Runge-Kutta 4th Order Method for Ordinary Differential Equations

What is the Runge-Kutta 4th order method?

Runge-Kutta 4^{th} order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4^{th} order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ \ y(0)=y_{0}\ \text{form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}},\ \ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1}=y_{i}+\left(a_{1}k_{1}+a_{2}k_{2}+a_{3}k_{3}+a_{4}k_{4}\right)h \tag{1}$$

where knowing the value of $y=y_{i}$ at x_{i}, we can find the value of $y=y_{i+1}$ at x_{i+1}, and

$$h=x_{i+1}-x_{i}$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1}=y_{i}+\frac{dy}{dx}\Big|_{x_{i},y_{i}}\left(x_{i+1}-x_{i}\right)+\frac{1}{2!}\frac{d^{2}y}{dx^{2}}\Big|_{x_{i},y_{i}}\left(x_{i+1}-x_{i}\right)^{2}+\frac{1}{3!}\frac{d^{3}y}{dx^{3}}\Big|_{x_{i},y_{i}}\left(x_{i+1}-x_{i}\right)^{3}+\frac{1}{4!}\frac{d^{4}y}{dx^{4}}\Big|_{x_{i},y_{i}}\left(x_{i+1}-x_{i}\right)^{4} \tag{2}$$

Knowing that $\frac{dy}{dx}=f(x,y)$ and $x_{i+1}-x_{i}=h$

$$y_{i+1}=y_{i}+f\left(x_{i},y_{i}\right)h+\frac{1}{2!}f^{'}\left(x_{i},y_{i}\right)h^{2}+\frac{1}{3!}f^{''}\left(x_{i},y_{i}\right)h^{3}+\frac{1}{4!}f^{'''}\left(x_{i},y_{i}\right)h^{4} \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1}=y_{i}+\frac{1}{6}\left(k_{1}+2k_{2}+2k_{3}+k_{4}\right)h \tag{4}$$

$$k_{1}=f\left(x_{i},y_{i}\right) \tag{5a}$$

$$k_{2}=f\left(x_{i}+\frac{1}{2}h,y_{i}+\frac{1}{2}k_{1}h\right) \tag{5b}$$

$$k_{3}=f\left(x_{i}+\frac{1}{2}h,y_{i}+\frac{1}{2}k_{2}h\right) \tag{5c}$$

$$k_{4}=f\left(x_{i}+h,y_{i}+k_{3}h\right) \tag{5d}$$

Example 3

A ball at 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200\text{K}$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method. Assume a step size of $h = 240$ seconds.

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200\text{K}$

$$\begin{aligned}
k_1 &= f(t_0,\theta_0) \\
&= f(0,1200) \\
&= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\
&= -4.5579
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)\times 240\right) \\
&= f(120, 653.05) \\
&= -2.2067\times10^{-12}\left(653.05^4 - 81\times10^8\right) \\
&= -0.38347
\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) \\
&= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)\times 240\right) \\
&= f(120, 1154.0) \\
&= -2.2067\times10^{-12}\left(1154.0^4 - 81\times10^8\right) \\
&= -3.8954
\end{aligned}$$

$$\begin{aligned}
k_4 &= f(t_0 + h, \theta_0 + k_3 h) \\
&= f(0 + 240, 1200 + (-3.894)\times 240) \\
&= f(240, 265.10) \\
&= -2.2067\times10^{-12}\left(265.10^4 - 81\times10^8\right)
\end{aligned}$$

$$= 0.0069750$$

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 1200 + \frac{1}{6}\left(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750)\right)240$$

$$= 1200 + (-2.1848) \times 240$$

$$= 675.65\text{K}$$

θ_1 is the approximate temperature at

$$t = t_1$$

$$= t_0 + h$$

$$= 0 + 240$$

$$= 240$$

$$\theta_1 = \theta(240)$$

$$\approx 675.65\text{K}$$

For $i = 1, t_1 = 240, \theta_1 = 675.65\text{K}$

$$k_1 = f(t_1, \theta_1)$$

$$= f(240, 675.65)$$

$$= -2.2067 \times 10^{-12}\left(675.65^4 - 81 \times 10^8\right)$$

$$= -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right)$$

$$= f(360, 622.61)$$

$$= -2.2067 \times 10^{-12}\left(622.61^4 - 81 \times 10^8\right)$$

$$= -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right)$$

$$= f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372) \times 240\right)$$

$$= f(360, 638.00)$$

$$= -2.2067 \times 10^{-12}\left(638.00^4 - 81 \times 10^8\right)$$

$$= -0.34775$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h)$$

$$= f(240 + 240, 675.65 + (-0.34775) \times 240)$$

$$= f(480, 592.19)$$

$$= 2.2067 \times 10^{-12}\left(592.19^4 - 81 \times 10^8\right)$$

$$= -0.25351$$

$$
\begin{aligned}
\theta_2 &= \theta_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 675.65 + \frac{1}{6}\left(-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351)\right) \times 240 \\
&= 675.65 + \frac{1}{6}(-2.0184) \times 240 \\
&= 594.91\text{K}
\end{aligned}
$$

θ_2 is the approximate temperature at

$$
\begin{aligned}
t &= t_2 \\
&= t_1 + h \\
&= 240 + 240 \\
&= 480
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= \theta(480) \\
&\approx 594.91\text{K}
\end{aligned}
$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method with different step sizes.

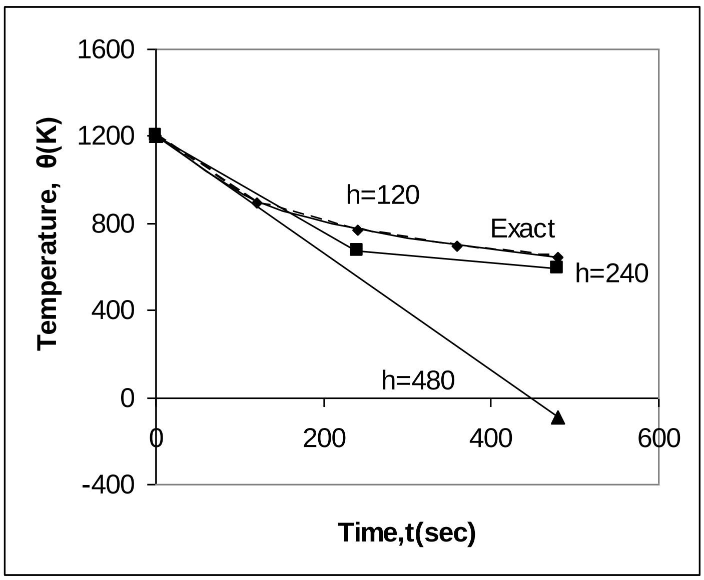

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 480$ seconds.

Table 1 Value of temperature at time, $t = 480\text{s}$ for different step sizes

Step size, h	$\theta(480)$	E_t	$\lvert\varepsilon_t\rvert$%
480	-90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

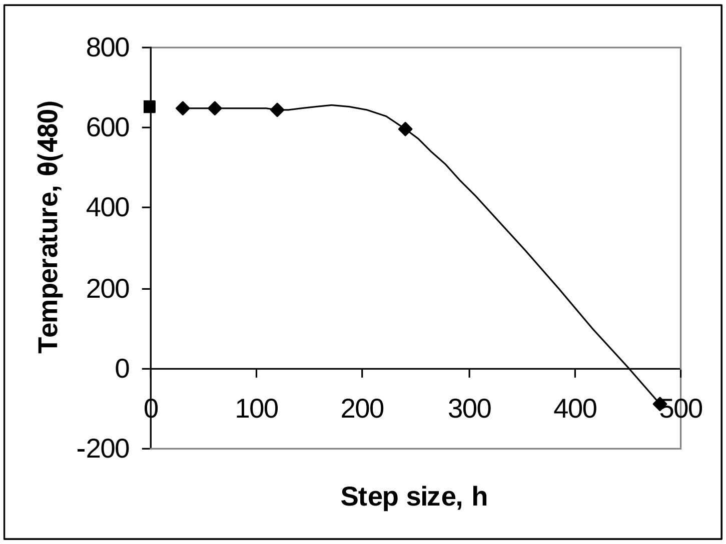

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method), and Runge-Kutta 4th order method.

The formula described in this chapter was developed by Runge. This formula is same as Simpson's 1/3 rule, if $f(x, y)$ were only a function of x. There are other versions of the 4th order method just like there are several versions of the second order methods. The formula developed by Kutta is

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h \tag{6}$$

where

$$k_1 = f(x_i, y_i) \tag{7a}$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right) \tag{7b}$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right) \tag{7c}$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3) \tag{7d}$$

This formula is the same as the Simpson's 3/8 rule, if $f(x, y)$ is only a function of x.

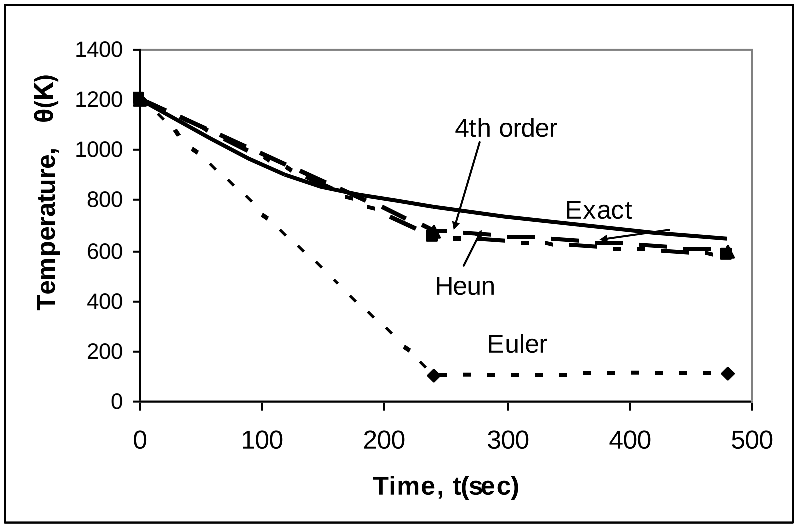

Figure 3 Comparison of Runge-Kutta methods of 1st (Euler), 2nd, and 4th order.

Reference

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
Major	General Engineering
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