Simpson’s 1/3 Rule of Integration

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson’s 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.
Here, we will discuss the Simpson’s 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)\,dx$$

where

$f(x)$ is called the integrand,
$a =$ lower limit of integration
$b =$ upper limit of integration

Simpson’s 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson’s 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

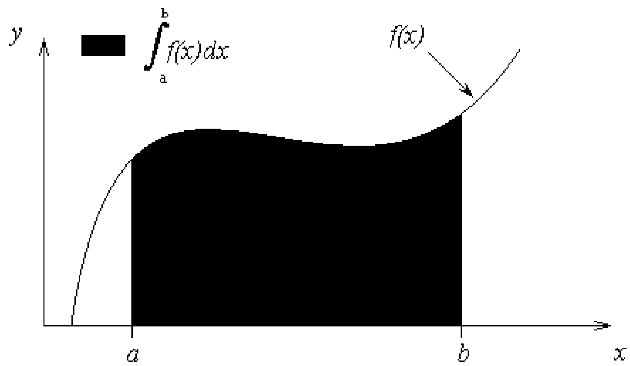

Figure 1 Integration of a function

<u>Method 1:</u>
Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2 .$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$
\begin{aligned}
I &\approx \int_a^b f_2(x)dx \\
&= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx \\
&= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b \\
&= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}
\end{aligned}
$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

<u>Method 2:</u>

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \dfrac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)\right] dx$$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right)$$

$$+ b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substituting values of b_0, b_1, and b_2 into this equation yields the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Method 3:

One could even use the Lagrange polynomial to derive Simpson’s formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \frac{a+b}{2}\right)(x-b)}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \frac{a+b}{2}\right)}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx = \left[\begin{array}{l} \dfrac{\frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \dfrac{\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) \\ + \dfrac{\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \end{array} \right]_a^b$$

$$= \frac{\dfrac{b^3-a^3}{3} - \dfrac{(a+3b)(b^2-a^2)}{4} + \dfrac{b(a+b)(b-a)}{2}}{\left(a-\dfrac{a+b}{2}\right)(a-b)} f(a)$$

$$+ \frac{\dfrac{b^3-a^3}{3} - \dfrac{(a+b)(b^2-a^2)}{2} + ab(b-a)}{\left(\dfrac{a+b}{2}-a\right)\left(\dfrac{a+b}{2}-b\right)} f\left(\frac{a+b}{2}\right)$$

$$+ \frac{\dfrac{b^3-a^3}{3} - \dfrac{(3a+b)(b^2-a^2)}{4} + \dfrac{a(a+b)(b-a)}{2}}{(b-a)\left(b-\dfrac{a+b}{2}\right)} f(b)$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

<u>Method 4:</u>

Simpson’s 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1dx$, $\int_a^b xdx$, and $\int_a^b x^2 dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1dx = b-a = c_1 + c_2 + c_3$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3-a^3}{3} = c_1 a^2 + c_2\left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above three equations for c_0, c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The integral from the first method

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2)dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

The distance covered by a rocket in meters from $t = 8s$ to $t = 30s$ is given by

$$x = \int_8^{30} \left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t \right) dt$$

a) Use Simpson's 1/3 rule to find the approximate value of x.
b) Find the true error, E_t.
c) Find the absolute relative true error, $|\in_t|$.

Solution

a) $$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$a = 8$
$b = 30$

$$\frac{a+b}{2} = 19$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000 - 2100(8)}\right] - 9.8(8) = 177.27 m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000 - 2100(30)}\right] - 9.8(30) = 901.67 m/s$$

$$f(19) = 2000\ln\left(\frac{140000}{140000 - 2100(19)}\right) - 9.8(19) = 484.75 m/s$$

$$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30)\right]$$
$$= \frac{22}{6}\left[177.27 + 4 \times 484.75 + 901.67\right]$$
$$=11065.72 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$
$$=11061.34 \text{ m}$$

So the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$=11061.34-11065.72$$
$$= -4.38\ m$$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{-4.38}{11061.34}\right| \times 100$$
$$= 0.0396\%$$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + ...$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, ..., n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + ...$$

$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + ... + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + ... + f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1\\i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\i=even}}^{n-2} f(x_i) + f(x_n)\right]}$$

Example 2

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s as given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

a) Use four segment Simpson's 1/3rd Rule to estimate x.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_t|$ for part (a).

Solution:

a) Using n segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0)+4\sum_{\substack{i=1\\i=odd}}^{n-1} f(t_i)+2\sum_{\substack{i=2\\i=even}}^{n-2} f(t_i)+f(t_n)\right]$$

$$n = 4$$
$$a = 8$$
$$b = 30$$
$$h = \frac{b-a}{n}$$
$$= \frac{30-8}{4}$$
$$= 5.5$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

So

$$f(t_0) = f(8)$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right]-9.8(8) = 177.27 m/s$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(13.5) = 2000\ln\left[\frac{140000}{140000-2100(13.5)}\right]-9.8(13.5) = 320.25 m/s$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right)-9.8(19) = 484.75 m/s$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(24.5) = 2000\ln\left[\frac{140000}{140000-2100(24.5)}\right]-9.8(24.5) = 676.05 m/s$$

$$f(t_4) = f(t_n) = f(30)$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67\,m/s$$

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(t_i) + f(t_n)\right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1\\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{2} f(t_i) + f(30)\right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30)\right]$$

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)\right]$$

$$= \frac{11}{6}\left[177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67\right]$$

$$= 11061.64\ m$$

b) The exact value of the above integral is

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt$$

=11061.34 m

So the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$E_t = 11061.34 - 11061.64$$

$$= -0.30\ m$$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{-0.3}{11061.34}\right| \times 100$$

= 0.0027%

Table 1 Values of Simpson’s 1/3 rule for Example 2 with multiple-segments

n	Approximate Value	E_t	$\|\in_t\|$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%

8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given[1] by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the *n* segments Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_2)$$

$$\vdots$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_i)$$

$$\vdots$$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

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[1] The $f^{(4)}$ in the true error expression stands for the fourth derivative of the function $f(x)$.

$$
\begin{aligned}
E_t &= \sum_{i=1}^{\frac{n}{2}} E_i \\
&= -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
&= -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
&= -\frac{(b-a)^5}{180n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}},
\end{aligned}
$$

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$ is an approximate average value of $f^{(4)}(x), a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{180n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$$

Reference

INTEGRATION	
Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	General Engineering
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Date	March 22, 2022