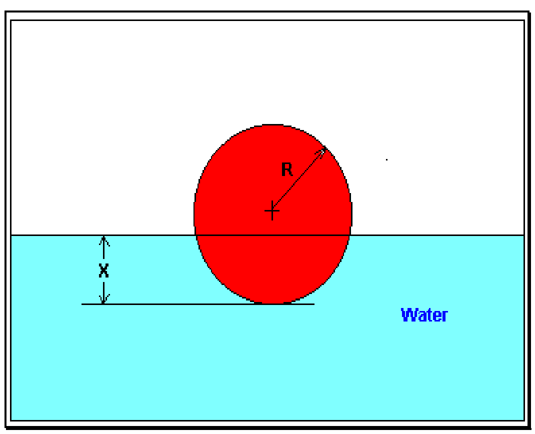

Figure 2 Floating ball problem.

Solution

From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$, where

$$R = \text{radius of the ball},$$

that is

$$0 \le x \le 2R$$
$$0 \le x \le 2(0.055)$$
$$0 \le x \le 0.11$$

Let us assume

$$x_L = 0,\ x_U = 0.11$$

Check if the function changes sign between x_L and x_U

$$f(x_L) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times 10^{-4} = 3.993\times 10^{-4}$$
$$f(x_U) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times 10^{-4} = -2.662\times 10^{-4}$$

Hence

$$f(x_L)f(x_U) = f(0)f(0.11) = \left(3.993\times 10^{-4}\right)\left(-2.662\times 10^{-4}\right) < 0$$

Therefore, there is at least one root between x_L and x_U, that is between 0 and 0.11.

Iteration 1

The estimate of the root is

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$
$$= \frac{0.11\times 3.993\times 10^{-4} - 0\times\left(-2.662\times 10^{-4}\right)}{3.993\times 10^{-4} - \left(-2.662\times 10^{-4}\right)}$$
$$= 0.0660$$
$$f(x_r) = f(0.0660)$$
$$= (0.0660)^3 - 0.165(0.0660)^2 + \left(3.993\times 10^{-4}\right)$$
$$= -3.1944\times 10^{-5}$$
$$f(x_L)f(x_r) = f(0)f(0.0660) = (+)(-) < 0$$

Hence, the root is bracketed between x_L and x_r, that is, between 0 and 0.0660. So, the lower and upper limits of the new bracket are $x_L = 0,\ x_U = 0.0660$, respectively.

Iteration 2

The estimate of the root is

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$
$$= \frac{0.0660\times 3.993\times 10^{-4} - 0\times\left(-3.1944\times 10^{-5}\right)}{3.993\times 10^{-4} - \left(-3.1944\times 10^{-5}\right)}$$
$$= 0.0611$$

The absolute relative approximate error for this iteration is

$$\in_a = \left|\frac{0.0611 - 0.0660}{0.0611}\right| \times 100 \cong 8\%$$

$$f(x_r) = f(0.0611)$$
$$= (0.0611)^3 - 0.165(0.0611)^2 + (3.993 \times 10^{-4})$$
$$= 1.1320 \times 10^{-5}$$
$$f(x_L)f(x_r) = f(0)f(0.0611) = (+)(+) > 0$$

Hence, the lower and upper limits of the new bracket are $x_L = 0.0611, x_U = 0.0660$, respectively.

Iteration 3

The estimate of the root is

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$
$$= \frac{0.0660 \times 1.132 \times 10^{-5} - 0.0611 \times (-3.1944 \times 10^{-5})}{1.132 \times 10^{-5} - (-3.1944 \times 10^{-5})}$$
$$= 0.0624$$

The absolute relative approximate error for this iteration is

$$\in_a = \left|\frac{0.0624 - 0.0611}{0.0624}\right| \times 100 \cong 2.05\%$$

$$f(x_r) = -1.1313 \times 10^{-7}$$
$$f(x_L)f(x_r) = f(0.0611)f(0.0624) = (+)(-) < 0$$

Hence, the lower and upper limits of the new bracket are $x_L = 0.0611, x_U = 0.0624$

All iterations results are summarized in Table 1. To find how many significant digits are at least correct in the last iterative value

$$|\in_a| \le 0.5 \times 10^{2-m}$$
$$2.05 \le 0.5 \times 10^{2-m}$$
$$m \le 1.387$$

The number of significant digits at least correct in the estimated root of 0.0624 at the end of 3rd iteration is 1.

Table 1 Root of $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$ for false-position method.

Iteration	x_L	x_U	x_r	$\|\in_a\|\%$	$f(x_m)$
1	0.0000	0.1100	0.0660	----	-3.1944×10^{-5}
2	0.0000	0.0660	0.0611	8.00	-1.1320×10^{-5}
3	0.0611	0.0660	0.0624	2.05	-1.1313×10^{-7}

Example 2

Find the root of $f(x)=(x-4)^2(x+2)=0$, using the initial guesses of $x_L=-2.5$ and $x_U=-1.0$, and a pre-specified tolerance of $\in_s = 0.1\%$.

Solution

The individual iterations are not shown for this example, but the results are summarized in Table 2. It takes five iterations to meet the pre-specified tolerance.

Table 2 Root of $f(x)=(x-4)^2(x+2)=0$ for false-position method.

Iteration	x_L	x_U	$f(x_L)$	$f(x_U)$	x_r	$\lvert\in_a\rvert\%$	$f(x_m)$
1	-2.5	-1	-21.13	25.00	-1.813	N/A	6.319
2	-2.5	-1.813	-21.13	6.319	-1.971	8.024	1.028
3	-2.5	-1.971	-21.13	1.028	-1.996	1.229	0.1542
4	-2.5	-1.996	-21.13	0.1542	-1.999	0.1828	0.02286
5	-2.5	-1.999	-21.13	0.02286	-2.000	0.02706	0.003383

To find how many significant digits are at least correct in the last iterative answer,

$$\lvert\in_a\rvert \le 0.5\times 10^{2-m}$$

$$0.02706 \le 0.5\times 10^{2-m}$$

$$m \le 3.2666$$

Hence, at least 3 significant digits can be trusted to be accurate at the end of the fifth iteration.

Reference

FALSE-POSITION METHOD OF SOLVING A NONLINEAR EQUATION	
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