On Solving Higher Order Equations for Ordinary Differential Equations

We have learned Euler's and Runge-Kutta methods to solve first order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \tag{1}$$

What do we do to solve simultaneous (coupled) differential equations, or differential equations that are higher than first order? For example an n^{th} order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_o y = f(x) \tag{2}$$

with $n-1$ initial conditions can be solved by assuming

$$y = z_1 \tag{3.1}$$

$$\frac{dy}{dx} = \frac{dz_1}{dx} = z_2 \tag{3.2}$$

$$\frac{d^2 y}{dx^2} = \frac{dz_2}{dx} = z_3 \tag{3.3}$$

$$\vdots$$

$$\frac{d^{n-1} y}{dx^{n-1}} = \frac{dz_{n-1}}{dx} = z_n \tag{3.n}$$

$$\begin{aligned} \frac{d^n y}{dx^n} &= \frac{dz_n}{dx} \\ &= \frac{1}{a_n}\left(-a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \ldots - a_1 \frac{dy}{dx} - a_0 y + f(x)\right) \\ &= \frac{1}{a_n}\left(-a_{n-1} z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right) \end{aligned} \tag{3.n+1}$$

The above Equations from (3.1) to (3.n+1) represent n first order differential equations as follows

$$\frac{dz_1}{dx} = z_2 = f_1(z_1, z_2, \ldots, x) \tag{4.1}$$

$$\frac{dz_2}{dx} = z_3 = f_2(z_1, z_2, \ldots, x) \tag{4.2}$$

$$\vdots$$

$$\frac{dz_n}{dx} = \frac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right) \tag{4.n}$$

Each of the n first order ordinary differential equations are accompanied by one initial condition. These first order ordinary differential equations are simultaneous in nature but can be solved by the methods used for solving first order ordinary differential equations that we have already learned.

Example 1

Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x},\ y(0) = 5,\ y'(0) = 7$$

Solution

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$

$$\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right)$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z,\ y(0) = 5$$

$$\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right),\ z(0) = 7.$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 2

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t},\ y(0) = 1,\ \frac{dy}{dt}(0) = 2,$$ find by Euler's method

a) $y(0.75)$

b) the absolute relative true error for part(a), if $y(0.75)\big|_{exact} = 1.668$

c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t,y,z),\ y(0) = 1 \qquad \text{(E2.1)}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t,y,z),\ \ z(0) = 2 \qquad \text{(E2.2)}$$

Using Euler's method on Equations (E2.1) and (E2.2), we get

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)h \qquad \text{(E2.3)}$$

$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)h \qquad \text{(E2.4)}$$

a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the value of $y(0.75)$.

For $i = 0, t_0 = 0,\ y_0 = 1,\ z_0 = 2$,

From Equation (E2.3)

$$\begin{aligned} y_1 &= y_0 + f_1(t_0, y_0, z_0)h \\ &= 1 + f_1(0,1,2)(0.25) \\ &= 1 + 2(0.25) \\ &= 1.5 \end{aligned}$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$

$$y_1 = y(0.25) \approx 1.5$$

From Equation (E2.4)

$$\begin{aligned} z_1 &= z_0 + f_2(t_0, y_0, z_0)h \\ &= 2 + f_2(0,1,2)(0.25) \\ &= 2 + \left(e^{-0} - 2(2) - 1\right)(0.25) \\ &= 1 \end{aligned}$$

z_1 is the approximate value of z (same as $\frac{dy}{dt}$) at $t = 0.25$

$$z_1 = z(0.25) \approx 1$$

For $i = 1,\ t_1 = 0.25,\ y_1 = 1.5,\ z_1 = 1$,

From Equation (E2.3)

$$\begin{aligned} y_2 &= y_1 + f_1(t_1, y_1, z_1)h \\ &= 1.5 + f_1(0.25, 1.5, 1)(0.25) \\ &= 1.5 + (1)(0.25) \\ &= 1.75 \end{aligned}$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$

$$y_2 = y(0.5) \approx 1.75$$

From Equation (E2.4)

$$\begin{aligned} z_2 &= z_1 + f_2(t_1, y_1, z_1)h \\ &= 1 + f_2(0.25, 1.5, 1)(0.25) \\ &= 1 + \left(e^{-0.25} - 2(1) - 1.5\right)(0.25) \\ &= 1 + (-2.7211)(0.25) \\ &= 0.31970 \end{aligned}$$

z_2 is the approximate value of z at

$$t = t_2 = 0.5$$

$$z_2 = z(0.5) \approx 0.31970$$

For $i = 2,\ t_2 = 0.5,\ y_2 = 1.75,\ z_2 = 0.31970$,

From Equation (E2.3)

$$\begin{aligned} y_3 &= y_2 + f_1(t_2, y_2, z_2)h \\ &= 1.75 + f_1(0.50, 1.75, 0.31970)(0.25) \\ &= 1.75 + (0.31970)(0.25) \\ &= 1.8299 \end{aligned}$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y(0.75) \approx 1.8299$$

From Equation (E2.4)

$$\begin{aligned} z_3 &= z_2 + f_2(t_2, y_2, z_2)h \\ &= 0.31972 + f_2(0.50, 1.75, 0.31970)(0.25) \\ &= 0.31972 + \left(e^{-0.50} - 2(0.31970) - 1.75\right)(0.25) \\ &= 0.31972 + (-1.7829)(0.25) \\ &= -0.1260 \end{aligned}$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \approx -0.12601$$

$$y(0.75) \approx y_3 = 1.8299$$

b) The exact value of $y(0.75)$ is

$$y(0.75)\big|_{exact} = 1.668$$

The absolute relative true error in the result from part (a) is

$$|\in_t| = \left|\frac{1.668-1.8299}{1.668}\right| \times 100$$
$$= 9.7062\%$$

c) $\frac{dy}{dx}(0.75) = z_3 \approx -0.12601$

Example 3

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2,$$

find by Heun's method

a) $y(0.75)$

b) $\frac{dy}{dx}(0.75)$.

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is rewritten as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), y(0) = 1 \quad \text{(E3.1)}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), z(0) = 2 \quad \text{(E3.2)}$$

Using Heun's method on Equations (1) and (2), we get

$$y_{i+1} = y_i + \frac{1}{2}\left(k_1^y + k_2^y\right)h \quad \text{(E3.3)}$$

$$k_1^y = f_1\left(t_i, y_i, z_i\right) \quad \text{(E3.4a)}$$

$$k_2^y = f_1\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \quad \text{(E 3.4b)}$$

$$z_{i+1} = z_i + \frac{1}{2}\left(k_1^z + k_2^z\right)h \quad \text{(E3.5)}$$

$$k_1^z = f_2\left(t_i, y_i, z_i\right) \quad \text{(E3.6a)}$$

$$k_2^z = f_2\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \tag{E3.6b}$$

For $i = 0,\ t_o = 0,\ y_o = 1,\ z_o = 2$

From Equation (E3.4a)

$$k_1^y = f_1(t_o, y_o, z_o)$$
$$= f_1(0,1,2)$$
$$= 2$$

From Equation (E3.6a)

$$k_1^z = f_2(t_0, y_0, z_0)$$
$$= f_2(0,1,2)$$
$$= e^{-0} - 2(2) - 1$$
$$= -4$$

From Equation (E3.4b)

$$k_2^y = f_1\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)$$
$$= f_1\left(0 + 0.25, 1 + (0.25)(2), 2 + (0.25)(-4)\right)$$
$$= f_1(0.25, 1.5, 1)$$
$$= 1$$

From Equation (E3.6b)

$$k_2^z = f_2\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)$$
$$= f_2\left(0 + 0.25, 1 + (0.25)(2), 2 + (0.25)(-4)\right)$$
$$= f_2(0.25, 1.5, 1)$$
$$= e^{-0.25} - 2(1) - 1.5$$
$$= -2.7212$$

From Equation (E3.3)

$$y_1 = y_0 + \frac{1}{2}\left(k_1^y + k_2^y\right)h$$
$$= 1 + \frac{1}{2}(2 + 1)(0.25)$$
$$= 1.375$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y(0.25) \cong 1.375$$

From Equation (E3.5)

$$z_1 = z_0 + \frac{1}{2}\left(k_1^z + k_2^z\right)h$$
$$= 2 + \frac{1}{2}(-4 + (-2.7212))(0.25)$$
$$= 1.1598$$

z_1 is the approximate value of z at

$$t = t_1 = 0.25$$
$$z_1 = z(0.25) \approx 1.1598$$

For $i=1,\ t_1=0.25,\ y_1=1.375,\ z_1=1.1598$

From Equation (E3.4a)

$$\begin{aligned} k_1^y &= f_1(t_1, y_1, z_1) \\ &= f_1(0.25, 1.375, 1.1598) \\ &= 1.1598 \end{aligned}$$

From Equation (E3.6a)

$$\begin{aligned} k_1^z &= f_2(t_1, y_1, z_1) \\ &= f_2(0.25, 1.375, 1.1598) \\ &= e^{-0.25} - 2(1.1598) - 1.375 \\ &= -2.9158 \end{aligned}$$

From Equation (E3.4b)

$$\begin{aligned} k_2^y &= f_1(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z) \\ &= f_1(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)) \\ &= f_1(0.50, 1.6649, 0.43087) \\ &= 0.43087 \end{aligned}$$

From Equation (E3.6b)

$$\begin{aligned} k_2^z &= f_2(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z) \\ &= f_2(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)) \\ &= f_2(0.50, 1.6649, 0.43087) \\ &= e^{-0.50} - 2(0.43087) - 1.6649 \\ &= -1.9201 \end{aligned}$$

From Equation (E3.3)

$$\begin{aligned} y_2 &= y_1 + \frac{1}{2}(k_1^y + k_2^y)h \\ &= 1.375 + \frac{1}{2}(1.1598 + 0.43087)(0.25) \\ &= 1.5738 \end{aligned}$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$

$$y_2 = y(0.50) \approx 1.5738$$

From Equation (E3.5)

$$\begin{aligned} z_2 &= z_1 + \frac{1}{2}(k_1^z + k_2^z)h \\ &= 1.1598 + \frac{1}{2}(-2.9158 + (-1.9201))(0.25) \\ &= 0.55533 \end{aligned}$$

z_2 is the approximate value of z at

$$t = t_2 = 0.50$$

$$z_2 = z(0.50) \approx 0.55533$$

For $i=2,\ t_2=0.50,\ y_2=1.57384,\ z_2=0.55533$

From Equation (E3.4a)

$$k_1^y = f_1(t_2, y_2, z_2)$$
$$= f_1(0.50, 1.5738, 0.55533)$$
$$= 0.55533$$

From Equation (E3.6a)

$$k_1^z = f_2(t_2, y_2, z_2)$$
$$= f_2(0.50, 1.5738, 0.55533)$$
$$= e^{-0.50} - 2(0.55533) - 1.5738$$
$$= -2.0779$$

From Equation (E3.4b)

$$k_2^y = f_2\left(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z\right)$$
$$= f_1(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779))$$
$$= f_1(0.75, 1.7126, 0.035836)$$
$$= 0.035836$$

From Equation (E3.6b)

$$k_2^z = f_2\left(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z\right)$$
$$= f_2(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779))$$
$$= f_2(0.75, 1.7126, 0.035836)$$
$$= e^{-0.75} - 2(0.035836) - 1.7126$$
$$= -1.3119$$

From Equation (E3.3)

$$y_3 = y_2 + \frac{1}{2}\left(k_1^y + k_2^y\right)h$$
$$= 1.5738 + \frac{1}{2}(0.55533 + 0.035836)(0.25)$$
$$= 1.6477$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.50 + 0.25 = 0.75$$
$$y_3 = y(0.75) \approx 1.6477$$

b) From Equation (E3.5)

$$z_3 = z_2 + \frac{1}{2}\left(k_1^z + k_2^z\right)h$$
$$= 0.55533 + \frac{1}{2}(-2.0779 + (-1.3119))(0.25)$$
$$= 0.13158$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$
$$z_3 = z(0.75) \cong 0.13158$$

The intermediate and the final results are shown in Table 1.

Table 1 Intermediate results of Heun's method.

i	0	1	2
t_i	0	0.25	0.50
y_i	1	1.3750	1.5738
z_i	2	1.1598	0.55533
k_1^y	2	1.1598	0.55533
k_1^z	-4	-2.9158	-2.0779
k_2^y	1	0.43087	0.035836
k_2^z	-2.7211	-1.9201	-1.3119
y_{i+1}	1.3750	1.5738	1.6477
z_{i+1}	1.1598	0.55533	0.13158

Reference

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Higher Order Equations
Summary	Textbook notes on higher order differential equations
Major	General Engineering
Authors	Autar Kaw
Last Revised	April 12, 2022