Runge-Kutta 2nd Order Method for Ordinary Differential Equations

What is the Runge-Kutta 2nd order method?

The Runge-Kutta 2nd order method is a numerical technique used to solve an ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Only first order ordinary differential equations can be solved by using the Runge-Kutta 2nd order method. In other sections, we will discuss how the Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x), \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y}, \ \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

Runge-Kutta 2nd order method

Euler's method is given by

$$y_{i+1} = y_i + f(x_i, y_i)h \quad (1)$$

where

$$x_0 = 0$$

$$y_0 = y(x_0)$$

$$h = x_{i+1} - x_i$$

To understand the Runge-Kutta 2nd order method, we need to derive Euler's method from the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2 y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3 y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$= y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots \quad (2)$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h$$

are Euler's method and hence can be considered to be the Runge-Kutta 1st order method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \quad (3)$$

So what would a 2nd order method formula look like. It would include one more term of the Taylor series as follows.

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 \quad (4)$$

Let us take a generic example of a first order ordinary differential equation

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5$$

$$f(x, y) = e^{-2x} - 3y$$

Now since y is a function of x,

$$f'(x, y) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} \quad (5)$$

$$= \frac{\partial}{\partial x}\left(e^{-2x} - 3y\right) + \frac{\partial}{\partial y}\left[\left(e^{-2x} - 3y\right)\right]\left(e^{-2x} - 3y\right)$$
$$= -2e^{-2x} + (-3)\left(e^{-2x} - 3y\right)$$
$$= -5e^{-2x} + 9y$$

The 2nd order formula for the above example would be

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$
$$= y_i + \left(e^{-2x_i} - 3y_i\right)h + \frac{1}{2!}\left(-5e^{-2x_i} + 9y_i\right)h^2$$

However, we already see the difficulty of having to find $f'(x, y)$ in the above method. What Runge and Kutta did was write the 2nd order method as

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \tag{6}$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \tag{7}$$

This form allows one to take advantage of the 2nd order method without having to calculate $f'(x, y)$.

So how do we find the unknowns a_1, a_2, p_1 and q_{11}. Without proof (see Appendix for proof), equating Equation (4) and (6) , gives three equations.

$$a_1 + a_2 = 1$$
$$a_2 p_1 = \frac{1}{2}$$
$$a_2 q_{11} = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally the value of a_2 is chosen to evaluate the other three constants. The three values generally used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, and are known as Heun's Method, the midpoint method and Ralston's method, respectively.

<u>Heun's Method</u>

Here $a_2 = \frac{1}{2}$ is chosen, giving

$$a_1 = \frac{1}{2}$$
$$p_1 = 1$$
$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \tag{8}$$

where

$$k_1 = f(x_i, y_i) \tag{9a}$$

$$k_2 = f(x_i + h, y_i + k_1 h) \tag{9b}$$

This method is graphically explained in Figure 1.

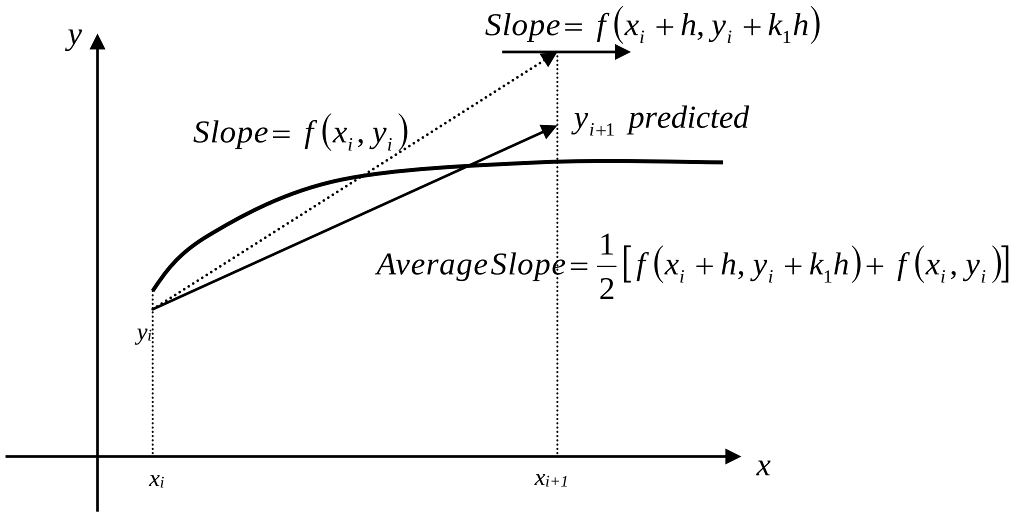

Figure 1 Runge-Kutta 2nd order method (Heun's method).

<u>Midpoint Method</u>

Here $a_2 = 1$ is chosen, giving

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h \tag{10}$$

where

$$k_1 = f(x_i, y_i) \tag{11a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \tag{11b}$$

<u>Ralston's Method</u>

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h \tag{12}$$

where

$$k_1 = f(x_i, y_i) \tag{13a}$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right) \tag{13b}$$

Example 3

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\ (\theta^4 - 81\times10^8)$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Runge-Kutta 2nd order method. Assume a step size of $h = 240$ seconds.

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

Per Heun's method given by Equations (8) and (9)

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(t_i, \theta_i)$$

$$k_2 = f(t_i + h, \theta_i + k_1 h)$$

$$i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$$

$$\begin{aligned} k_1 &= f(t_0, \theta_o) \\ &= f(0, 1200) \\ &= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\ &= -4.5579 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\ &= f(0 + 240, 1200 + (-4.5579)240) \\ &= f(240, 106.09) \\ &= -2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right) \\ &= 0.017595 \end{aligned}$$

$$\theta_1 = \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240$$

$$= 1200 + (-2.2702)240$$

$$= 655.16\text{K}$$

$$i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16\text{K}$$

$$k_1 = f(t_1, \theta_1)$$

$$= f(240, 655.16)$$

$$= -2.2067 \times 10^{-12}(655.16^4 - 81 \times 10^8)$$

$$= -0.38869$$

$$k_2 = f(t_1 + h, \theta_1 + k_1 h)$$

$$= f(240 + 240, 655.16 + (-0.38869)240)$$

$$= f(480, 561.87)$$

$$= -2.2067 \times 10^{-12}(561.87^4 - 81 \times 10^8)$$

$$= -0.20206$$

$$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240$$

$$= 655.16 + (-0.29538)240$$

$$= 584.27\text{K}$$

$$\theta_2 = \theta(480) = 584.27\text{K}$$

The results from Heun's method are compared with exact results in Figure 2.

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at $t = 480\text{s}$ is

$$\theta(480) = 647.57 \text{ K}$$

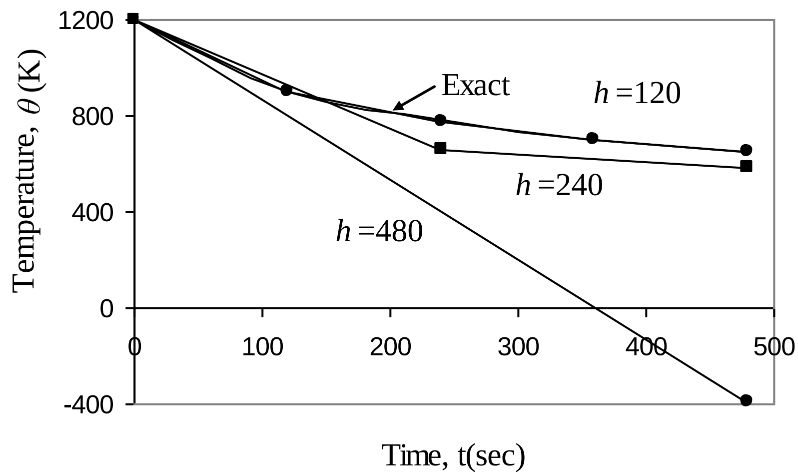

Figure 2 Heun's method results for different step sizes.

Using a smaller step size would increase the accuracy of the result as given in Table 1 and Figure 3 below.

Table 1 Effect of step size for Heun's method

Step size, h	$\theta(480)$	E_t	$\lvert\in_t\rvert\%$
480	-393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	-3.7762	0.58313
60	649.91	-2.3406	0.36145
30	648.21	-0.63219	0.097625

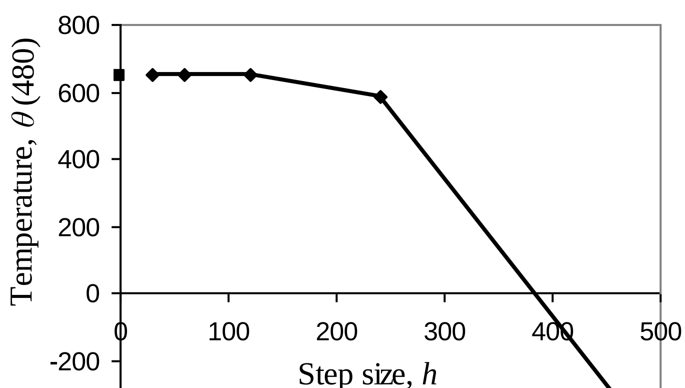

Figure 3 Effect of step size in Heun's method.

In Table 2, Euler's method and the Runge-Kutta 2nd order method results are shown as a function of step size,

Table 2 Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	-987.84	-393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

while in Figure 4, the comparison is shown over the range of time.

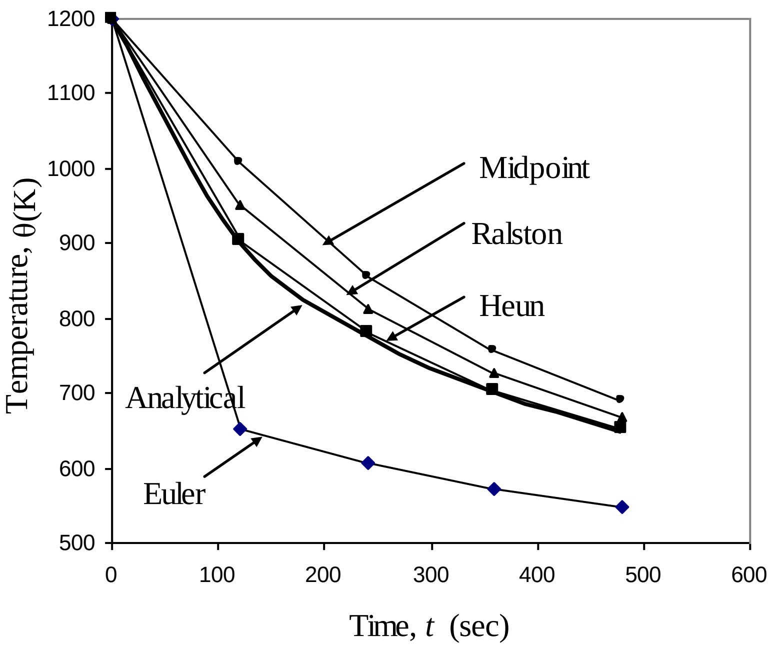

Figure 4 Comparison of Euler and Runge Kutta methods with exact results over time.

How do these three methods compare with results obtained if we found $f'(x, y)$ directly?

Of course, we know that since we are including the first three terms in the series, if the solution is a polynomial of order two or less (that is, quadratic, linear or constant), any of the three methods are exact. But for any other case the results will be different.

Let us take the example of

$$\frac{dy}{dx} = e^{-2x} - 3y,\ y(0) = 5.$$

If we directly find $f'(x, y)$, the first three terms of the Taylor series gives

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$

where

$$f(x,y) = e^{-2x} - 3y$$
$$f'(x,y) = -5e^{-2x} + 9y$$

For a step size of $h = 0.2$, using Heun's method, we find

$$y(0.6) = 1.0930$$

The exact solution

$$y(x) = e^{-2x} + 4e^{-3x}$$

gives

$$y(0.6) = e^{-2(0.6)} + 4e^{-3(0.6)}$$
$$= 0.96239$$

Then the absolute relative true error is

$$|\in_t| = \left|\frac{0.96239 - 1.0930}{0.96239}\right| \times 100$$
$$= 13.571\%$$

For the same problem, the results from Euler's method and the three Runge-Kutta methods are given in Table 3.

Table 3 Comparison of Euler's and Runge-Kutta 2nd order methods

	y(0.6)					
	Exact	Euler	Direct 2nd	Heun	Midpoint	Ralston
Value	0.96239	0.4955	1.0930	1.1012	1.0974	1.0994
$\|\in_t\|$ %		48.514	13.571	14.423	14.029	14.236

Reference

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge 2nd Order Method for Ordinary Differential Equations
Summary	Textbook notes on Runge 2nd order method for ODE
Major	General Engineering
Authors	Autar Kaw